formulas for pi

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abstract

In this note we give some formulas for pi constant:

$$\pi = 3.14159265\ ...$$

Introducción

En esta nota mostramos una colección de fórmulas para la constante pi :

$$\frac{1}{\pi} = \sqrt{7}\ F(1/2,\ 1/2\ ;\ 1\ ;\ x)\, F(-1/2,\ 1/2\ ;\ 1\ ;\ x) - \left(\frac{2+\sqrt{7}}{2}\right)(F(1/2,\ 1/2\ ;\ 1\ ;\ x))^2 \tag{1}$$

donde

$$x = \frac{8 - 3\sqrt{7}}{16} \tag{2}$$

$$F(a,\ b\ ;\ c;\ z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n}\,\frac{z^n}{n!},\ \ |z| < 1 \tag{3}$$

$F(a,\ b\ ;\ c\ ;\ z)$ es la función hipergeométrica de Gauss.

fórmula 1.

Sean $m,\ n\ \in \mathbb{N} \bigcup \{0\},\ \ x > 1$:

$$\pi = f(m,\ n)\,(\mathrm{S1}(m,\ n,\ x) + \mathrm{S2}(m,\ n,\ x)) \tag{4}$$

$$f(m,\ n) = \frac{m!\, n!\, (m+n)!\, 2^{2m+2n+1}}{(2m)!\,(2n)!} \tag{5}$$

$$\mathrm{S1}(m,\ n,\ x) = x^{2m+1}\left(\frac{2}{2+x^2}\right)^{m+n+1} \sum_{k=0}^{\infty}\binom{m+n+k}{k}\left(\frac{x^2}{2+x^2}\right)^k g(m,\ k) \tag{6}$$

$$g(m,\, k) = \sum_{s=0}^{k} \binom{k}{s} \frac{(-2)^s}{2\,m + 2\,s + 1} \tag{7}$$

$$\mathrm{S2}(m,\, n,\, x) = x^{-2\,n-1} \sum_{k=0}^{\infty} (-1)^k \binom{m+n+k}{k} \frac{x^{-2\,k}}{2\,n + 2\,k + 1} \tag{8}$$

Ejemplo 1 : $x = 3/2,\; m = n = 0$:

$$\pi = \frac{24}{17} \sum_{k=0}^{\infty} \left(\frac{9}{17}\right)^k \sum_{s=0}^{k} \binom{k}{s} \frac{(-2)^s}{2\,s+1} + 2 \sum_{k=0}^{\infty} \frac{(-1)^k\,(2/3)^{2\,k+1}}{2\,k+1} \tag{9}$$

Ejemplo 2 : $x = \sqrt{2}\,,\; m = n = 0$:

$$\pi = \sqrt{2} \sum_{k=0}^{\infty} 2^{-k} \sum_{s=0}^{k} \binom{k}{s} \frac{(-2)^s}{2\,s+1} + \sqrt{2} \sum_{k=0}^{\infty} \frac{(-1)^k\, 2^{-k}}{2\,k+1} \tag{10}$$

fórmula 2.

$$\pi = 8 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{2\sqrt{2} - 2}{\sqrt{2} + \sqrt{3} - 1} \right)^n \sum_{k=0}^{n} \binom{n}{k} \operatorname{sen}\left(\frac{(3\,n + k)\,\pi}{12} \right) \tag{11}$$

$$\pi = 12 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{4 - 2\sqrt{3}}{\sqrt{6} + 2\sqrt{3} - \sqrt{2} - 2} \right)^n \sum_{k=0}^{n} \binom{n}{k} \operatorname{sen}\left(\frac{(3\,n + k)\,\pi}{12} \right) \tag{12}$$

fórmula 3.

$$\frac{(\Gamma(1/3))^3}{12\sqrt{3}\,\sqrt[3]{2}\,\pi} = \frac{1}{3\sqrt{3}} + \int_u^{1/4} \sqrt[3]{\sqrt{9 + x^{-2}} - 5}\; dx \tag{13}$$

$$u = \frac{1}{3\sqrt{3}}$$

$\Gamma(x)$ es la función gamma usual.

fórmula 4.

