Gelfond ' s Constant using MKB constant like integrals.

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Abstract

First we will follow the path the author took to find out that for

$$\int_1^{\infty}\text{Cos}[\text{Pi}*I*x]\,x^{1/x}\,dx \text{ and } \int_1^{\infty}\text{Sin}[\text{Pi}*I*x]\,x^{1/x}\,dx,$$

the limit of the ratio of *a* to *a* – 1, as *a* goes to infinity is Gelfond ' *s* Constant, (*e* ^ pi).

We will consider that the hypothesis and provide hints for *a* proof using *L*' Hospital' *s* Rule (since we have indeterminate forms as *a* goes to infinity).

We find we there is no limit of the ratio of the previous forms of integrals when the "I" is left out, and give *a* small proof for them.

Preliminaries

Below, and throughout this paper, we will show the best known (to the author) Mathematica 11 options, found to give the desired results. Many of the computations took several minutes and produced a few warning messages which are not displayed.
In 1999[1], the constant referred to at https://oeis.org/A037077,
Limit[Sum[(-1)^n n^(1/n),{n,1,2x}],x->Infinity] [2], was named the MRB constant, (after its original investigator), by Simon Plouffe[3]. Then on Feb 23, 2009, Marvin Ray Burns named https://oeis.org/A157852 "MKB constant" (MKB) after his wife at the time. Technically, A157852 is the integer sequence of the digits of MKB, (the integral analog of the MRB constant[4]). Hence MKB was named after one with a close relationship to the person the MRB constant was named after.
MKB= $\left|\lim_{n\to\infty}\int_1^{2n} e^{i\pi x}\,x^{1/x}\,dx\right|$.

It can be proved that

$\lim_{n\to\infty}\int_1^{2n} e^{i\pi x}\, x^{1/x}\, dx ==$
$\lim_{n\to\infty}\int_1^{2n}\mathrm{Cos}[\mathrm{Pi}*x]\; x^{1/x}\, dx + I*\lim_{n\to\infty}\int_1^{2n}\mathrm{Sin}[\mathrm{Pi}*x]\; x^{1/x}\, dx.$

This is shown to be true up to 23 digits of precision in the result to the following Mathematica code. (Some unknown [to this author] form of regularization is used, [at least for the operation with sine in it. See work below as to why the cosine operation might involve convergent integrals.])

We will need the["NumericalCalculus`"] package.

```
Needs["NumericalCalculus`"]
```

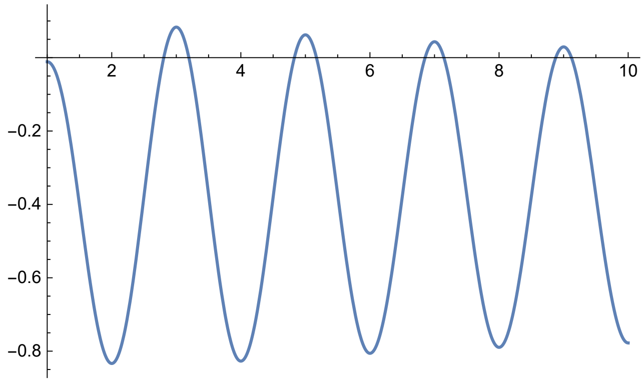
```
digits = 50;
Rationalize[NLimit[NIntegrate[Exp[Pi * I * x] x^ (1 / x), {x, 1, 2 a},
    WorkingPrecision → 50, PrecisionGoal → 50, MaxRecursion → 50], a → Infinity,
   WorkingPrecision → 50, Terms → 15, Method → SequenceLimit, WynnDegree → 6], 0] -
 (NIntegrate[x^ (1 / x) * Cos[Pi * x], {x, 1, Infinity}, WorkingPrecision → 2 * digits] +
   I NIntegrate[x^ (1 / x) Sin[Pi * x], {x, 1, Infinity}, WorkingPrecision → 2 * digits] - I / Pi)
```

```
9.799242913445728747553048568494203547251606464399860365576612315013382138315720 × 10^-23 -
 3.422998270568720522742953726075580777060144547611096469893522817755464792449 89 × 10^-23 ⅈ
```

Here are what $\int_1^a$`Cos[Pi * I * x]` $x^{1/x}$ and $\int_1^a$`Sin[Pi * I * x]` $x^{1/x}$ for a from 1 to 10 look like. They are periodic (like the trig functions they come from).

