Solutions for Euler and Navier-Stokes Equations in Finite and Infinite Series of Time

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Abstract – We present solutions for the Euler and Navier-Stokes equations in finite and infinite series of time, in spatial dimension $N = 3$, firstly based on expansion in Taylor's series of time and then, in special case, solutions for velocity given by irrotational vectors, for incompressible flows and conservative external force, the Bernoulli's law. A little description of the Lamb's solution for Euler equations is done.

Keywords – Lagrange, Mécanique Analitique, exact differential, Euler's equations, Navier-Stokes equations, Taylor's series, series of time, Cauchy, Mémoire sur la Théorie des Ondes, Lagrange's theorem, Bernoulli's law, non-uniqueness solutions, Millenium Problem, velocity potential, Liouville's theorem, Helmholtz decomposition, Hodge decomposition.

§ 1

Let p, q, r be the three components of velocity of an element of fluid in the 3-D orthogonal Euclidean system of spatial coordinates (x, y, z) and t the time in this system.

Lagrange in his *Mécanique Analitique*, firstly published in 1788, proved that if the quantity $(p\,dx + q\,dy + r\,dz)$ is an exact differential when $t = 0$ it will also be an exact differential when t has any other value. If the quantity $(p\,dx + q\,dy + r\,dz)$ is an exact differential at an arbitrary instant, it should be such for all other instants. Consequently, if there is one instant during the motion for which it is not an exact differential, it cannot be exact for the entire period of motion. If it were exact at another arbitrary instant, it should also be exact at the first instant.[1]

To prove it Lagrange used

$$(1.1)\qquad \begin{cases} p = p^I + p^{II}t + p^{III}t^2 + p^{IV}t^3 + \cdots \\ q = q^I + q^{II}t + q^{III}t^2 + q^{IV}t^3 + \cdots \\ r = r^I + r^{II}t + r^{III}t^2 + r^{IV}t^3 + \cdots \end{cases}$$

in which the quantities p^I, p^{II}, p^{III}, etc., q^I, q^{II}, q^{III}, etc., r^I, r^{II}, r^{III}, etc., are functions of x, y, z but without t.

Here we will finally solve the equations of Euler and Navier-Stokes using this representation of the velocity components in infinite series, as pointed by Lagrange. We assume satisfied the condition of incompressibility, for brevity.

Without it the resulting equations are more complicated, as we know, but the method of solution is essentially the same in both cases. We focus our attention in the general case of the Navier-Stokes equations, with $\nu \geq 0$ constant, and for the Euler equations simply set the viscosity coefficient as $\nu = 0$.

To facilitate and abbreviate our writing, we represent the fluid velocity by its three components in indicial notation, i.e., $u = (u_1, u_2, u_3)$, as well as the external force will be $f = (f_1, f_2, f_3)$ and the spatial coordinates $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$. The pressure, a scalar function, will be represented as p. As frequently used in mathematics approach, the density mass will be $\rho = 1$. We consider all functions belonging to C^∞, being valid the use of $\frac{\partial^2 u_i}{\partial x_j \partial x_k} = \frac{\partial^2 u_i}{\partial x_k \partial x_j}$ and other inversions in order of derivatives, so much in relation to space as to time.

The representation (1.1) is as the expansion of the velocity in a Taylor´s series in relation to time around $t = 0$, considering x, y, z as constant, i.e., for $1 \leq i \leq 3$,

(1.2) $$u_i = u_i|_{t=0} + \frac{\partial u_i}{\partial t}|_{t=0}\, t + \frac{\partial^2 u_i}{\partial t^2}|_{t=0}\, \frac{t^2}{2} + \frac{\partial^3 u_i}{\partial t^3}|_{t=0}\, \frac{t^3}{6} + \cdots + \frac{\partial^k u_i}{\partial t^k}|_{t=0}\, \frac{t^k}{k!} + \cdots$$

or

(1.3) $$u_i = u_i^0 + \sum_{k=1}^{\infty} \frac{\partial^k u_i}{\partial t^k}|_{t=0}\, \frac{t^k}{k!}.$$

For the calculation of $\frac{\partial u_i}{\partial t}, \frac{\partial^2 u_i}{\partial t^2}, \frac{\partial^3 u_i}{\partial t^3}, \ldots$ we use the values that are obtained directly from the Navier-Stokes equations and its derivatives in relation to time, i.e.,

(1.4) $$\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} + \nu\, \nabla^2 u_i + f_i,$$

and therefore

(1.5) $$\frac{\partial^2 u_i}{\partial t^2} = -\frac{\partial^2 p}{\partial t\, \partial x_i} - \sum_{j=1}^{3} \left(\frac{\partial u_j}{\partial t} \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} \right) + \nu\, \nabla^2 \frac{\partial u_i}{\partial t} + \frac{\partial f_i}{\partial t},$$

(1.6) $$\frac{\partial^3 u_i}{\partial t^3} = -\frac{\partial^3 p}{\partial t^2\, \partial x_i} - \sum_{j=1}^{3} \left(\frac{\partial^2 u_j}{\partial t^2} \frac{\partial u_i}{\partial x_j} + 2 \frac{\partial u_j}{\partial t} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} \frac{\partial^2 u_i}{\partial t^2} \right) + \nu\, \nabla^2 \frac{\partial^2 u_i}{\partial t^2} + \frac{\partial^2 f_i}{\partial t^2},$$

(1.7) $$\frac{\partial^4 u_i}{\partial t^4} = -\frac{\partial^4 p}{\partial t^3\, \partial x_i} - \sum_{j=1}^{3} N_j^3 + \nu\, \nabla^2 \frac{\partial^3 u_i}{\partial t^3} + \frac{\partial^3 f_i}{\partial t^3},$$

$$N_j^3 = \frac{\partial}{\partial t} N_j^2,\ N_j^2 = \frac{\partial^2 u_j}{\partial t^2}\frac{\partial u_i}{\partial x_j} + 2\frac{\partial u_j}{\partial t}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t} + u_j\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2},$$

$$N_j^3 = \frac{\partial^3 u_j}{\partial t^3}\frac{\partial u_i}{\partial x_j} + 3\frac{\partial^2 u_j}{\partial t^2}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t} + 3\frac{\partial u_j}{\partial t}\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2} + u_j\frac{\partial}{\partial x_j}\frac{\partial^3 u_i}{\partial t^3},$$

(1.8)
$$\frac{\partial^5 u_i}{\partial t^5} = -\frac{\partial^5 p}{\partial t^4\,\partial x_i} - \sum_{j=1}^3 N_j^4 + \nu\,\nabla^2\frac{\partial^4 u_i}{\partial t^4} + \frac{\partial^4 f_i}{\partial t^4},$$

$$N_j^4 = \frac{\partial}{\partial t}N_j^3 = \frac{\partial^4 u_j}{\partial t^4}\frac{\partial u_i}{\partial x_j} + 4\frac{\partial^3 u_j}{\partial t^3}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t} + 6\frac{\partial^2 u_j}{\partial t^2}\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2} +$$

$$+4\frac{\partial u_j}{\partial t}\frac{\partial}{\partial x_j}\frac{\partial^3 u_i}{\partial t^3} + u_j\frac{\partial}{\partial x_j}\frac{\partial^4 u_i}{\partial t^4},$$

and using induction we come to

(1.9)
$$\frac{\partial^k u_i}{\partial t^k} = -\frac{\partial^k p}{\partial t^{k-1}\,\partial x_i} - \sum_{j=1}^3 N_j^{k-1} + \nu\,\nabla^2\frac{\partial^{k-1} u_i}{\partial t^{k-1}} + \frac{\partial^{k-1} f_i}{\partial t^{k-1}},$$

$$N_j^{k-1} = \frac{\partial}{\partial t}N_j^{k-2} = \sum_{l=0}^{k-1}\binom{k-1}{l}\partial_t^{k-1-l}u_j\,\frac{\partial}{\partial x_j}\partial_t^l u_i,$$

$$\partial_t^0 u_n = u_n,\ \partial_t^m u_n = \frac{\partial^m u_n}{\partial t^m},\ \binom{k-1}{l} = \frac{(k-1)!}{(k-1-l)!\,l!}.$$

