# Attacking Legendre’s Conjecture Using Moivre-Stirling Approximation


Bijoy Rahman Arif

Sonargaon University, Dhaka

bijoyarif71@yahoo.com



## Abstract:

In this paper, we are going to prove a famous problem concerning prime numbers. Legendre’s conjecture states that there is always a prime $p$ with $n^2 < p < (n+1)^2$, if $n > 0$. In 1912, Landau called this problem along with other three problems “unattackable at the presesnt state of mathematics.” Our approach to solve this problem is very simple. We will find a lower bound of the difference of second Chebyshev functions, $\psi((n+1)^2)-\psi(n^2)$ using a better Moiver-Stirling approximation and finally, we transfer it to the difference of first Chebyshev functions, $\upsilon((n+1)^2)-\upsilon(n^2)$. The final difference is always greater than zero will prove Legendre’s conjecture.


## Definition

According to Moivre-Stirling Approximation of factorial [1]:

$$\int_1^n \log(x)dx \ < \ \log(n!) \ < \ \int_1^{n+1} \log(x)dx \ , \ \int \log(x)dx \ = \ xlog(x)-x+C \quad , \ldots.. (1)$$

If we assume $\Delta_2=\frac{1}{2}\log(\frac{2}{1})$ , $\Delta_3=\frac{1}{2}\log(\frac{3}{2})$ ,…, $\Delta_{n-1}=\frac{1}{2}\log(\frac{n-1}{n-2})$ , $\Delta_n=\frac{1}{2}\log(\frac{n}{n-1})$ , we can get better Moivre-Stirling Approximation using simple geometric arguments from figure below [1]:

$$\int_{n-1}^n \log(n)dn-\log(n-1) \ = \ nlog(\frac{n}{n-1})-1 \ = \ 2n\Delta_n-1 \ \le \ \Delta_{n-1} \quad \text{for } n > 2$$

$$\text{because } \frac{d}{dn}(2n\Delta_n-\Delta_{n-1}) \ > 0 \text{ and } \lim_{n\to\infty}(2n\Delta_n-\Delta_{n-1}) \ = 1$$

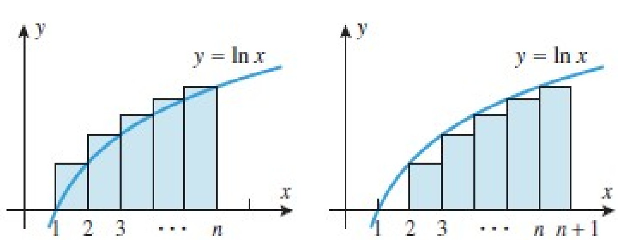
so,
$$\int_2^{n+1} \log(x)dx \ - \ (\Delta_2+\Delta_3+\ldots+\Delta_{n-1}+\Delta_n) \ < \ \log(n!)$$

$$< \ \int_1^n \log(x)dx \ + \ (\Delta_2+\Delta_3+\ldots+\Delta_{n-1}+\Delta_n)$$

hence,
$$\int_2^{n+1} \log(x)dx \ - \ \frac{1}{2}\log(n) \ < \ \log(n!) \ < \ \int_1^n \log(x)dx \ + \ \frac{1}{2}\log(n)$$

or,
$$(n+1)\log(n+1)-n-\frac{1}{2}\log(n)-b+1 \ < \ \log(n!) \ < \ nlog(n)-n+\frac{1}{2}\log(n)+1 \quad , \ldots.. (2)$$

where $b = log(4)$




▲ Figure Ex-30

We define function $\upsilon(x)$ and $\psi(x)$ conventionally as [2]:

$$\upsilon(x) = \sum_{p\le x} \log(p) = \log \prod_{p\le x} p \quad , \quad \psi(x) = \sum_{p^m\le x} \log(p)$$

Since $p^2 \le x, p^3 \le x, \ldots$ are equivalent to $p \le x^{1/2}, p \le x^{1/3}, \ldots$ , we have [2], [3] :

$$\psi(x) = \upsilon(x)+\upsilon(x^{1/2})+\upsilon(x^{1/3})+\ldots = \sum_{m\ge 1} \upsilon(x^{1/m}) \quad , \ldots.. (3)$$

and so, $\psi(2n) = \upsilon(2n)+\upsilon((2n)^{1/2})+\upsilon((2n)^{1/3})+\ldots = \sum_{m\ge 1} \upsilon((2n)^{1/m})$

## Proof

We know [3]: $\log((2n)!) = \psi(2n)+\psi(n)+\psi(\frac{2n}{3})+\ldots$ , ….. (4)