$$\int_0^1 \frac{\psi(x)\,\Gamma(x)}{1 + (\Gamma(x))^2}\, dx = -\frac{\pi}{4} \tag{14}$$

$$\int_1^\infty \frac{\psi(x)\,\Gamma(x)}{1 + (\Gamma(x))^2}\, dx = \frac{\pi}{4} \tag{15}$$

$$\int_2^\infty \frac{\psi(x)\,\Gamma(x)}{1 + (\Gamma(x))^2}\, dx = \frac{\pi}{4} \tag{16}$$

$$\int_0^2 \frac{\psi(x)\,\Gamma(x)}{1 + (\Gamma(x))^2}\, dx = -\frac{\pi}{4} \tag{17}$$

$\Gamma(x)$ es la función gamma usual , $\psi(x)$ es la función Psi.

fórmula 5.

$$\pi = \frac{4}{\sqrt[3]{14}} \sum_{n=0}^{\infty} \frac{(-1)^n\, 14^{-2n}}{6n+1} + \frac{4}{\sqrt[3]{14^2}} \sum_{n=0}^{\infty} \frac{(-1)^n\, 14^{-2n-1}}{6n+5} - 4 \sum_{n=0}^{\infty} \frac{(-1)^n\, 14^{-2n-1}}{6n+3} + 4 \sum_{n=0}^{\infty} \frac{(-1)^n\, 13^{2n+1}}{(2n+1)\left(15 + 2\sqrt[3]{14} + 2\sqrt[3]{14^2}\right)^{2n+1}} \tag{18}$$

Observaciones :

$$x = \frac{\sqrt[3]{14} - 1}{\sqrt[3]{14} + 1} = \frac{13 + 2\sqrt[3]{14} - 2\sqrt[3]{14^2}}{15} = \frac{13}{15 + 2\sqrt[3]{14} + 2\sqrt[3]{14^2}} \tag{19}$$

$$15x^3 - 39x^2 + 45x - 13 = 0 \tag{20}$$

$$\left(15 + 2\sqrt[3]{14} + 2\sqrt[3]{14^2}\right)^{2n+1} = a_n + b_n \sqrt[3]{14} + c_n \sqrt[3]{14^2} \tag{21}$$

$$a_n,\ b_n,\ c_n \in \mathbb{N} \tag{22}$$

$$\begin{gathered} a_{n+1} = 337\, a_n + 896\, b_n + 1624\, c_n \\ b_{n+1} = 116\, a_n + 337\, b_n + 896\, c_n \\ c_{n+1} = 64\, a_n + 116\, b_n + 337\, c_n \\ a_0 = 15\,,\ \ b_0 = 2\,,\ \ c_0 = 2 \end{gathered} \tag{23}$$

$$a_n = \sum_{k=0}^{[(4n+2)/3]} \sum_{m=0}^{[(3k)/2]} \binom{2n+1}{3k-m} \binom{3k-m}{m} 15^{2n+1-3k+m}\, 2^{3k-m}\, 14^k \tag{24}$$

$$b_n = \sum_{k=0}^{[(4n+1)/3]} \sum_{m=0}^{[(3k+1)/2]} \binom{2n+1}{3k+1-m} \binom{3k+1-m}{m} 15^{2n-3k+m}\, 2^{3k+1-m}\, 14^k \tag{25}$$

$$c_n = \sum_{k=0}^{[(4n)/3]} \sum_{m=0}^{[(3k+2)/2]} \binom{2n+1}{3k+2-m} \binom{3k+2-m}{m} 15^{2n-3k-1+m}\, 2^{3k+2-m}\, 14^k \tag{26}$$

$$\pi = 4\tan^{-1}\left(\frac{1}{\sqrt[3]{14}}\right) + 4\tan^{-1}\left(\frac{\sqrt[13]{14} - 1}{\sqrt[13]{14} + 1}\right) \tag{27}$$

$$\pi = 4\tan^{-1}\left(\frac{p_n}{q_n}\right) + 4\tan^{-1}\left(\frac{q_n - p_n \sqrt[3]{14}}{p_n + q_n \sqrt[3]{14}}\right) + 4\tan^{-1}\left(\frac{u_n}{v_n}\right) + 4\tan^{-1}\left(\frac{(v_n - u_n)\sqrt[3]{14} - (v_n + u_n)}{(v_n - u_n)\sqrt[3]{14} + (v_n - u_n)}\right) \tag{28}$$

$$p_n\,,\ q_n\,,\ u_n\,,\ v_n \in \mathbb{N} \tag{29}$$

$$\frac{p_n}{q_n} \text{ es una aproximación racional de } \frac{1}{\sqrt[3]{14}} \tag{30}$$