In (1.2) and (1.3) it is necessary to know the values of the derivatives $\frac{\partial u_i}{\partial t}, \frac{\partial^2 u_i}{\partial t^2}, \dots, \frac{\partial^k u_i}{\partial t^k}$ in $t = 0$ then we must to calculate, from (1.4) to (1.9),

(1.10)
$$\frac{\partial u_i}{\partial t}\Big|_{t=0} = -\frac{\partial p^0}{\partial x_i} - \sum_{j=1}^3 u_j^0\frac{\partial u_i^0}{\partial x_j} + \nu\,\nabla^2 u_i^0 + f_i^0,$$

the superior index 0 meaning the value of the respective function at $t = 0$, and

(1.11)
$$\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0} = -\frac{\partial^2 p}{\partial t\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^3 N_j^1\Big|_{t=0} +$$

$$+\nu\,\nabla^2\frac{\partial u_i}{\partial t}\Big|_{t=0} + \frac{\partial f_i}{\partial t}\Big|_{t=0},$$

$$N_j^1\Big|_{t=0} = \sum_{j=1}^3\left(\frac{\partial u_j}{\partial t}\Big|_{t=0}\frac{\partial u_i^0}{\partial x_j} + u_j^0\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}\Big|_{t=0}\right),$$

(1.12)
$$\frac{\partial^3 u_i}{\partial t^3}\Big|_{t=0} = -\frac{\partial^3 p}{\partial t^2\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^3 N_j^2\Big|_{t=0} +$$

$$+\nu\,\nabla^2\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0} + \frac{\partial^2 f_i}{\partial t^2}\Big|_{t=0},$$

$$N_j^2\Big|_{t=0} = \frac{\partial^2 u_j}{\partial t^2}\Big|_{t=0}\frac{\partial u_i^0}{\partial x_j} + 2\frac{\partial u_j}{\partial t}\Big|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}\Big|_{t=0} +$$

$$+u_j^0\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0},$$

(1.13)
$$\frac{\partial^4 u_i}{\partial t^4}\Big|_{t=0} = -\frac{\partial^4 p}{\partial t^3\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^3 N_j^3\Big|_{t=0} +$$

$$+\nu\,\nabla^2\frac{\partial^3 u_i}{\partial t^3}|_{t=0}+\frac{\partial^3 f_i}{\partial t^3}|_{t=0},$$

$$N_j^3|_{t=0}=\frac{\partial^3 u_j}{\partial t^3}|_{t=0}\frac{\partial u_i^0}{\partial x_j}+3\frac{\partial^2 u_j}{\partial t^2}|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}|_{t=0}+$$

$$+3\frac{\partial u_j}{\partial t}|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2}|_{t=0}+u_j^0\frac{\partial}{\partial x_j}\frac{\partial^3 u_i}{\partial t^3}|_{t=0},$$

(1.14)
$$\frac{\partial^5 u_i}{\partial t^5}|_{t=0}=-\frac{\partial^5 p}{\partial t^4\,\partial x_i}|_{t=0}-\sum_{j=1}^3 N_j^4|_{t=0}+$$

$$+\nu\,\nabla^2\frac{\partial^4 u_i}{\partial t^4}|_{t=0}+\frac{\partial^4 f_i}{\partial t^4}|_{t=0},$$

$$N_j^4|_{t=0}=\frac{\partial^4 u_j}{\partial t^4}|_{t=0}\frac{\partial u_i^0}{\partial x_j}+4\frac{\partial^3 u_j}{\partial t^3}|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}+$$

$$+6\frac{\partial^2 u_j}{\partial t^2}|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2}|_{t=0}+4\frac{\partial u_j}{\partial t}|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial^3 u_i}{\partial t^3}|_{t=0}+$$

$$+u_j^0\frac{\partial}{\partial x_j}\frac{\partial^4 u_i}{\partial t^4}|_{t=0},$$

and of generic form,

(1.15)
$$\frac{\partial^k u_i}{\partial t^k}|_{t=0}=-\frac{\partial^k p}{\partial t^{k-1}\,\partial x_i}|_{t=0}-\sum_{j=1}^3 N_j^{k-1}|_{t=0}+$$

$$+\nu\,\nabla^2\frac{\partial^{k-1} u_i}{\partial t^{k-1}}|_{t=0}+\frac{\partial^{k-1} f_i}{\partial t^{k-1}}|_{t=0},$$

$$N_j^{k-1}|_{t=0}=\sum_{l=0}^{k-1}\binom{k-1}{l}\partial_t^{k-1-l}u_j|_{t=0}\,\frac{\partial}{\partial x_j}\partial_t^l u_i|_{t=0},$$

$$\partial_t^0 u_n|_{t=0}=u_n^0,\ \partial_t^m u_n|_{t=0}=\frac{\partial^m u_n}{\partial t^m}|_{t=0}.$$

If the external force is conservative there is a scalar potential U such as $f=\nabla U$ and the pressure can be calculated from this potential U, i.e.,

(1.16)
$$\frac{\partial p}{\partial x_i}=f_i=\frac{\partial U}{\partial x_i},$$

and then

(1.17)
$$p=U+\theta(t),$$

$\theta(t)$ a generic function of time of class C^∞, so it is not necessary the use of the pressure p and external force f, and respective derivatives, in (1.4) to (1.15) if the external force is conservative. In this case, the velocity can be independent of the both pressure and external force, otherwise it will be necessary to use both the pressure and external force derivatives to calculate the velocity in powers of time.

The result that we obtain here in this development in Taylor's series seems to me a great advance in the search of the solutions of the Euler's and Navier-Stokes equations. It is possible now to know on the possibility of non-uniqueness solutions as well as breakdown solution respect to unbounded energy of another manner. We now can choose previously an infinity of different pressures such that the calculation of $\frac{\partial u}{\partial t}$ and derivatives can be done, for a given initial velocity and external force, although such calculation can be very hard.

It is convenient say that Cauchy[2] in his memorable and admirable *Mémoire sur la Théorie des Ondes*, winner of the Mathematical Analysis award, year 1815, firstly does a study on the equations to be obeyed by three-dimensional molecules in a homogeneous fluid in the initial instant $t = 0$, coming to the conclusion which the initial velocity must be irrotational, i.e., a potential flow. Of this manner, after, he comes to conclusion that the velocity is always irrotational, potential flow, if the external force is conservative, which is essentially the Lagrange's theorem described in the begin of this article, but it is shown without the use of series expansion (a possible exception to the theorem occurs if one or two components of velocity are identically zero, when the reasonings on 3-D molecular volume are not valid). The solution obtained by Cauchy for Euler's equations is the Bernoulli's law, as almost always happens. Now at first a more generic solution is obtained, in special when it is possible a solution be expanded in polynomial series of time. Though not always a function can be expanded in Taylor's series, there is certainly an infinity of possible cases of solutions where this is possible.