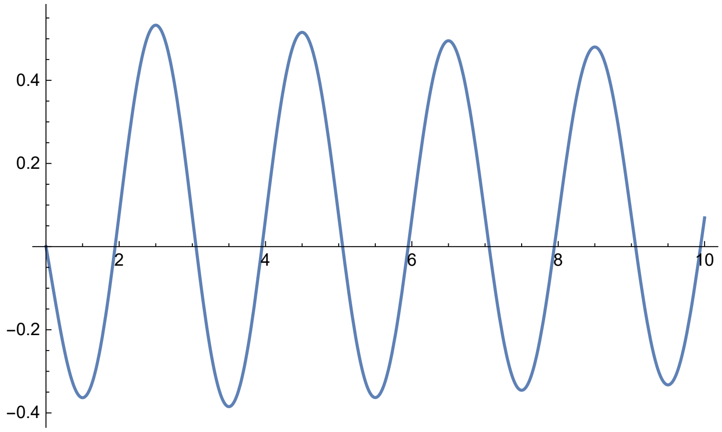
```
In[30] := Plot[Integrate[Sin[Pi * x] x^(1/x), {x, 1, a}], {a, 1, 10}]
```

Out[30] =



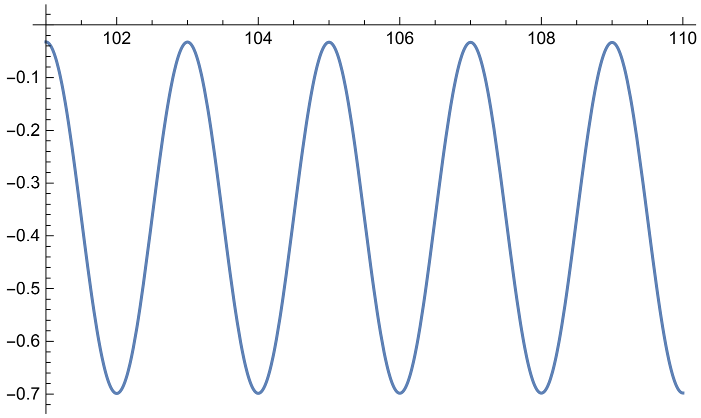
```
In[25] := Plot[Integrate[Cos[Pi * x] x^(1/x), {x, 1, a}], {a, 1, 10}]
```

Out[25] =

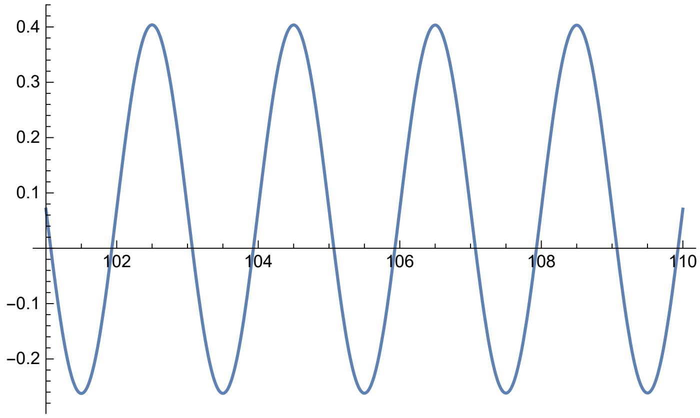


Here are plots of the same operations from 101 to110.

```
Plot[Integrate[Sin[Pi * x] x^(1/x), {x, 1, a}], {a, 101, 110}]
```



```
Plot[Integrate[Cos[Pi * x] x^(1/x), {x, 1, a}], {a, 101, 110}]
```



For more precise measurements and a general scheme for calculation of the digits of MKB, by flattening out the oscillatory integral, see Mathar’s work[5], where he also used many integration methods to compute MKB. He called it M1, published many error-bounds of methods, and compared M1 to what he called the MRB constant, M. R. Burns’ constant, and M.

Tools

We will use many integration and limit options available in Mathematica 11.0. We also use a Intel 6 core 3.5 GH extreme edition desktop for the computations.

