Geodesic Integration in the Schwarzschild Gravitational Field

Christopher A. Laforet
claforet@gmail.com

Abstract

It is demonstrated in the present work that the standard integration of the proper time along a freefalling geodesic in Schwarzschild spacetime does not properly account for the coordinate curvature in the vicinity of the event horizon. It is shown that that the condition given by the metric, namely that the change in proper time of the freefalling observer per change of coordinate time goes to zero at the horizon can be maintained while still allowing for an infinite proper time to the horizon. With the aid of a transformation of the radial Schwarzschild coordinate and analysis of light signals, we find that observers at rest will see the freefalling observer slow exponentially as she approaches the horizon, while the freefaller will see rest observers slow asymptotically as she approaches the horizon.

Freefall in the Schwarzschild Field

The well-known Schwarzschild metric is given in (1) below (note we will be using units where the Schwarzschild radius is 1 and we will drop the angular term of the metric since we will only be examining radial freefall):

$$d\tau^2 = \left[1 - \frac{1}{r}\right] dt^2 - \left[1 - \frac{1}{r}\right]^{-1} dr^2 \tag{1}$$

These coordinates are quite useful for describing the spacetime for observers at rest in the gravitational field, particularly the observer at infinity in asymptotically flat spacetime. The *r* coordinate represents some notion of distance from the center of the gravitational source, where the units of *r* are in units of Schwarzschild radius of the source. Thus, this radial coordinate gives circles around the source where, in a top-down view of the source, the circle radii increase linearly as one moves away from the center. Let's now consider the coordinate speed of a freefalling observer (who starts to fall from rest at infinity) in the frame of an observer at infinity in Schwarzschild coordinates [1]:

$$\frac{dr}{dt} = -\sqrt{\frac{1}{r}}\left[1 - \frac{1}{r}\right] \tag{2}$$

As can be seen in (2), for large *r*, the velocity is very small, converging to zero as *r* goes to infinity. This is sensible since the gravitation field is weak at large *r* and therefore we expect that at the beginning of the fall, the velocity is slow. What is interesting is that as *r* goes to 1, representing the observer approaching the event horizon, the velocity goes to zero again.

Let us now substitute (2) into (1) to examine the proper time of the freefalling observer in the frame of the observer at infinity:

$$d\tau_{ff} = \left[1 - \frac{1}{r_{ff}}\right] dt \tag{3}$$

We see that as r goes to 1, (3) goes to zero. This means that the freefalling observer's velocity is approaching the speed of light in the frame of the infinite observer. This is also supported by the fact that in the frame of observers at rest in the gravitational field (and therefore also at rest relative to the infinite observer), the relative velocity of the freefalling observer relative to the observer at rest at r as the freefalling observer passes by is given by [1]: $V = \sqrt{\frac{1}{r}}$, which approaches the speed of light as r goes to 1.

But it is also well known that the observer at infinity will see the freefalling observer's clock slow to a stop and signals from the observer will be infinitely redshifted as she approaches the horizon, which is more evidence that the true relative velocity between the freefalling and infinite observer increases to light speed as she approaches the horizon. This light speed condition combined with the coordinate velocity going to zero at the horizon give us the clues needed to show how the Schwarzschild coordinates are actually related to the curved manifold. The relationship is shown in Figure 1 below:

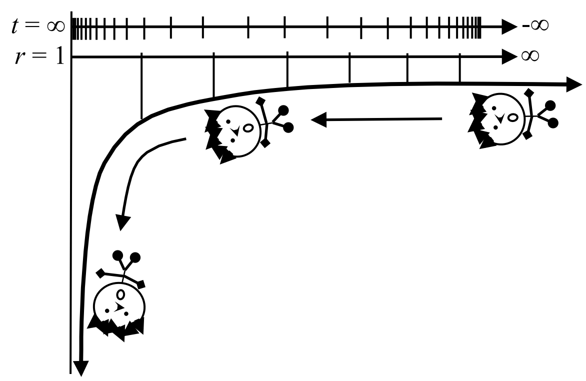

Figure 1- Relationship Between Schwarzschild Coordinates and the Curved Manifold

In Figure 1, we see our intrepid explorer, Scout, freefalling along a radial geodesic in the Schwarzschild gravitational field in the frame of observers at rest in the field. The infinite observer would be off to the right on this diagram where the geodesic (the dark black line) would be horizontal. Since, in this particular depiction, the tangent to the manifold is horizontal at the infinite observer who is inertial in flat space, the acceleration needed for an observer to remain at rest at a given point is proportional to the slope of the tangent at that point.