If the mentioned series is divergent in some point or region may be an indicative of that the correspondent velocity and its square diverge, again going to the case of breakdown solution due to unbounded energy. With the three functions initial velocity, pressure and external force belonging to Schwartz Space is expected that the solution for velocity also belongs to Schwartz Space, obtaining physically reasonable and well-behaved solution throughout the space.

The method presented here in this first section can also be applied in other equations, of course, for example in the heat equation, Schrödinger equation, wave equation and many others. Always will be necessary that the remainder in the Taylor's series goes to zero when the order k of the derivative tends to infinity (Courant[3], chap. VI). Applying this concept in (1.3) and (1.9), substituting t by τ, the remainder $R_{i,k}$ of order k for velocity component i is

(1.18) $$R_{i,k} = \frac{1}{k!}\int_0^t (t-\tau)^k \frac{\partial^{k+1} u_i}{\partial t^{k+1}} d\tau,$$

which can be estimated by Lagrange's remainder,

(1.19) $$R_{i,k} = \frac{t^{k+1}}{(k+1)!}\frac{\partial^{k+1} u_i}{\partial t^{k+1}}(\xi),$$

or by Cauchy's remainder,

(1.20) $$R_{i,k} = \frac{t^{k+1}}{k!}(1-\theta)^k \frac{\partial^{k+1} u_i}{\partial t^{k+1}}(\xi),$$

with $0 \le \xi \le t$ and $0 \le \theta \le 1$.

§ 2

In this section we will build a series of powers of time solving the Navier-Stokes equations, differently than that used in the previous section. From theorem of uniqueness of series of powers (*A function $f(x)$ can be represented by a power series in x in only one way, if it all, i.e., the representation of a function by a power series is "unique"; Every power series which converges for points other than $x = 0$ is the Taylor series of the function which it represents* (Courant[3], chap. VIII)), both solutions need be the same, for a same initial velocity, pressure, external force, compressibility condition and all boundary conditions.

Defining

(2.1) $$u_i = u_i^0 + X_{i,1}t + X_{i,2}t^2 + \cdots + X_{i,n}t^n + \cdots = \sum_{n=0}^{\infty} X_{i,n}t^n,$$
$$X_{i,0} = u_i^0 = u_i(x_1, x_2, x_3, 0),$$

where each $X_{i,n}$ is a function of position (x_1, x_2, x_3), without t, and

(2.2) $$\frac{\partial p}{\partial x_i} = q_i^0 + q_{i,1}t + q_{i,2}t^2 + \cdots + q_{i,n}t^n + \cdots = \sum_{n=0}^{\infty} q_{i,n}t^n,$$
$$q_{i,0} = q_i^0 = \frac{\partial p^0}{\partial x_i},\ p^0 = p(x_1, x_2, x_3, 0),$$

(2.3) $$f_i = f_i^0 + f_{i,1}t + f_{i,2}t^2 + \cdots + f_{i,n}t^n + \cdots = \sum_{n=0}^{\infty} f_{i,n}t^n,$$
$$f_{i,0} = f_i^0 = f_i(x_1, x_2, x_3, 0),$$

we can put these series in the Navier-Stokes equation

(2.4) $$\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} + \nu\, \nabla^2 u_i + f_i.$$

The velocity derivative in relation to time is

(2.5) $$\frac{\partial u_i}{\partial t} = X_{i,1} + 2X_{i,2}t + 3X_{i,3}t^2 + \cdots + nX_{i,n}t^{n-1} + \cdots =$$
$$= \sum_{n=0}^{\infty}(n+1)X_{i,n+1}t^n,$$

the nonlinear terms are, of order zero (constant in time),

(2.6) $$\sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j},$$

of order 1,

$$(2.7) \qquad \sum_{j=1}^{3}\left(u_j^0\frac{\partial X_{i,1}}{\partial x_j}+X_{j,1}\frac{\partial u_i^0}{\partial x_j}\right)t,$$

of order 2,

$$(2.8) \qquad \sum_{j=1}^{3}\left(u_j^0\frac{\partial X_{i,2}}{\partial x_j}+X_{j,1}\frac{\partial X_{i,1}}{\partial x_j}+X_{j,2}\frac{\partial u_i^0}{\partial x_j}\right)t^2,$$

of order 3,

$$(2.9) \qquad \sum_{j=1}^{3}\left(u_j^0\frac{\partial X_{i,3}}{\partial x_j}+X_{j,1}\frac{\partial X_{i,2}}{\partial x_j}+X_{j,2}\frac{\partial X_{i,1}}{\partial x_j}+X_{j,3}\frac{\partial u_i^0}{\partial x_j}\right)t^3,$$

and of order n, of generic form, equal to

$$(2.10) \qquad \sum_{j=1}^{3}\sum_{k=0}^{n}X_{j,k}\frac{\partial X_{i,n-k}}{\partial x_j}t^n,$$

with $X_{j,0}=u_j^0,\ \frac{\partial X_{i,0}}{\partial x_j}=\frac{\partial u_i^0}{\partial x_j}$.

Applying these sums in (2.4) we have

$$(2.11) \qquad \begin{aligned}&\sum_{n=0}^{\infty}(n+1)X_{i,n+1}t^n=-\sum_{n=0}^{\infty}q_{i,n}t^n-\\&-\sum_{n=0}^{\infty}\sum_{j=1}^{3}\sum_{k=0}^{n}X_{j,k}\frac{\partial X_{i,n-k}}{\partial x_j}t^n+\nu\sum_{n=0}^{\infty}\nabla^2X_{i,n}\,t^n+\\&+\sum_{n=0}^{\infty}f_{i,n}t^n,\end{aligned}$$

and then

$$(2.12) \qquad \begin{aligned}(n+1)X_{i,n+1}=-q_{i,n}-\sum_{j=1}^{3}\sum_{k=0}^{n}X_{j,k}\frac{\partial X_{i,n-k}}{\partial x_j}+\\+\nu\nabla^2X_{i,n}+f_{i,n},\end{aligned}$$

which allows us to obtain, by recurrence, $X_{i,1},\ X_{i,2},\ X_{i,3}$, etc., that is, for $1\le i\le 3$ and $n\ge 0$,

$$(2.13) \qquad \begin{aligned}&X_{i,n+1}=\frac{1}{n+1}S_n,\\&S_n=-q_{i,n}-\sum_{j=1}^{3}\sum_{k=0}^{n}X_{j,k}\frac{\partial X_{i,n-k}}{\partial x_j}+\nu\nabla^2X_{i,n}+f_{i,n}.\end{aligned}$$

You can see how much will become increasingly difficult calculate the terms $X_{i,n}$ with increasing the values of n, for example, will appear terms in $\nu^n, \nabla^2\nabla^2\ldots\nabla^2u_i^0$, etc. If $\nu>1$ certainly there is a specific problem to be studied with relation to convergence of the series, which of course also occurs in the representation given in section § 1. The same can be said for $t\to\infty$.