Operations and Results

Realizing his mistake of confusing integral(Sin[Pi*I*x]x^(1/x),{x,1,a}) with integral(Sin[Pi*x]x^(1/x),{x,1,a}), through the following operations, on Monday, Aug 8, 2016 at 2:00PM, Marvin Ray Burns began to see that the limit of the ratio of a to a-1, as a goes to infinity, in integral(Sin[Pi*I*x]x^(1/x),{x,1,a})/ integral(Sin[Pi*I*x]x^(1/x),{x,1,a-1}) and integral(-Cos[Pi*I*x]x^(1/x),{x,1,a})/integral(Cos[Pi*I*x]x^(1/x),{x,1,a-1}) is Gelfond’s Constant, (e^pi).

.

```
i1 = Table[
  NIntegrate[Sin[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 20], {a, 9990, 10001}]
```

```
{2.0989175549269083147 × 10^13629 ⅈ, 4.8570402004948497061 × 10^13630 ⅈ,
 1.1239526514136629501 × 10^13632 ⅈ, 2.6009040701590741249 × 10^13633 ⅈ,
 6.0186716707026764644 × 10^13634 ⅈ, 1.3927621974178786927 × 10^13636 ⅈ,
 3.2229479272454941452 × 10^13637 ⅈ, 7.4581241228073160071 × 10^13638 ⅈ,
 1.7258614376558644163 × 10^13640 ⅈ, 3.9937625775425609291 × 10^13641 ⅈ,
 9.2418424666174747439 × 10^13642 ⅈ, 2.1386261832231106579 × 10^13644 ⅈ}
```

```
i2 = Ratios[i1]
```

```
{23.140690729341148011, 23.140690729698949373,
 23.140690730056647960, 23.140690730414243812, 23.140690730771736967,
 23.140690731129127467, 23.140690731486415350, 23.140690731843600657,
 23.140690732200683426, 23.140690732557663699, 23.140690732914541513}
```

```
N[E^Pi, 20]
```

```
23.140692632779269006
```

```
N[i2 - E^Pi, 20]
```

```
{-1.903438120995 × 10^-6, -1.903080319633 × 10^-6, -1.902722621046 × 10^-6, -1.902365025194 × 10^-6,
 -1.902007532038 × 10^-6, -1.901650141539 × 10^-6, -1.901292853656 × 10^-6, -1.900935668349 × 10^-6,
 -1.900578585579 × 10^-6, -1.900221605307 × 10^-6, -1.899864727493 × 10^-6}
```

Larger a: (Both sine and cosine in the this operation give Gelfond’s Constant.)

```
i11 = Table[NIntegrate[Cos[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 30],
  {a, 999 990, 1 000 000}]
```

```
{8.16402704299180404095259449988 × 10^1 364 361,
 1.88921240445149992403409339252 × 10^1 364 363, 4.37176835688857590343446623243 × 10^1 364 364,
 1.01165747807172488243183607690 × 10^1 364 366, 2.34104547494103980098507246551 × 10^1 364 367,
 5.41734137742750968168005476577 × 10^1 364 368, 1.25361031700280490130583526513 × 10^1 364 370,
 2.90094110266711182266711546873 × 10^1 364 371, 6.71297864017550964458508807654 × 10^1 364 372,
 1.55342975360723204911512762411 × 10^1 364 374, 3.59474404543783042332075392032 × 10^1 364 375}
```

```
i12 = Ratios[i11]
```

```
{23.1406926324827036186672961841, 23.1406926324827041886616877912,
 23.1406926324827047586544156856, 23.1406926324827053286454798737,
 23.1406926324827058986348803621, 23.1406926324827064686226171574,
 23.1406926324827070386086902659, 23.1406926324827076085930996944,
 23.1406926324827081785758454491, 23.1406926324827087485569275368}
```

```
N[E^Pi, 20]
```

```
23.140692632779269006
```

```
N[i12 - E^Pi, 20]
```

```
{-2.965653870617901839 × 10^-10, -2.965648170673985768 × 10^-10,
 -2.965642470746706824 × 10^-10, -2.965636770836064943 × 10^-10, -2.965631070942060058 × 10^-10,
 -2.965625371064692106 × 10^-10, -2.965619671203961020 × 10^-10, -2.965613971359866736 × 10^-10,
 -2.965608271532409188 × 10^-10, -2.965602571721588312 × 10^-10}
```