In the frame of the infinite observer (let's call him Bob), Scout is accelerated along this geodesic and this is why she approaches the speed of light as she approaches the horizon. At large r, the linear spacing of the r coordinate is useful because the space there is approximately flat. But if we look at the region between $r = 1$ and $r = 2$, a single unit of the coordinate distance is used to label an infinite amount of coordinate time. This is why the coordinate velocity goes to zero near the horizon. One can see that as time increases near the horizon, Scout covers less and less coordinate distance, giving rise to the 'coordinate singularity' of the Schwarzschild coordinates. The coordinate speed of light is also known to slow near the horizon, which is also explained by Figure 1. Light signals travel along the geodesic at a fixed velocity, but since the geodesic is asymptotic to the radial coordinate near the horizon, its coordinate velocity there is decreased (to zero at the horizon).

Nonetheless, Scout's velocity measured by rest observers increases to the speed of light as she approaches the horizon, such that reaching the horizon would mean reaching luminal speed. Moving at the speed of light in one frame means that you are moving at the speed of light in all frames, and timelike observers cannot become lightlike observers over a finite distance/time. Since the Schwarzschild spacetime is static, it cannot be that space or time is 'flowing' into the horizon, such that the apparent luminal speed is caused by the underlying spatial dynamics. But this is in contradiction to the common view that Scout passes the horizon in a finite time according to her clock without noticing that she has even reached the horizon. To resolve this, we must investigate how the proper time of Scout's geodesic is calculated.

Integration over Curved Coordinates

The proper time along Scout's geodesic is found by integrating (3). A plot of the result of this integral near the horizon is given in Figure 2 below.

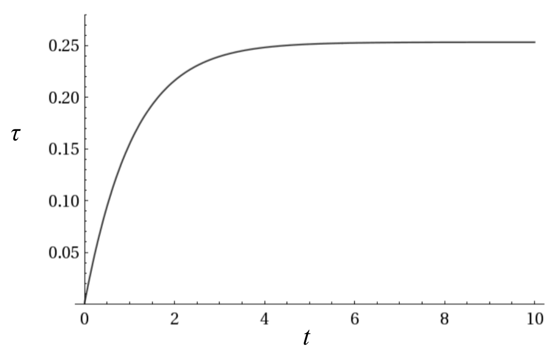

Figure 2 – Proper Time vs. Coordinate Time

We see in Figure 2 that as coordinate time goes to infinity, the event horizon being at infinite coordinate time, the proper time stays constant. This is why it appears as though it only takes a finite amount of proper time for Scout to reach the horizon. But there is a flaw in this integration. Note that the t and τ coordinates are evenly spaced at all times. This integration is applicable to flat spacetime, but is not valid for curved spacetime. Figure 2 shows Scout's proper time if she were accelerating in Minkowski spacetime. The Schwarzschild metric does not tell us the relationship between the values of coordinate and proper time, it only tells us the relationship between *changes* in coordinate and proper time. With this in mind consider Figure 3, which shows τ vs. t for various observers at rest:

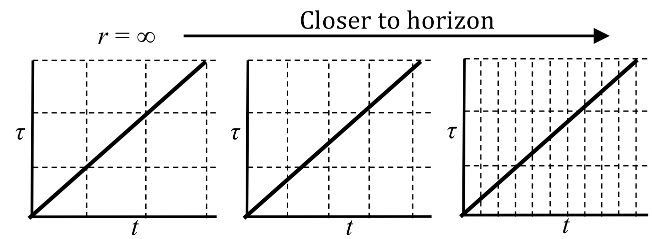

Figure 3 – Time Curvature for Rest Observers

On the left of Figure 3, we see the observer at infinity where changes in proper time equal changes in coordinate time. The center and right diagrams show observers at rest at some finite distance from the horizon. Figure 3 underscores that the time dilation is due to the coordinate curvature of t relative to the observer at infinity. If you were to stretch out the center and right diagrams in Figure 3 such that the time coordinate is equally spaced in all three diagrams, the slopes of the lines relating τ and t would become shallower as a result of the stretching. Now, there is no problem integrating those figures because the coordinate spacings are *constant over the regions in question*. It is interesting to compare these cases to Special Relativity. In SR, $\frac{d\tau}{dt} = \sqrt{1 - \left(\frac{dr}{dt}\right)^2}$, so the slope of the lines in Figure 3 in an SR context would be shallower as relative velocity increases. The GR version of this relationship is very similar in that the relationship is given by $\frac{d\tau}{dt} = \sqrt{1 - \frac{1}{r}} = \sqrt{1 - V(r)^2}$, where, as we have found, $V(r)$ is the relative velocity between the observer falling from infinity and the observer at rest at r (where we know that this relative velocity is caused by coordinate curvature).

An equivalent way to depict the cases in Figure 3 would be to leave the t spacing constant for all three cases, while increasing the τ coordinate spacing for observers closer to the horizon. It is this depiction that we will use to view the correct τ vs. t plot for the freefalling observer in Figure 4:

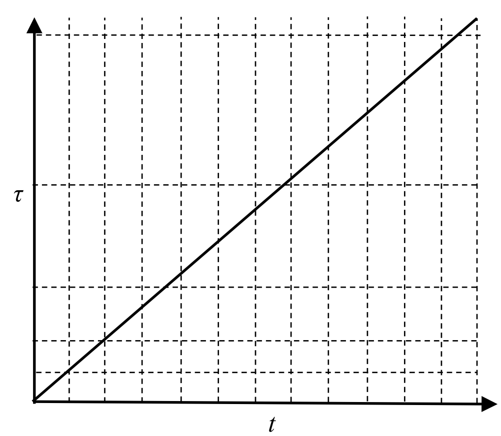

Figure 4 – Proper Time vs. t for the Freefalling Observer

We know that the acceleration of the freefalling observer is the result of manifold curvature and not because the observer is an accelerated reference frame. Since t goes to infinity at the horizon, we can see that τ also goes to infinity while falling; the decrease in $\frac{d\tau}{dt}$ in (3) over time is simply a result of the coordinate curvature (this derivative is maintained in Figure 4, but the line extends infinitely). We can see in Figure 4 that if we were to compress the τ coordinate axis as t increases (such that the τ and t spacing is always the same in the vicinity of the worldline), the straight line in Figure 4 would take the form of the curve in Figure 2. Again, this is simply the result of the fact that these accelerations in General Relativity are caused by the underlying manifold curvature and not the typical Newtonian changes in motion over time, and we therefore cannot naively integrate the differentials from the Schwarzschild metric without accounting for this curvature.

Radial Coordinate Transformation

It is desirable at this point to make a coordinate change for the radial coordinate such that it is better able to capture the curvature near the horizon similar to the way the time coordinate does. We will choose coordinate R such that $\frac{dR}{dr} = \frac{r}{r-1}$. This coordinate varies identically to the r coordinate for large r (this is good because r is a good physical coordinate at large r) and then diverges from it at the horizon. Integrating the expression gives:

$$R = r + \ln(r-1),\ r = W(e^{R-1}) + 1 \tag{4}$$

Where W is the product-log function. Note that $R \to \infty$ as $r \to \infty$ and $R \to -\infty$ as $r \to 1$. R is zero in the region of the elbow of the geodesic pictured in Figure 1. Making this coordinate substitution in (2) gives:

$$\frac{dR}{dt} = -\sqrt{\frac{1}{W(e^{R-1})+1}} = -V \tag{5}$$

This coordinate choice is also useful because the speed of light is 1 independent of R and t. The Schwarzschild metric with the new coordinate becomes:

$$d\tau^2 = \frac{W(e^{R-1})}{W(e^{R-1})+1}[dt^2 - dR^2] \tag{6}$$

A portion of Scout's worldline is plotted on the t-R plane is shown in Figure 5 below:

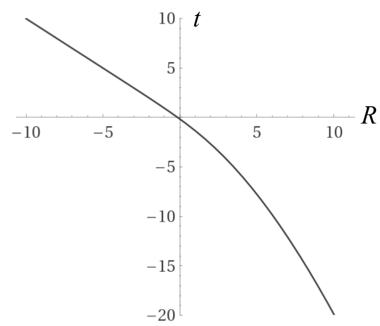

Figure 5 – t vs. R

The slope of the worldline is close to but less than 1 in the upper right quadrant for all finite R and t. To understand how the proper time relates to the worldline, it is easiest to imagine a 3rd axis perpendicular to the page representing proper time. In this dimension, the worldline rises to infinity as t increases (note that the spacing on that axis will be non-linear – a consequence of the intrinsic spacetime curvature is that either the τ dimension or the R and t dimensions must have non-linear spacing). Another benefit of this coordinate change is in its relationship to the proper time of the freefalling observer. Combining (3) and (5) gives:

$$d\tau = -dR\,\frac{W(e^{R-1})}{\sqrt{W(e^{R-1})+1}} \tag{7}$$

Far from the black hole, the R coordinate behaves in the same way as the r coordinate in that $\frac{d\tau}{dR} \to \frac{d\tau}{dr} \to \infty$ as $R \to r \to \infty$ and it behaves like the t coordinate $\left(\frac{d\tau}{dR} \to \frac{d\tau}{dt} \to 0\right)$ near the horizon. Figure 6 shows Scout's worldline depicted with all relevant quantities such that all the Schwarzschild differential relationships are captured:

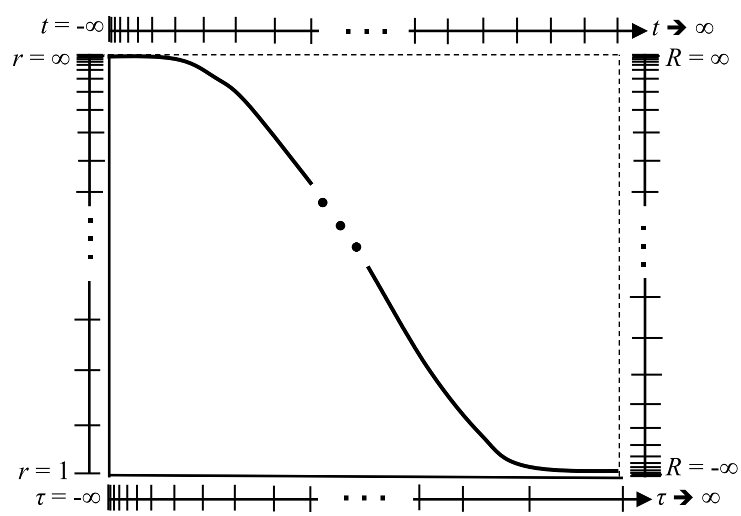

Figure 6 – Freefalling Geodesic Plotted against Multiple Coordinates

Kruskal-Szekeres Coordinates

Another commonly used coordinate system to analyze black hole geodesics are the Kruskal-Szekeres (KS) coordinates. Outside the horizon, these are defined in terms of Schwarzschild coordinates as [2]:

$$X = \sqrt{(r-1)e^r}\cosh\left(\frac{t}{2}\right)$$
$$T = \sqrt{(r-1)e^r}\sinh\left(\frac{t}{2}\right) \tag{8}$$

The differentials of these coordinates are given by:

$$dX = \frac{\partial X}{\partial r}dr + \frac{\partial X}{\partial t}dt = \frac{re^r}{2\sqrt{(r-1)e^r}}\left[\cosh\left(\frac{t}{2}\right)dr + \frac{(r-1)}{r}\sinh\left(\frac{t}{2}\right)dt\right]$$
$$dT = \frac{\partial T}{\partial r}dr + \frac{\partial T}{\partial t}dt = \frac{re^r}{2\sqrt{(r-1)e^r}}\left[\sinh\left(\frac{t}{2}\right)dr + \frac{(r-1)}{r}\cosh\left(\frac{t}{2}\right)dt\right] \tag{9}$$