§ 3

The previous solutions show us that we need to have, for all integers $1 \leq i \leq 3$ and $n \geq 0$,

(3.1) $$\frac{1}{n!}\frac{\partial^n u_i}{\partial t^n}|_{t=0} = X_{i,n},$$

and both members of this relation are very difficult to be calculated, either equation (1.15) as well as (2.13). Add to this difficulty the fact that besides the main Navier-Stokes equations (1.4)-(2.4) must be included the condition of incompressibility,

(3.2) $$\nabla \cdot u = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} u_i = 0.$$

Using (2.1) in (3.2) we have

(3.3) $$\nabla \cdot u = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \sum_{n=0}^{\infty} X_{i,n} t^n = \sum_{n=0}^{\infty} \left(\sum_{i=1}^{3} \frac{\partial}{\partial x_i} X_{i,n} \right) t^n = 0.$$

As this equation need be valid for all $t \geq 0$ we have

(3.4) $$\sum_{i=1}^{3} \frac{\partial}{\partial x_i} X_{i,n} = \nabla \cdot X_n = 0,$$

defining $X_n = (X_{1,n},\ X_{2,n},\ X_{3,n})$, i.e., all coefficients X_n must obey the condition of incompressibility in the vector representation of velocity,

(3.5) $$u = \sum_{n=0}^{\infty} X_n\, t^n.$$

As we realized that it is possible infinite solutions for a same initial condition for velocity then we can try choose a more easier solution, whose maximum value of n is finite, in special $n = 1$, i.e.,

(3.6) $$u_i = u_i^0 + X_{i,1} t,$$

where

(3.7) $$X_{i,1} = \frac{\partial u_i}{\partial t}|_{t=0}.$$

Then of (3.4), for $n = 0$, it is necessary that

(3.8) $$\nabla \cdot u^0 = 0,$$

which is also an initial condition, for $n = 1$ it is necessary that

(3.9) $$\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(-q_{i,0} - \sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j} + \nu \nabla^2 u_i^0 + f_i^0 \right) = 0,$$

$q_{i,0} = \frac{\partial p}{\partial x_i}(x_1, x_2, x_3, 0)$, and for $n \geq 2$ do not have no term for velocity, by definition in (3.6), but we need that the nonlinear terms of second order in time vanishes in the sum $\sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}$, i.e.,

(3.10) $$\sum_{j=1}^{3} X_{j,1} \frac{\partial X_{i,1}}{\partial x_j} = 0.$$

The two last conditions are obviously satisfied when $X_{i,1}$ is zero, any other numerical constant or a generic smooth function of time $\tau_i(t)$, i.e.,

(3.11) $$X_{i,1} = -q_{i,0} - \sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j} + \nu \nabla^2 u_i^0 + f_i^0 = \tau_i(t),$$

and then

(3.12) $$u_i = u_i^0 + T_i(t)$$

may be our solution obtained for velocity, with $T_i(0) = 0$ and $T = \tau\, t$, still lacking the calculation of pressure.

As our initial hypothesis was that $X_{i,1}$ was not time dependent we need review the solution or simply choose $T_i(t) = c_i t$, c_i a numerical constant. Note that for compliance with the Millenium Problem[4], case (A), only if $T_i(t)$ is identically zero we can have the condition of bounded energy satisfied, i.e.,

(3.13) $$\int_{\mathbb{R}^3} |u|^2 dx < C \text{ finite}, x \in \mathbb{R}^3,$$

what force us to choose $T_i(t) \equiv 0$ and thus the final solution for velocity will be $u = u^0$ for any $t \geq 0$, since that u^0 obey to the necessary conditions of case (A). For case (B), related with periodical spatially solutions, where the condition of bounded energy in whole space is not necessary, it is possible $T_i(t) \neq 0$, and in fact this is a promising and well behaved solution for Euler and Navier-Stokes equations in periodic spatially solutions cases, choosing $T(t) \in C^\infty$ limited, as well as u^0 and consequently u.

Note that the solution (3.12) also is compatible with the condition (3.4) of divergence free for series of form

(3.14) $$u_i = u_i^0 + \sum_{n=1}^{\infty} c_{i,n}\, t^n,$$

i.e.,

(3.15) $$T_i(t) = \sum_{n=1}^{\infty} c_{i,n}\, t^n, \qquad X_{i,n} = c_{i,n},$$

supposing that (3.8) is valid. The same is said if the maximum value of n is finite, obviously. All $c_{i,n}$ are numerical constants for $n \geq 1$.

§ 4

How calculate the pressure value, if it is not given nor previously chosen?

From Navier-Stokes equations, it is equal to the line integral

$$(4.1) \qquad p = \int_L \left(-\frac{\partial u}{\partial t} - (u \cdot \nabla)u + \nu \nabla^2 u + f\right) \cdot dl,$$

where L is any sectionally smooth curve going from a point (x^0, y^0, z^0) to (x, y, z), for a fixed time t, and for this calculation it is necessary that the vector

$$(4.2) \qquad S = -\frac{\partial u}{\partial t} - (u \cdot \nabla)u + \nu \nabla^2 u + f$$

is gradient, i.e.,

$$(4.3) \qquad S = \nabla p, \qquad p = \int_L S \cdot dl,$$

so the condition

$$(4.4) \qquad \frac{\partial S_i}{\partial x_j} = \frac{\partial S_j}{\partial x_i},$$

need be satisfied for all integer $1 \leq i, j \leq 3$ in order that (4.1) can be calculated, where $S = (S_1,\ S_2,\ S_3)$ and

$$(4.5) \qquad S_i = -\frac{\partial u_i}{\partial t} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} + \nu\, \nabla^2 u_i + f_i.$$

Obviously, even if the pressure is given or chosen previously, as indicated in sections § 1 and § 2, the equations (4.1) and (4.4) need to be fulfilled.

The condition (4.4) is a very hard condition to be satisfied, instead the incompressibility condition

$$(4.6) \qquad \nabla \cdot u = \nabla \cdot u^0 = 0.$$

Following Lagrange[1], getting two differentiable and continuous functions α and β of class C^2 and defining

$$(4.7.1) \qquad u_1 = \frac{\partial \alpha}{\partial z}, \qquad u_2 = \frac{\partial \beta}{\partial z}, \qquad u_3 = -\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y}\right),$$

$$(4.7.2) \qquad u_1^0 = \frac{\partial \alpha^0}{\partial z}, \qquad u_2^0 = \frac{\partial \beta^0}{\partial z}, \qquad u_3^0 = -\left(\frac{\partial \alpha^0}{\partial x} + \frac{\partial \beta^0}{\partial y}\right),$$

with $\alpha^0 = \alpha(t=0)$ and $\beta^0 = \beta(t=0)$, we have satisfied the condition (4.6), which it is easy to see. Other manner is when u is derived from a vector potential A, i.e.,

(4.8.1) $$u = \nabla \times A,$$

(4.8.2) $$u^0 = \nabla \times A^0,$$

with $A^0 = A(t=0)$.

The relations (4.7) are very useful and easy to be implemented. Given any continuous, differentiable and integrable vector components u_1 and u_2 then

(4.9.1) $$\alpha = \int u_1\, dz,$$

(4.9.2) $$\beta = \int u_2\, dz,$$

and thus u_3 and u_3^0 need to be according

(4.10.1) $$u_3 = -\int \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right) dz = -\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y}\right),$$

(4.10.2) $$u_3^0 = -\int \left(\frac{\partial u_1^0}{\partial x} + \frac{\partial u_2^0}{\partial y}\right) dz = -\left(\frac{\partial \alpha^0}{\partial x} + \frac{\partial \beta^0}{\partial y}\right),$$

which reminds us that the components of the velocity vector maintains conditions to be complied to each other, i.e., it is not any initial velocity which can be used for solution of Euler and Navier-Stokes equations in incompressible flows case.