Let, $N_{2n}$= $\frac{(2n)!}{n!n!}$ , then from (4) [3]: $\log(N_{2n}) = \psi(2n)-\psi(n)+\psi(\frac{2n}{3})-\ldots$ , ….. (5)

As $\psi(x)$ is a steadily increasing function,

$$\log(\frac{N_{(n+1)^2}}{N_{n^2}}) = (\psi((n+1)^2)-\psi(n^2))-(\psi(\frac{(n+1)^2}{2})-\psi(\frac{n^2}{2}))+(\psi(\frac{(n+1)^2}{3})-\psi(\frac{n^2}{3}))-\ldots$$

$$< (\psi((n+1)^2)-\psi(n^2)) \quad , \ldots.. (6)$$

From (2) and (5), we get:

$$\log(\frac{N_{(n+1)^2}}{N_{n^2}}) >$$

$$((n+1)^2+1)\log((n+1)^2+1)-(n+1)^2-\log(n+1)-b+1-n^2\log(n^2)+n^2-\log(n)-1$$

$$-(n+1)^2\log(\frac{(n+1)^2}{2})+(n+1)^2-\log(\frac{(n+1)^2}{2})-2+(n^2+2)\log(\frac{n^2}{2}+1)-n^2-\log(\frac{n^2}{2})-2b+2 =$$

$$(n+1)^2\log(\frac{2(n+1)^2+2}{(n+1)^2})-n^2\log(\frac{2n^2}{n^2+2})+2\log(n^2+2)+\log((n+1)^2+1)-3\log(n+1)-3\log(n)-3b$$

, ….. (7)

Again, form relation of $\upsilon(x)$ and $\psi(x)$ , we get from (3):

$$\psi((n+1)^2)-\psi(n^2) = (\upsilon((n+1)^2)-\upsilon(n^2))+(\upsilon(n+1)-\upsilon(n))+(\upsilon((n+1)^{\frac{2}{3}})-\upsilon(n^{\frac{2}{3}}))+\ldots$$

We assume $2^m = (n+1)$ , then $(\upsilon(n+1)-\upsilon(n))+(\upsilon((n+1)^{\frac{2}{3}})-\upsilon(n^{\frac{2}{3}}))+\ldots$ $<$

$$m log(2)+(\frac{2m}{3})\log(2)+\ldots+\log(2) \;=\; 2m log(2)(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2m}) \;<\; 2m log(2)\int_{1}^{(2m)} \frac{1}{x}dx \;=$$

$$2m log(2)\log(2m) \;=\; 2\log(n+1)\log(2\,lg(n+1)) \qquad \text{where } \; lg(x)=\frac{\log(x)}{\log(2)}$$

because 1 = $(n+1)-n \;>\; (n+1)^{\frac{2}{3}}-n^{\frac{2}{3}} \;>\; (n+1)^{\frac{1}{2}}-n^{\frac{1}{2}} \;>\ldots$

hence, $(\psi((n+1)^2)-\psi(n^2)) \;<\; (\upsilon((n+1)^2)-\upsilon(n^2))+2\log(n+1)\log(2\,lg(n+1))$ , ….. (8)

Finally, we get from (6), (7) and (8):

$$(\upsilon((n+1)^2)-\upsilon(n^2)) \;>$$

$$(n+1)^2\log(\frac{2(n+1)^2+2}{(n+1)^2})-n^2\log(\frac{2n^2}{n^2+2})+2\log(n^2+2)+\log((n+1)^2+1)-3\log(n+1)-3\log(n)-3b$$
$$-2\log(n+1)\log(2\,lg(n+1)) \qquad ,\ldots.. (9)$$

As $(\upsilon((n+1)^2)-\upsilon(n^2)) > 0$ for $n > 3$, and the first derivative of right hand side is:

$$2n log(\frac{(n^2+2)((n+1)^2+1)}{n^2(n+1)^2})+2\log(\frac{2(n+1)^2+2}{(n+1)^2})-\frac{3}{n+1}-\frac{3}{n}$$

$$-\frac{2}{n+1}(\log(2\,lg(n+1)))-\frac{2}{n+1} \;> 0 \text{ for } n \geq 3$$

As we can easily verify the conjecture for $1 \leq n \leq 3$, we have actually proved Legendre's conjecture.

## Reference


[1] H. Anton, I. Bivens and S. Davis, "Calculus," John Wiley & Sons, Inc, NY, p. 528, p. 664, 2002.

[2] G. H. Hardy and E. M. Wright, "An Introduction to the Theory of Numbers," Oxford University Press, NY, pp. 340-341, 1979.

[3] Edited: G. H. Hardy, P. V. S. Aiyar and B. M. Wilson, "Collected Papers of Srinivasa Ramanujan," Cambridge University Press, pp. 208-209, 1927.