$$\frac{u_n}{v_n} \text{ es una aproximación racional de } \frac{\sqrt[13]{14} - 1}{\sqrt[13]{14} + 1} \tag{31}$$

$$\frac{p_n}{q_n} = \left\{0,\ \frac{1}{2},\ \frac{2}{5},\ \frac{5}{12},\ \frac{17}{41},\ \frac{22}{53},\ \frac{39}{94},\ \ldots\right\} \tag{32}$$

$$\frac{u_n}{v_n} = \left\{0,\ \frac{1}{2},\ \frac{2}{5},\ \frac{5}{12},\ \frac{7}{17},\ \frac{12}{29},\ \frac{43}{104},\ \ldots\right\} \tag{33}$$

$$\pi = 8\tan^{-1}\left(\frac{1}{2}\right) + 4\tan^{-1}\left(\frac{2 - \sqrt[3]{14}}{1 + 2\sqrt[3]{14}}\right) - 4\tan^{-1}\left(\frac{3 - \sqrt[3]{14}}{1 + 3\sqrt[3]{14}}\right) \tag{34}$$

fórmula 6.

$$\pi = 8\sum_{n=1}^{\infty}\frac{\cosh n}{n}\left(\sqrt{1+(\cosh 1)^2}-\cosh 1\right)^n \operatorname{sen}\left(\frac{n\pi}{2}\right) \tag{35}$$

$$\pi = 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cosh(2n-1)}{2n-1}\left(\sqrt{1+(\cosh 1)^2}-\cosh 1\right)^{2n-1} \tag{36}$$

$$\pi = 12\sum_{n=1}^{\infty}\frac{\cosh n}{n}\left(\sqrt{1+3(\cosh 1)^2}-\sqrt{3}\cosh 1\right)^n \operatorname{sen}\left(\frac{n\pi}{2}\right) \tag{37}$$

$$\pi = 12\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cosh(2n-1)}{2n-1}\left(\sqrt{1+3(\cosh 1)^2}-\sqrt{3}\cosh 1\right)^{2n-1} \tag{38}$$

$$\pi = 12\sum_{n=1}^{\infty}\frac{\operatorname{senh} n}{n}\left(\sqrt{3}\operatorname{senh} 1-\sqrt{3(\operatorname{senh} 1)^2-1}\right)^n \operatorname{sen}\left(\frac{n\pi}{2}\right) \tag{39}$$

$$\pi = 12\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\operatorname{senh}(2n-1)}{2n-1}\left(\sqrt{3}\operatorname{senh} 1-\sqrt{3(\operatorname{senh} 1)^2-1}\right)^{2n-1} \tag{40}$$

$$\pi = 8\sum_{n=1}^{\infty}\frac{\cosh n}{n}\left(\sqrt{2}\cosh 1-\sqrt{2(\cosh 1)^2-1}\right)^n \operatorname{sen}\left(\frac{n\pi}{4}\right) \tag{41}$$

$$\pi = 8\sum_{n=1}^{\infty}\frac{\operatorname{senh} n}{n}\left(\frac{e-\sqrt{e^2-2}}{\sqrt{2}}\right)^n \operatorname{sen}\left(\frac{n\pi}{4}\right) \tag{42}$$

$$\pi = 12\sum_{n=1}^{\infty}\frac{\operatorname{senh} n}{n}\, a^n \operatorname{sen}\left(\frac{n\pi}{4}\right) \tag{43}$$

$$a = \frac{r-\sqrt{r^2-48e^2}}{4\sqrt{3}\,e} \tag{44}$$

$$r = 3\sqrt{2}\left(e^2-1\right)+\sqrt{6}\left(e^2+1\right) \tag{45}$$

fórmula 7.

Sean $p, q \in \mathbb{N}$, $0 < p < q$, se tiene.

$$\pi = 2\left(p^2-q^2\right)\ln\left(\frac{(p+q)q+1}{(p+q)q-1}\right)+4\sum_{k=1}^{\infty}(-1)^k A_k(p,q)\,B_k(p,q) \tag{46}$$

donde

$$A_k(p,q) = \left(p^2+pq\right)^{2k-1}\left(\left(p^2+pq\right)^2+1\right)+\left(q^2-pq\right)^{2k-1}\left(\left(q^2-pq\right)^2+1\right) \tag{47}$$

$$B_k(p,q) = \frac{1}{2}\ln\left(\frac{(p+q)q+1}{(p+q)q-1}\right)-\sum_{n=0}^{k-1}\frac{(q^2+pq)^{-2n-1}}{2n+1} \tag{48}$$