Following these transformations, in the equations of the sections § 1 and § 2, instead u_1 we will use $\frac{\partial \alpha}{\partial z}$, instead u_2 will be $\frac{\partial \beta}{\partial z}$, and $-\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y}\right)$ instead u_3, as well as the correspondents initial values, replacing u_1^0 by $\frac{\partial \alpha^0}{\partial z}$, u_2^0 by $\frac{\partial \beta^0}{\partial z}$, and u_3^0 by $-\left(\frac{\partial \alpha^0}{\partial x} + \frac{\partial \beta^0}{\partial y}\right)$. Of this manner, we will be developing series for $\frac{\partial \alpha}{\partial z}$, $\frac{\partial \beta}{\partial z}$ and $-\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y}\right)$, so that $\nabla \cdot u = 0$. Then this is a preliminary problem to be solved, the calculation of α^0 and β^0 giving u_1^0, u_2^0 and u_3^0 when $\nabla \cdot u^0 = 0$ and it is necessary that $\nabla \cdot u = 0$, i.e.,

(4.11.1) $$\alpha^0 = \int u_1^0\, dz,$$

(4.11.2) $$\beta^0 = \int u_2^0\, dz,$$

with the validity of (4.10.2). Done this, the exact solution for the principal problem can be calculated from reasoning exposed here, if there is not an equivalent solution described in a most simplified formulation, for example, according Bernoulli's law and Laplace's equation.

Another relation need be obeyed for obtaining the mentioned solution, both in compressible as incompressible flows, described below.

For obtaining a solution for the system

$$(4.12)\qquad \begin{cases} \frac{\partial p}{\partial x} = S_1 \\ \frac{\partial p}{\partial y} = S_2 \\ \frac{\partial p}{\partial z} = S_3 \end{cases}$$

representing the Euler ($\nu = 0$) and Navier-Stokes equations, with S_i given by (4.5) using $x \equiv x_1,\ y \equiv x_2,\ z \equiv x_3$, it is necessary that $\nabla \times S = 0,\ S = (S_1, S_2, S_3)$. This condition is equivalent to follow system

$$(4.13)\qquad \begin{cases} \frac{\partial S_1}{\partial y} = \frac{\partial S_2}{\partial x} \\ \frac{\partial S_1}{\partial z} = \frac{\partial S_3}{\partial x} \\ \frac{\partial S_2}{\partial z} = \frac{\partial S_3}{\partial y} \end{cases}$$

which is the mentioned condition (4.4).

The first of these equations leads to

$$(4.14)\qquad \frac{\partial}{\partial y}\frac{\partial u_1}{\partial t} = \frac{\partial}{\partial x}\frac{\partial u_2}{\partial t}$$

respect to equality of temporal derivatives of velocity components 1 and 2, or

$$(4.15)\qquad \frac{\partial}{\partial t}\frac{\partial u_1}{\partial y} = \frac{\partial}{\partial t}\frac{\partial u_2}{\partial x}.$$

Repeating this reasoning for the second and third equations of (4.13) we come to

$$(4.16.1)\qquad \frac{\partial}{\partial t}\frac{\partial u_1}{\partial z} = \frac{\partial}{\partial t}\frac{\partial u_3}{\partial x},$$

$$(4.16.2)\qquad \frac{\partial}{\partial t}\frac{\partial u_2}{\partial z} = \frac{\partial}{\partial t}\frac{\partial u_3}{\partial y},$$

or

$$(4.17)\qquad \frac{\partial}{\partial t}\frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial t}\frac{\partial u_j}{\partial x_i}, \qquad 1 \le i, j \le 3,$$

i.e., $\nabla \times \frac{\partial}{\partial t} u = \frac{\partial}{\partial t} \nabla \times u = 0$ for all $t \ge 0$, which contains $\nabla \times u = 0$ and $\frac{\partial u}{\partial t} = 0$ as solutions.

Continuing the reasoning for the Laplacian terms, we have for the first equation of (4.13)

$$\frac{\partial}{\partial y}\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u_1=\frac{\partial}{\partial x}\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u_2, \tag{4.18}$$

where we can have

$$\frac{\partial}{\partial y}\frac{\partial^2 u_1}{\partial x^2}=\frac{\partial^3 u_2}{\partial x^3},\qquad \frac{\partial}{\partial y}\frac{\partial^2 u_1}{\partial y^2}=\frac{\partial^3 u_2}{\partial x\partial y^2},\qquad \frac{\partial}{\partial y}\frac{\partial^2 u_1}{\partial z^2}=\frac{\partial^3 u_2}{\partial x\partial z^2}, \tag{4.19}$$

for the second equation of (4.13)

$$\frac{\partial}{\partial z}\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u_1=\frac{\partial}{\partial x}\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u_3, \tag{4.20}$$

with the possibility

$$\frac{\partial}{\partial z}\frac{\partial^2 u_1}{\partial x^2}=\frac{\partial^3 u_3}{\partial x^3},\qquad \frac{\partial}{\partial z}\frac{\partial^2 u_1}{\partial y^2}=\frac{\partial^3 u_3}{\partial x\partial y^2},\qquad \frac{\partial}{\partial z}\frac{\partial^2 u_1}{\partial z^2}=\frac{\partial^3 u_3}{\partial x\partial z^2}, \tag{4.21}$$

and for the third equation of (4.13)

$$\frac{\partial}{\partial z}\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u_2=\frac{\partial}{\partial y}\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u_3, \tag{4.22}$$

again as the respective previous equalities, equaling each parcel of the left side to the respective parcel of the right side, we can have

$$\frac{\partial}{\partial z}\frac{\partial^2 u_2}{\partial x^2}=\frac{\partial^3 u_3}{\partial y\partial x^2},\qquad \frac{\partial}{\partial z}\frac{\partial^2 u_2}{\partial y^2}=\frac{\partial^3 u_3}{\partial y^3},\qquad \frac{\partial}{\partial z}\frac{\partial^2 u_2}{\partial z^2}=\frac{\partial^3 u_3}{\partial y\partial z^2}. \tag{4.23}$$