Even larger a,(about as big as Mathematica will tolerate)!

```
i21 = Table[NIntegrate[Cos[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 40],
   {a, 9 999 990, 10 000 008}]
```

```
{1.248750590784479766789088244974362158446 × 10^13 643 749,
 2.889695359634080199319377313714578172209 × 10^13 643 750,
 6.686955211965069049415827804642155765176 × 10^13 643 751,
 1.547407752092216146387935592164153432162 × 10^13 643 753,
 3.580808716874046215569412546222015226236 × 10^13 643 754,
 8.286239389394650135109617259085898574115 × 10^13 643 755,
 1.917493187915811799795191512425441918062 × 10^13 643 757,
 4.437212048700115233728772579760159563694 × 10^13 643 758,
 1.026801601654186402948091558294664519330 × 10^13 643 760,
 2.376090025872139277296579342903098022129 × 10^13 643 761,
 5.498436895651140346939155692172340668487 × 10^13 643 762,
 1.272376381629768218143854086101177504926 × 10^13 643 764,
 2.944367076049786971400355256642342436912 × 10^13 643 765,
 6.813469350493284237431892945563446297757 × 10^13 643 766,
 1.576684000026034571600891047822655334778 × 10^13 643 768,
 3.648555982361789107489894116297179355389 × 10^13 643 769,
 8.443011254130941753836957370586199018495 × 10^13 643 770,
 1.953771283269108031257718779721784119022 × 10^13 643 772,
 4.521162074087431137089658317298126596488 × 10^13 643 773}
```

```
i22 = Ratios[i21]
```

```
{23.14069263277577056704002372095483993866, 23.14069263277577056771657128578128293096,
 23.14069263277577056839311865227146209701, 23.14069263277577056906966582042545538291,
 23.14069263277577056974621279024334073474, 23.14069263277577057042275956172519609852,
 23.14069263277577057109930613487109942026, 23.14069263277577057177585250968112864590,
 23.14069263277577057245239868615536172137, 23.14069263277577057312894466429387659253,
 23.14069263277577057380549044409675120524, 23.14069263277577057448203602556406350528,
 23.14069263277577057515858140869589143843, 23.14069263277577057583512659349231295041,
 23.14069263277577057651167157995340598690, 23.14069263277577057718821636807924849356,
 23.14069263277577057786476095786991841598, 23.14069263277577057854130534932549369975}
```

```
N[E^Pi, 20]
```

```
23.140692632779269006
```

```
N[i22 - E^Pi, 20]
```

```
{-3.4984386890626469937 × 10^-12, -3.4984380125150821673 × 10^-12,
 -3.4984373359677156771 × 10^-12, -3.4984366594205475231 × 10^-12,
 -3.4984359828735777052 × 10^-12, -3.4984353063268062234 × 10^-12,
 -3.4984346297802330774 × 10^-12, -3.4984339532338582674 × 10^-12,
 -3.4984332766876817932 × 10^-12, -3.4984326001417036547 × 10^-12,
 -3.4984319235959238518 × 10^-12, -3.4984312470503423845 × 10^-12,
 -3.4984305705049592527 × 10^-12, -3.4984298939597744562 × 10^-12,
 -3.4984292174147879951 × 10^-12, -3.4984285408699998693 × 10^-12,
 -3.4984278643254100786 × 10^-12, -3.4984271877810186231 × 10^-12}
```

When we switch forms of the operations, Mathematica can give at least 42 digits of accuracy here. Again we will need the [“NumericalCalculus`”] package.

```
Needs["NumericalCalculus`"]

N[E^Pi -
   Rationalize[NLimit[NIntegrate[Cos[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 50,
       PrecisionGoal → 50, MaxRecursion → 50] / NIntegrate[Cos[Pi * I * x] x^(1/x), {x, 1,
        a - 1}, WorkingPrecision → 50, PrecisionGoal → 50, MaxRecursion → 50], a → Infinity,
     WorkingPrecision → 50, Terms → 15, Method → SequenceLimit, WynnDegree → 6], 0], 50]
```

```
-5.9702708245087317429519268920016913700154824404879 × 10^-42
```

```
N[E^Pi -
   Rationalize[NLimit[NIntegrate[Sin[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 50,
       PrecisionGoal → 50, MaxRecursion → 50] / NIntegrate[Sin[Pi * I * x] x^(1/x), {x, 1,
        a - 1}, WorkingPrecision → 50, PrecisionGoal → 50, MaxRecursion → 50], a → Infinity,
     WorkingPrecision → 50, Terms → 15, Method → SequenceLimit, WynnDegree → 6], 0], 50]
```

```
-5.9702708245087317429519268920016913700154824404879 × 10^-42
```