With some manipulation and substituting (2) into (9) to get the differentials of the freefalling worldline we get:

$$\frac{dr}{dX} = \left[\frac{re^r}{2\sqrt{(r-1)e^r}}\left[1-\sqrt{r}\tanh\left(\frac{t(r)}{2}\right)\right]\cosh\left(\frac{t(r)}{2}\right)\right]^{-1}$$
$$\frac{dr}{dT} = \left[\frac{re^r}{2\sqrt{(r-1)e^r}}\left[\tanh\left(\frac{t(r)}{2}\right)-\sqrt{r}\right]\cosh\left(\frac{t(r)}{2}\right)\right]^{-1} \tag{10}$$

Where *t*(*r*) is found from integrating (2), giving: $t(r) = -\frac{2}{3}\sqrt{r}(3+r) + \ln\left[\frac{\sqrt{r}+1}{\sqrt{r}-1}\right]$. What we find here is that as *r* goes to 1, the derivatives in (10) go to zero for the freefalling worldline. From this, we see that either the worldline is displaced by $dX \to dT \to \infty$ at the horizon, meaning that the worldline discontinuously spikes parallel to the horizon there, or the worldline simply terminates there. The metric in KS coordinates is given by $d\tau^2 = \frac{4}{re^r}[dT^2 - dX^2]$, or equivalently $d\tau^2 = \frac{4}{re^r}\left[\left(\frac{dT}{dr}\right)^2 dr^2 - \left(\frac{dX}{dr}\right)^2 dr^2\right]$ and what (10) shows is that $\left(\frac{dT}{dr}\right)^2 \to \left(\frac{dX}{dr}\right)^2$ for the freefaller at the horizon meaning that $d\tau^2$ there goes to zero. This is what we also see in (6) and in Figure 5, namely that the freefalling observer approaches the speed of light on approach of the horizon. We must therefore conclude that the worldline on the KS chart terminates at the horizon since a timelike observer cannot be moving at light speed in any frame and crossing the horizon would result in a discontinuity in the worldline at the horizon.

Conclusion

It has been shown that when accounting for curved spacetime while integrating the freefall geodesic, the freefaller experiences an infinite amount of proper time before reaching the horizon. We also know that the freefalling observer will not receive all infalling signals from rest observers as her worldline will approach a final light signal asymptotically as can be deduced from Figure 5. Therefore, we must conclude that in the frame of the freefalling observer near the horizon, when she looks out to signals coming from the rest observers, those observers will appear to her to be slowing down since she experiences infinite proper time in her frame while receiving a finite number of light signals from the rest observers. What we find is that the rest observers will see the freefalling observer slow *exponentially* as their times go to infinity, while the freefaller will see the rest observers slow *asymptotically* as her time goes to infinity. This means that in the rest observer frame, the freefaller will have an open future, unfolding at an exponentially slower rate over time, while in the freefalling frame the rest observers will have a closed future, where the rest observers will appear to evolve toward a finite future time at an asymptotically slower rate over time. These features are shown in Figure 7 below:

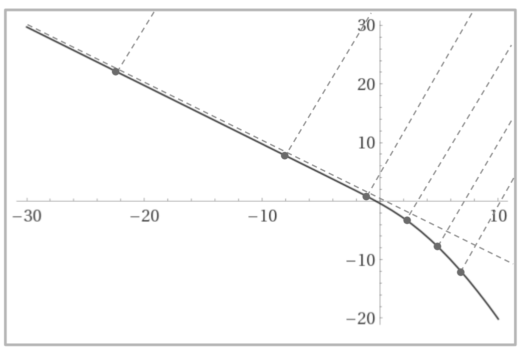

Figure 7 – Light Signals on *t*-*R* Chart

Figure 7 is a *t*-*R* chart that shows a single infalling signal representing the signal to which the freefall worldline is asymptotic. The freefalling observer will receive this signal after an infinite time and will receive no signals lying above that one on the chart. The dots represent intervals of equal proper time along the worldline and we can see that since the worldline is infinite (with tangents always below the speed of light) on this chart, there will be an infinite number of dots on the line spaced increasingly far apart and rest observers will receive an infinite number of signals from the freefalling observer.

References

[1] Raine, D., Thomas, E.: Black Holes: A Student Text. Imperial College Press, (2015).

[2] Cheng T. P.: Relativity, Gravitation and Cosmology. A Basic Introduction. Oxford University Press, New York, (2010).