For the nonlinear terms of (4.13) we have, for the first equation

$$\frac{\partial}{\partial y}\left(u_1\frac{\partial u_1}{\partial x}+u_2\frac{\partial u_1}{\partial y}+u_3\frac{\partial u_1}{\partial z}\right)=\frac{\partial}{\partial x}\left(u_1\frac{\partial u_2}{\partial x}+u_2\frac{\partial u_2}{\partial y}+u_3\frac{\partial u_2}{\partial z}\right) \tag{4.24}$$

$$\begin{aligned}&\frac{\partial u_1}{\partial x}\frac{\partial u_1}{\partial y}+u_1\frac{\partial^2 u_1}{\partial x\partial y}+\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial y}+u_2\frac{\partial^2 u_1}{\partial y^2}+\frac{\partial u_1}{\partial z}\frac{\partial u_3}{\partial y}+u_3\frac{\partial^2 u_1}{\partial y\partial z}=\\&=\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x}+u_1\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial u_2}{\partial x}\frac{\partial u_2}{\partial y}+u_2\frac{\partial^2 u_2}{\partial x\partial y}+\frac{\partial u_3}{\partial x}\frac{\partial u_2}{\partial z}+u_3\frac{\partial^2 u_2}{\partial x\partial z}\end{aligned} \tag{4.25}$$

for the second equation

$$\frac{\partial}{\partial z}\left(u_1\frac{\partial u_1}{\partial x}+u_2\frac{\partial u_1}{\partial y}+u_3\frac{\partial u_1}{\partial z}\right)=\frac{\partial}{\partial x}\left(u_1\frac{\partial u_3}{\partial x}+u_2\frac{\partial u_3}{\partial y}+u_3\frac{\partial u_3}{\partial z}\right) \tag{4.26}$$

$$\begin{aligned}&\frac{\partial u_1}{\partial x}\frac{\partial u_1}{\partial z}+u_1\frac{\partial^2 u_1}{\partial x\partial z}+\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial z}+u_2\frac{\partial^2 u_1}{\partial y\partial z}+\frac{\partial u_1}{\partial z}\frac{\partial u_3}{\partial z}+u_3\frac{\partial^2 u_1}{\partial z^2}=\\&=\frac{\partial u_1}{\partial x}\frac{\partial u_3}{\partial x}+u_1\frac{\partial^2 u_3}{\partial x^2}+\frac{\partial u_2}{\partial x}\frac{\partial u_3}{\partial y}+u_2\frac{\partial^2 u_3}{\partial x\partial y}+\frac{\partial u_3}{\partial x}\frac{\partial u_3}{\partial z}+u_3\frac{\partial^2 u_3}{\partial x\partial z}\end{aligned} \tag{4.27}$$

and for the last equation

$$\frac{\partial}{\partial z}\left(u_1\frac{\partial u_2}{\partial x}+u_2\frac{\partial u_2}{\partial y}+u_3\frac{\partial u_2}{\partial z}\right)=\frac{\partial}{\partial y}\left(u_1\frac{\partial u_3}{\partial x}+u_2\frac{\partial u_3}{\partial y}+u_3\frac{\partial u_3}{\partial z}\right) \tag{4.28}$$

$$\begin{aligned}\frac{\partial u_1}{\partial z}\frac{\partial u_2}{\partial x}+u_1\frac{\partial^2 u_2}{\partial x\partial z}+\frac{\partial u_2}{\partial y}\frac{\partial u_2}{\partial z}+u_2\frac{\partial^2 u_2}{\partial y\partial z}+\frac{\partial u_2}{\partial z}\frac{\partial u_3}{\partial z}+u_3\frac{\partial^2 u_2}{\partial z^2}&=\\=\frac{\partial u_1}{\partial y}\frac{\partial u_3}{\partial x}+u_1\frac{\partial^2 u_3}{\partial x\partial y}+\frac{\partial u_2}{\partial y}\frac{\partial u_3}{\partial y}+u_2\frac{\partial^2 u_3}{\partial y^2}+\frac{\partial u_3}{\partial y}\frac{\partial u_3}{\partial z}+u_3\frac{\partial^2 u_3}{\partial y\partial z}&.\end{aligned} \tag{4.29}$$

All these equations, from (4.17) to (4.29), admit for solution the condition

$$\frac{\partial u_i}{\partial x_j}=\frac{\partial u_j}{\partial x_i},\quad 1\le i,j\le 3, \tag{4.30}$$

in which case u is an irrotational vector, $\nabla\times u=0$, and so there is a velocity potential ϕ such that $u=\nabla\phi$. We will use this condition that u is irrotational and that also it is incompressible for the calculation of pressure in this special situation, coming to the known Bernoulli's law. For this also it is necessary to consider that the external force is conservative, i.e., it has a potential U such that $f=\nabla U$ and $\nabla\times f=0$, because then we will have satisfied the system (4.13) completely, when

$$\frac{\partial f_i}{\partial x_j}=\frac{\partial f_j}{\partial x_i},\quad 1\le i,j\le 3. \tag{4.31}$$

If $\nabla\times u=0$ and $\nabla\cdot u=0$ then

$$\nabla^2 u=\nabla(\nabla\cdot u)-\nabla\times(\nabla\times u)=0, \tag{4.32}$$

i.e., the Laplacian in the Navier-Stokes equations vanishes for any viscosity coefficient and the Navier-Stokes reduced to Euler equations.

If $\nabla\times u=0$ then the nonlinear term in vector form is simplified, according

$$(u\cdot\nabla)u=(\nabla\times u)\times u+\tfrac{1}{2}\nabla|u|^2=\tfrac{1}{2}\nabla|u|^2, \tag{4.33}$$

thus, using (4.32) and (4.33) and more the potentials of the velocity and external force, the Navier-Stokes (and Euler) equations reduced to

$$\nabla p+\frac{\partial}{\partial t}\nabla\phi+\tfrac{1}{2}\nabla|u|^2=\nabla U, \tag{4.34}$$

therefore the solution for pressure is

$$p=-\frac{\partial\phi}{\partial t}-\tfrac{1}{2}|u|^2+U+\theta(t), \tag{4.35}$$

the Bernoulli's law, where $\theta(t)$ is a generic time function, let's suppose $\theta(t) \in C^\infty$ a limited time function, a numeric constant or even zero.

If $u = \nabla\phi$ and $\nabla \cdot u = 0$, according we are admitting, then from incompressibility condition

$$\nabla \cdot u = \frac{\partial}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial \phi}{\partial y} + \frac{\partial}{\partial z}\frac{\partial \phi}{\partial z} = 0 \tag{4.36}$$

we come to the Laplace's equation

$$\nabla^2\phi = 0, \tag{4.37}$$

where each possible solution gives the respective values of velocity components, such that

$$u_1 = \frac{\partial \phi}{\partial x}, \qquad u_2 = \frac{\partial \phi}{\partial y}, \qquad u_3 = \frac{\partial \phi}{\partial z}, \tag{4.38}$$

and the pressure is given by (4.35), with

$$|u|^2 = u_1^2 + u_2^2 + u_3^2 = \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2. \tag{4.39}$$

According Courant[5] (p.241), for $n = 2$ the "general solution" of the potential equation (or Laplace's equation) is the real part of any analytic function of the complex variable $x + iy$. For $n = 3$ one can also easily obtain solutions which depend on arbitrary functions. For example, let $f(w, t)$ be analytic in the complex variable w for fixed real t. Then, for arbitrary values of t, both the real and imaginary parts of the function

$$u = f(z + ix\cos t + iy\sin t\,, t) \tag{4.40}$$

of the real variables x, y, z are solutions of the equation $\nabla^2 u = 0$. Further solutions may be obtained by superposition:

$$u = \int_a^b f(z + ix\cos t + iy\sin t\,, t)dt. \tag{4.41}$$

For example, if we set

$$f(w, t) = w^n e^{iht}, \tag{4.42}$$

where n and h are integers, and integrate from $-\pi$ to $+\pi$, we get homogeneous polynomials

$$u = \int_{-\pi}^{\pi}(z + ix\cos t + iy\sin t)^n\, e^{iht}dt \tag{4.43}$$

in x, y, z, following example given by Courant. Introducing polar coordinates $z = r\cos\theta, x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi$, we obtain

$$u = 2r^n e^{ih\phi} \int_0^{\pi} (\cos\theta + i\sin\theta\cos t)^n \cos ht \; dt \\ = r^n e^{ih\phi} P_{n,h}(\cos\theta), \tag{4.44}$$

where $P_{n,h}(\cos\theta)$ are the associated Legendre functions.