The following should help in a proof of the hypothesis: Cos[Pi*I*x] == Cosh[Pi*x], Sin[Pi*I*x] == I Sinh[Pi*x], and Limit[x^(1/x),x->Infinity]==1.
Using L’Hospital’s Rule, we have the following:

In[41]:=
$$e^{\pi}-\left(\lim_{a\to\infty}\left(\frac{x^{1/x}\cos(\pi\, i\, x)\ /.\ x\to a}{x^{1/x}\cos(\pi\, i\, x)\ /.\ x\to a-1}\right)\right)$$

Out[41]= 0

In[46]:=
$$e^{\pi}-\left(\lim_{a\to\infty}\left(\frac{x^{1/x}\sinh(\pi\, x)\ /.\ x\to a}{x^{1/x}\sinh(\pi\, x)\ /.\ x\to a-1}\right)\right)$$

Out[46]= 0

If we perform the same operations on only the integrals without the "I," (the later integrals), we get the following results.

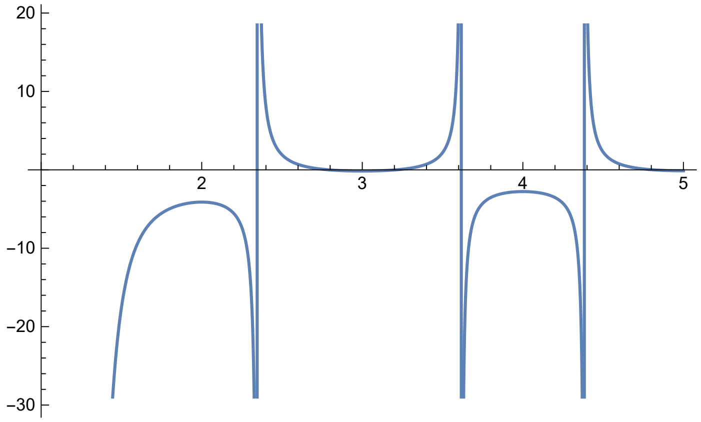
There ' s no point in working on the
Limit of $\int_1^a \mathrm{Sin}[\mathrm{Pi} * x]\, x^{1/x}\, dx \Big/ \int_1^{a-1} \mathrm{Sin}[\mathrm{Pi} * x]\, x^{1/x}\, dx$, because it is clearly divergent.

Here is a look at

$$\int_1^a \text{Sin}[\text{Pi}*x]\, x^{1/x}\, dx \Big/ \int_1^{a-1} \text{Sin}[\text{Pi}*x]\, x^{1/x}\, dx.$$

How does one say, take the limit as a->infinity, here?

```
Plot[Integrate[Sin[Pi * x] x^(1/x), {x, 1, a}] /
  Integrate[Sin[Pi * x] x^(1/x), {x, 0, a - 1}], {a, 1, 5}]
```



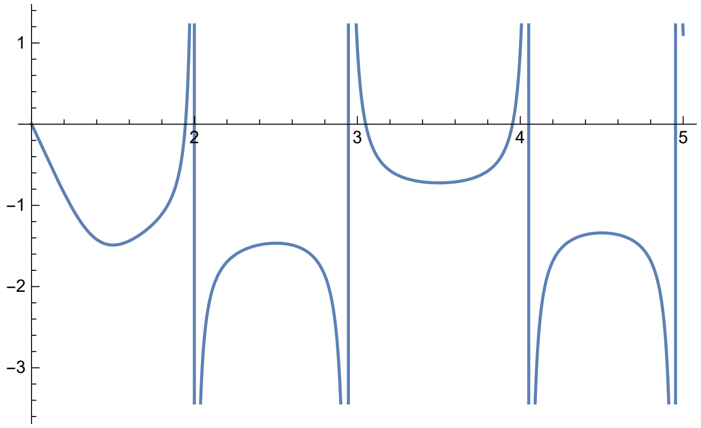
Sin[Pi∗*x*] == 0 → Sin[Pi∗*x*] *x*^(1/*x*) == 0 –> Integrate[Sin[Pi∗*x*] *x*^(1/*x*), {*x*, 0, *a* − 1}] == 0, all periodically as *a* → Infinity.

