Newton (~1666) , π formulas

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September 10 ,2016

abstract

In this note we give some formulas for pi constant :

$\pi = 3.1415926535 \ldots$

Introducción

1. Newton (~1666):

$$\pi = \frac{3\sqrt{3}}{4} + 24\int_0^{1/4} \sqrt{x(1-x)}\, dx \tag{1}$$

$$\pi = \frac{3\sqrt{3}}{4} + 2 - \frac{3}{4}\sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-4n}}{(n+1)(2n+5)} \tag{2}$$

Series alternativas para π

2. Serie n°1:

$$I = \int_0^{1/4} \sqrt{x\,(1-x)}\; dx \tag{3}$$

$$\sqrt{x\,(1-x)} = \frac{\sqrt{x}\;(1-x)}{\sqrt{1-x}} = \sqrt{x}\;(1-x) \sum_{n=0}^{\infty} \binom{2\,n}{n} 2^{-2\,n}\, x^n \tag{4}$$

$$\pi = \frac{3\sqrt{3}}{4} + \frac{3}{2} \sum_{n=0}^{\infty} \binom{2\,n}{n} \frac{2^{-4\,n}\,(6\,n+17)}{(2\,n+3)\,(2\,n+5)} \tag{5}$$

3. Serie n°2:

$$I = \int_0^{1/4} \sqrt{x\,(1-x)}\; dx \tag{6}$$

$$y = \sqrt{x\,(1-x)}\;,\; 0 \le x \le \frac{1}{4} \Longrightarrow x = \frac{1-\sqrt{1-4\,y^2}}{2}\;,\; 0 \le y \le \frac{\sqrt{3}}{4} \tag{7}$$

$$I = \frac{\sqrt{3}}{16} - \int_0^{v} \frac{1-\sqrt{1-4\,y^2}}{2}\; dy\;,\; v = \frac{\sqrt{3}}{4} \tag{8}$$

$$\sqrt{1-4\,y^2} = \left(1-4\,y^2\right) \sum_{n=0}^{\infty} \binom{2\,n}{n} y^{2\,n} \tag{9}$$

$$\pi = -\frac{3\sqrt{3}}{4} + \frac{3\sqrt{3}}{4} \sum_{n=0}^{\infty} \binom{2\,n}{n} \left(\frac{3}{16}\right)^n \frac{(2\,n+9)}{(2\,n+1)\,(2\,n+3)} \tag{10}$$

4. Serie n°3:

$$I = \int_0^{1/4} \sqrt{x\,(1-x)}\; dx \tag{11}$$

$$y=\sqrt{x(1-x)}\ ,\ 0\le x\le\frac{1}{4}\Longrightarrow x=\frac{1-\sqrt{1-4y^2}}{2}\ ,\ 0\le y\le\frac{\sqrt{3}}{4}\tag{12}$$

$$I=\frac{\sqrt{3}}{16}-\int_0^v\frac{1-\sqrt{1-4y^2}}{2}\,dy\ ,\ v=\frac{\sqrt{3}}{4}\tag{13}$$

$$\sqrt{1-4y^2}=1-2\sum_{n=0}^{\infty}\binom{2n}{n}\frac{y^{2n+2}}{n+1}\tag{14}$$

$$\pi=\frac{9\sqrt{3}}{4}-\frac{9\sqrt{3}}{8}\sum_{n=0}^{\infty}\binom{2n}{n}\left(\frac{3}{16}\right)^n\frac{1}{(n+1)(2n+3)}\tag{15}$$

5. Serie n°4:

$$I=\int_0^{1/4}\sqrt{x(1-x)}\,dx\tag{16}$$

$$\sqrt{x(1-x)}=\sqrt{x}\sqrt{\frac{3}{4}}\sqrt{1+\frac{1-4x}{3}}\tag{17}$$

$$\pi=\frac{7\sqrt{3}}{4}+\frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-1)^n\,12^{-n}}{(n+1)}\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{(-1)^k}{2k+3}\tag{18}$$

6. Serie n°5:

$$\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{(-1)^k}{2k+3}=\frac{(n+1)!}{4\,(1/2)_{n+3}}\tag{19}$$

$$(1/2)_n=\binom{2n}{n}n!\,2^{-2n}\tag{20}$$

Combinando (18), (19), (20) , se tiene :

$$\pi=\frac{7\sqrt{3}}{4}+\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n\,3^{-n}}{(2n+1)(2n+3)(2n+5)}\tag{21}$$

7. Serie n°6:

$$I = \int_0^{1/4} \sqrt{x\,(1-x)}\; dx \tag{22}$$

$$\sqrt{x\,(1-x)} = \frac{\sqrt{x}\;(1-x)}{\sqrt{1-x}} = \frac{\sqrt{x}\;(1-x)}{\sqrt{\frac{3}{4}}\;\sqrt{1+\frac{1-4\,x}{3}}} \tag{23}$$

$$\pi = \frac{3\sqrt{3}}{4} + \sqrt{3}\sum_{n=0}^{\infty}\binom{2\,n}{n}(-1)^n\,12^{-n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k\,(6\,k+17)}{(2\,k+3)\,(2\,k+5)} \tag{24}$$

8. Serie n°7:

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k\,(6\,k+17)}{(2\,k+3)\,(2\,k+5)} = \frac{(8\,n+17)\,n!}{8\,(1/2)_{n+3}} \tag{25}$$

$$(1/2)_n = \binom{2\,n}{n}\,n!\,2^{-2\,n} \tag{26}$$

Combinando (24), (25), (26) , se tiene :

$$\pi = \frac{3\sqrt{3}}{4} + \sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n\,3^{-n}\,(8\,n+17)}{(2\,n+1)\,(2\,n+3)\,(2\,n+5)} \tag{27}$$

9. Serie n°8:

$$4\binom{2\,n}{n} - \binom{2\,n+2}{n+1} = \binom{2\,n}{n}\frac{2}{n+1} \tag{28}$$

Combinando (2), (28) , se tiene :

$$\pi = \frac{3\sqrt{3}}{4} + 2 - \frac{3}{8}\sum_{n=0}^{\infty}\left(4\binom{2\,n}{n} - \binom{2\,n+2}{n+1}\right)\frac{2^{-4\,n}}{(2\,n+5)} \tag{29}$$

10. Serie n°9:

$$\frac{1}{\sqrt{3}} = \frac{1}{2} \sum_{n=0}^{\infty} \binom{2n}{n} 2^{-4n} \tag{30}$$

Combinando (2) , (30) , se tiene :

$$\pi = 2 + \frac{3}{8} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-4n} \left(6n^2 + 21n + 13\right)}{(n+1)(2n+5)} \tag{31}$$

11. Serie n°10:

$$\sqrt{3} = 2 - \frac{1}{4} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-4n}}{n+1} \tag{32}$$

Combinando (2) , (32) , se tiene :

$$\pi = \frac{7}{2} - \frac{3}{16} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-4n}(2n+9)}{(n+1)(2n+5)} \tag{33}$$

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