§ 5

The series obtained in two first sections admitted that the incompressibility condition is satisfied for any $t \geq 0$, but we saw how difficult are the expressions (1.15) and (2.13) for that this can really occur for $t > 0$. In $t = 0$ this can be satisfied without great problems because the terms in t, t^2, t^3, etc. vanish. We can construct a solution using the indicated in equations (4.7), more general than (4.8), but the easier solution is to consider all coefficients, since the order zero, the free time power coefficient, as components of an irrotational and incompressible vector, this when the initial velocity is compatible with these conditions, i.e., our solutions for velocity in series of time (finite and also infinite) are, in this case, of a generic form

$$u(x, y, z, t) = \sum_{k=0}^{m} X_k(x, y, z)\, T_k(t), \tag{5.1}$$

where all $X_k(x, y, z)$ are irrotational and incompressible vectors, i.e., solutions of Laplace's equation in vector form, they are harmonic functions, according superposition principle, as well as the respective velocity potentials are the scalar functions $\phi_k(x, y, z)$ such that

$$\phi(x, y, z, t) = \sum_{k=0}^{m} \phi_k(x, y, z)\, T_k(t), \tag{5.2}$$

solutions of

$$\nabla^2 \phi_k = 0, \tag{5.3}$$

where

$$X_{i,k} = \frac{\partial}{\partial x_i} \phi_k, \tag{5.4}$$

with $X_k = (X_{1,k},\ X_{2,k},\ X_{3,k})$, $X_0 = u^0$ the initial velocity, $T_0(0) = 1$ and $T_k(0) = 0$ if $k \geq 1$. The functions $T_k(t) \in C^\infty(\mathbb{R})$ are limited for t finite, by our convention.

Resorting again to the mentioned theorem of uniqueness of series of powers in § 2 and using the Taylor's theorem (Courant[3], chap. VI), we can choose $T_k(t) = t^k$ and $m \to \infty$ in (5.1), i.e.,

$$u(x, y, z, t) = \sum_{k=0}^{\infty} X_k(x, y, z)\, t^k, \tag{5.5}$$

and conclude that the coefficients of series of time given by (1.15) and (2.13) are, when u^0 is irrotational, solutions of Laplace's equation, at least in cases of conservative external forces and incompressible flows, for a same initial velocity, pressure, external force, compressibility condition and all boundary conditions, without contradictions with Lagrange[1] and Cauchy[2], and for this reason in these cases the Bernoulli's law is the correct solution for pressure in Euler and Navier-Stokes equations. We are assuming, but it is possible to prove in more detail, that always there some solution for Euler and Navier-Stokes equations in series of power (even, for example, $u = 0$), that it is analytical in a non-empty region for all $t \geq 0$ finite, and even not existing uniqueness of solutions, for each possible solution u it can be put in the form (5.5) using (1.15) or (2.13) or, for irrotational and incompressible flows, (5.3) and (5.4), existing the relation of equivalence (3.1), i.e.,

$$(5.6) \qquad \frac{1}{k!}\frac{\partial^k u_i}{\partial t^k}\Big|_{t=0} = X_{i,k},$$

for $i = 1, 2, 3$. Note that if it is not possible to make a series around $t = 0$ (for example, to the functions $\log t$, $\sqrt[3]{t}$, e^{-1/t^2}, according Courant[3], chap. VI) an other instant t_0 of convergence and remainder $R_{i,k\to\infty}$ zero must be found, and then replacing t^k by $(t - t_0)^k$ and the calculations in $t = 0$ by $t = t_0$ in previous equations.

It is not necessary the use of viscosity coefficient for smooth and incompressible fluids with conservative external force (or without any force). For non-stationary flows it is knows that the Lagrange's theorem[6],[7], as well as the Kelvin's circulation theorem[7],[8], is not valid for Navier-Stokes equations, but here it is implied that $\nu\nabla^2 u \neq 0$, the general case. The necessity of smooth velocity in whole space leads us to exclude all obstacles and regions without velocity of the fluid in study, which naturally occur using boundaries. The vorticity $\omega = \nabla \times u \neq 0$ is generated at solid boundaries[9], thus without boundaries (spatial domain $\Omega = \mathbb{R}^3$) no generation of vorticity, and without vorticity there is potential flow and vanishes the Laplacian of velocity if $\nabla \cdot u = 0$, then it is possible again the validity of Lagrange's theorem in an unlimited domain without boundaries and with both smooth and irrotational initial velocities and external forces, for incompressible fluids, because thus $\nabla^2 u = 0$, independently of viscosity coefficient.

Note that according Liouville's theorem[10], a harmonic function which is limited is constant, and equal to zero if it tends to zero at infinity. How the Millennium Problem requires a limited solution in all space for velocity and a limited initial velocity which goes to zero at infinity (in cases (A) and (C)), then we are forced to choose only $u^0 = 0$ for case (A) if $\nabla^2 u = 0$, what automatically implies the occurrence of case (C) due to infinite examples of prohibited u^0 and using any conservative external force f. If $\nabla^2 u = 0$ and $u^0 = 0$ then the unique possible solution for case (A), where it is necessary that $f=0$, is $u=0$ otherwise u would not be limited or u would be equal to constant greater than zero or any not null function of time, that violated the condition of bounded energy, equation (3.13).

If $\nabla^2 u \neq 0$ then the suitable general solution for Navier-Stokes equations is as indicated in sections § 1 and § 2 using (4.7), for an infinity of possible pressures of C^∞ class.

§ 6

I finish this article mentioning that Lamb[7] (chap. VII) gives a complete solution for velocity in Euler equations when the velocity vanishes at infinity.

He said that "no irrotational motion is possible in an incompressible fluid filling infinite space, and subject to the condition that the velocity vanishes at infinity." This is equivalent to the unique possible solution $u=0$.

From this result he proved the following theorem: "The motion of a fluid which fills infinite space, and is at rest at infinity, is determinate when we know the values of the expansion (θ, say) and of the component angular velocities ξ, η, ζ, at all points of the region.", where

(6.1) $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\theta$$

is the named expansion and

(6.2.1) $$\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}=2\xi$$

(6.2.2) $$\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=2\eta$$

(6.2.3) $$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=2\zeta$$

are the equations for angular velocities. The components of the velocity are u, v, w, and vanish at infinity as well as $\theta,\ \xi,\ \eta,\ \zeta$.

Lamb obtain the solution for velocity

(6.3.1) $$u=-\frac{\partial \Phi}{\partial x}+\frac{\partial H}{\partial y}-\frac{\partial G}{\partial z}$$

(6.3.2) $$v=-\frac{\partial \Phi}{\partial y}+\frac{\partial F}{\partial z}-\frac{\partial H}{\partial x}$$

(6.3.3) $$w=-\frac{\partial \Phi}{\partial z}+\frac{\partial G}{\partial x}-\frac{\partial F}{\partial y}$$

where

(6.4.1) $$\Phi = \frac{1}{4\pi}\iiint \frac{\theta\prime}{r} dx'dy'dz'$$

(6.4.2) $$F = \frac{1}{2\pi}\iiint \frac{\xi\prime}{r} dx'dy'dz'$$

(6.4.3) $$G = \frac{1}{2\pi}\iiint \frac{\eta\prime}{r} dx'dy'dz'$$

(6.4.4) $$H = \frac{1}{2\pi}\iiint \frac{\zeta\prime}{r} dx'dy'dz'$$

the accents attached to θ, ξ, η, ζ are used to distinguish the values of these quantities at the point (x', y', z'), r denoting the distance

(6.5) $$r = \{(x - x')^2 + (y - y')^2 + (z - z')^2\}^{1/2}$$

and the integrations including all places which θ', ξ', η', ζ' differ from zero, respectively.