The same is true for

$$\int_1^a \text{Cos}[\text{Pi}*x]\, x^{1/x}\, dx \Big/ \int_1^{a-1} \text{Cos}[\text{Pi}*x]\, x^{1/x}\, dx.$$

It diverges at some points to :

```
Plot[Integrate[Cos[Pi * x] x^(1/x), {x, 1, a}] /
  Integrate[Cos[Pi * x] x^(1/x), {x, 1, a - 1}], {a, 1, 5}]
```

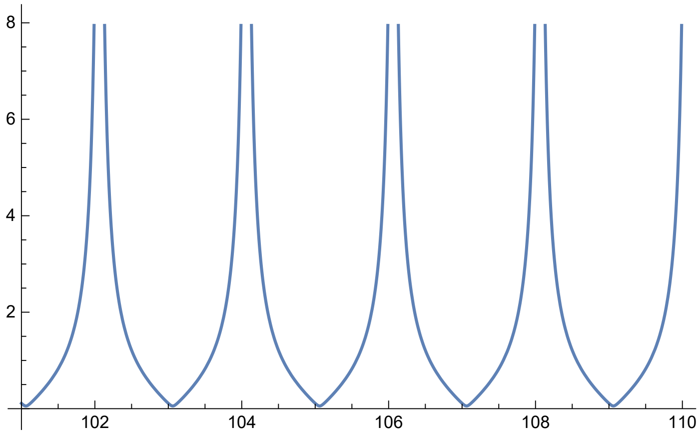


Cos[Pi∗*x*] == 0 → Cos[Pi∗*x*] *x*^(1/*x*) == 0 –> Integrate[Cos[Pi∗*x*] *x*^(1/*x*), {*x*, 0, *a* − 1}] == 0, all periodically as *a* → Infinity.

Looking at Limit of abs $\int_1^a \text{Exp}[\text{Pi}*x]\, x^{1/x}\, dx \Big/ \text{abs} \int_1^{a-1} \text{Exp}[\text{Pi}*x]\, x^{1/x}\, dx$,

it seems to diverge so we won ' *t* try to calculate it :

```
Plot[Abs[NIntegrate[Exp[Pi * I * x] x^ (1 / x), {x, 1, a}, WorkingPrecision → 30]] / Abs[
   NIntegrate[Exp[Pi * I * x] x^ (1 / x), {x, 1, a - 1}, WorkingPrecision → 30]], {a, 101, 110}]
```



Not sure how to prove this.

Open Questions

If we concern our self’s only with the domain of ratio of the later integrals for which they are defined, do they then have limits as a-> infinity? And what are they?

Sumary

We followed the path the author took to find out that for

$$\int_1^\infty \text{Cos}[\text{Pi}*I*x]\, x^{1/x}\, dx \text{ and } \int_1^\infty \text{Sin}[\text{Pi}*I*x]\, x^{1/x}\, dx,$$

**the limit of the ratio of *a* to *a* − 1, as *a* goes to infinity, is Gelfond ' *s* Constant,
(*e* ^ pi). We will considered that the hypothesis and provided hints for *a* proof using *L*’ Hospital’ *s* Rule, since we have indeterminate forms as *a* goes to infinity.**

We struggled with numeric evaluation to compare the limit of the ratios of

$$\int_1^\infty \text{Cos}[\text{Pi}*I*x]\, x^{1/x}\, dx \text{ and } \int_1^\infty \text{Sin}[\text{Pi}*I*x]\, x^{1/x}\, dx \text{ with } \int_1^\infty \text{Cos}[\text{Pi}*x]\, x^{1/x}\, dx \text{ and } \int_1^\infty \text{Sin}[\text{Pi}*x]\, x^{1/x}\, dx.$$

Finally, through some simple plotting and taking into consideration oscillatory nature of the sine and cosine functions, we make judgments on whether we should invest in finding their limit.

References

1 S. R. Finch, Mathematical Constants, Cambridge, 2003, pp 450, 452

2 http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.695.5959&rep=rep1&type=pdf page 28.

3 http://www.plouffe.fr/simon/articles/Tableofconstants.pdf

4 www.people.fas.harvard.edu/~sfinch/csolve/erradd.pdf p. 64.

5 http://arxiv.org/abs/0912.3844