The following relations are valid:

(6.6) $$u_1 = -\frac{\partial\Phi}{\partial x}, \qquad v_1 = -\frac{\partial\Phi}{\partial y}, \qquad w_1 = -\frac{\partial\Phi}{\partial z},$$

(6.7) $$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = -\nabla^2\Phi = \theta$$

for solution of (6.1), and

(6.8) $$u_2 = \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}, \qquad v_2 = \frac{\partial F}{\partial z} - \frac{\partial H}{\partial x}, \qquad w_2 = \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y},$$

(6.9) $$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial w_2}{\partial z} = 0$$

(6.10) $$2\xi = \frac{\partial w_2}{\partial y} - \frac{\partial v_2}{\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z}\right) - \nabla^2 F$$

(6.11) $$2\eta = \frac{\partial u_2}{\partial z} - \frac{\partial w_2}{\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z}\right) - \nabla^2 G$$

(6.12) $$2\zeta = \frac{\partial v_2}{\partial x} - \frac{\partial u_2}{\partial y} = \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z}\right) - \nabla^2 H$$

(6.13) $$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = 0$$

(6.14) $$\nabla^2 F = -2\xi, \qquad \nabla^2 G = -2\eta, \qquad \nabla^2 H = -2\zeta,$$

for solution of (6.2).

The solution (u, v, w) given by (6.3) is obtained by superposition

(6.15.1) $$u = u_1 + u_2$$

(6.15.2) $$v = v_1 + v_2$$

(6.15.3) $$w = w_1 + w_2$$

From the reasoning of Lamb, derived from von Helmholtz, and following your calculations, we cannot understand *a priori* that the equations (6.3) are the correct solutions of Euler equations because the equations (6.2) are not the Euler equations and the pressure is not mentioned, i.e., the relation (4.4) is not verified.

The equations (6.3) are a form of representation of any vector $\boldsymbol{u} = (u, v, w)$, a fluid flow or not, satisfying appropriate smoothness and decay conditions, in a sum of one gradient vector ($\boldsymbol{u_\Phi} = -\nabla\Phi$), the velocity potential, and one rotational vector ($\boldsymbol{u_\omega} = \nabla \times (F, G, H)$, with $\nabla \cdot (F, G, H) = 0$), which is the know Helmholtz or Hodge decomposition[11]. Adopting the minus sign of Lamb in $\nabla\Phi$,

(6.16) $$\boldsymbol{u} = \boldsymbol{u_\Phi} + \boldsymbol{u_\omega} = -\nabla\Phi + \nabla \times \boldsymbol{\psi},$$

where Φ is the scalar potential and $\boldsymbol{\psi} = (F, G, H)$ is the vector potential, with

(6.17) $$\nabla^2\boldsymbol{\psi} = -\boldsymbol{u_\omega}.$$

The solution given by Lamb in a sum derived of one scalar potential and one vector potential can be expressed as a single vector, gradient of scalar potential, in case of incompressible flow.

If $u = (u_1, u_2, u_3)$ and $A = (A_1, A_2, A_3)$ are vectors, ϕ is a scalar function, u, A, ϕ are smooth functions and we define

(6.18) $$u = \nabla \times A = \nabla\phi$$

then we have

(6.19) $$\nabla \cdot u = 0, \qquad \nabla \times u = 0, \qquad \nabla^2 u = 0,$$

and

(6.20) $$\phi = \int_L (\nabla \times A) \cdot dl = \int_L u \cdot dl,$$

since that $\nabla \times A$ is a gradient vector function, as well as the velocity u.

For that $\nabla \times A$ is gradient it is necessary that, for $1 \le i, j \le 3$,

(6.21) $$\frac{\partial}{\partial x_i}(\nabla \times A)_j = \frac{\partial}{\partial x_j}(\nabla \times A)_i.$$

Developing we have, with $x_1 \equiv x, x_2 \equiv y, x_3 \equiv z$,

$$(6.22.1) \qquad \frac{\partial^2 A_3}{\partial x^2} + \frac{\partial^2 A_3}{\partial y^2} = \frac{\partial}{\partial x}\frac{\partial A_1}{\partial z} + \frac{\partial}{\partial y}\frac{\partial A_2}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y}\right)$$

$$(6.22.2) \qquad \frac{\partial^2 A_2}{\partial x^2} + \frac{\partial^2 A_2}{\partial z^2} = \frac{\partial}{\partial x}\frac{\partial A_1}{\partial y} + \frac{\partial}{\partial z}\frac{\partial A_3}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_3}{\partial z}\right)$$

$$(6.22.3) \qquad \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2} = \frac{\partial}{\partial y}\frac{\partial A_2}{\partial x} + \frac{\partial}{\partial z}\frac{\partial A_3}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

When $\nabla \cdot A = 0$ then comes

$$(6.23.1) \qquad \frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2} = 0$$

$$(6.23.2) \qquad \frac{\partial^2 A_2}{\partial x^2} + \frac{\partial^2 A_2}{\partial y^2} + \frac{\partial^2 A_2}{\partial z^2} = 0$$

$$(6.23.3) \qquad \frac{\partial^2 A_3}{\partial x^2} + \frac{\partial^2 A_3}{\partial y^2} + \frac{\partial^2 A_3}{\partial z^2} = 0$$

i.e., each component of the vector A is a harmonic function and so

$$(6.24) \qquad \nabla^2 A = 0.$$

We see then that it is possible to have simultaneously a potential flow ($u = \nabla\phi$) and a vortex motion ($u = \nabla \times A$), since that $\nabla \cdot A = 0$, without be necessary that neither $\nabla \times A = 0$ nor $u = 0$. In this case the equation (6.16) can be rewritten as

$$(6.25) \qquad u = u_\phi = u_\omega = \nabla\phi = \nabla \times \mathrm{A},$$

where we use $\phi = -\Phi$ and $A = \psi$, without use of bold characters for indicate vectors. As we saw in section § 4 for incompressible and potential flow, the pressure is given by Bernoulli's law, equation (4.34),

$$(6.26) \qquad p = -\frac{\partial\phi}{\partial t} - \frac{1}{2}|u|^2 + U + \theta(t),$$

because here too $\nabla \cdot u = 0$ and $u = \nabla\phi$, even though $u = \nabla \times A$ (due to lack of a better name I also called *vortex motion* the not null velocity generated by a curl of a not null vector).

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October-07,20,25,27-2016
November-03-2016
December-20-2016

Lagrange, grande matemático.

A Matemática é um desafio quando se começa,
uma alegria quando pensamos estar certos pela 1ª vez,
uma vergonha quando se erra,
tortura quando o problema é difícil,
esporte gostoso quando o problema é fácil,
um alívio quando se termina,
um luxo quando se prova tudo.
Acima de tudo é grande beleza.

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