Ejemplo :

$$\pi = 10\ln\left(\frac{7}{5}\right)+2\sum_{k=1}^{\infty}(-1)^k\left(5\cdot 2^{2k-1}+10\cdot 3^{2k-1}\right)\left(\ln\left(\frac{7}{5}\right)-\sum_{n=0}^{k-1}\frac{6^{-2n}}{6n+3}\right) \tag{49}$$

fórmula 8.

$$\pi = 48\sqrt{2}\sum_{k=1}^{\infty}(-1)^{k-1}\left(\sqrt{2}-1\right)^{2k}\sum_{n=0}^{k-1}\frac{\left(\sqrt{3}-\sqrt{2}\right)^{2n+1}}{2n+1} \tag{50}$$

$$\pi = 48\sqrt{3}\sum_{k=1}^{\infty}(-1)^{k-1}\left(\sqrt{3}-\sqrt{2}\right)^{2k}\sum_{n=0}^{k-1}\frac{\left(\sqrt{2}-1\right)^{2n+1}}{2n+1} \tag{51}$$

fórmula 9.

$$\frac{2\sqrt{3}}{9}\pi = \operatorname{senh} 1 + \sum_{k=1}^{\infty}(k!)^2\, k\,(k+2)\left(\operatorname{senh} 1 - \sum_{n=0}^{k}\frac{1}{(2n+1)!}\right) \tag{52}$$

$$\frac{2\sqrt{3}}{27}\pi + \frac{4}{3} = \cosh 1 + \sum_{k=1}^{\infty}(k!)^2\, k\,(k+2)\left(\cosh 1 - \sum_{n=0}^{k}\frac{1}{(2n)!}\right) \tag{53}$$

$$\pi\sqrt{3} = 9(\cosh 1 - 1) + 9\sum_{k=1}^{\infty}k!\,(k+1)!\left(k^2+k-1\right)\left(\cosh 1 - 1 - \sum_{n=0}^{k-1}\frac{1}{(2n+2)!}\right) \tag{54}$$

fórmula 10.

$$\pi = 6\ln\left(\frac{5}{3}\right) + 3\sum_{k=1}^{\infty}\binom{2k}{k}\left(\frac{3k-2}{2k-1}\right)\left(\ln\left(\frac{5}{3}\right) - \sum_{n=0}^{k-1}\frac{2^{-4n}}{4n+2}\right) \tag{55}$$

fórmula 11.

$$\pi = 4G + 8\sum_{k=1}^{\infty}\left(G - \sum_{n=0}^{k-1}\frac{(-1)^n}{(2n+1)^2}\right) \tag{56}$$

$G = 0.91596559\ldots$, constante de Catalan

fórmula 12.

sean $x = \frac{\sqrt{6}-2}{2}$, $y = \frac{1}{\sqrt{2}}$, $H_n = \sum_{k=1}^{n}\frac{1}{k}$, se tiene :

$$\pi^2 = 6\ln\left(20-8\sqrt{6}\right)\ln\left(\frac{4-\sqrt{2}-\sqrt{6}}{4+\sqrt{2}-\sqrt{6}}\right) + 96\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n+1}H_{2n}\operatorname{Im}\left((x+iy)^{2n+1}\right) \tag{57}$$

$$\pi\left(3\ln\left(20-8\sqrt{6}\right) + 2\ln\left(\frac{4-\sqrt{2}-\sqrt{6}}{4+\sqrt{2}-\sqrt{6}}\right)\right) = 96\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n+1}H_{2n}\operatorname{Re}\left((x+iy)^{2n+1}\right) \tag{58}$$

fórmula 13.

$$\pi = 4\sqrt{3}\sum_{n=0}^{\infty}\left(\frac{3-\sqrt{6}}{3}\right)^n\sum_{k=0}^{n}\binom{n}{k}\frac{(-2)^{-k}\left(1-\left(2-\sqrt{2}\right)^{n+k+1}\right)}{n+k+1} \tag{59}$$

fórmula 14.

Sean $m \in \mathbb{N} \bigcup \{0\}$, $p, q \in \mathbb{N}$, $s_m = 1 - 3\left(\frac{p}{q}\right)^2$, $|s_m| < 3^{-m-1}$, $r_m = 4\, s^{-1}\, 3^{-m-1}$, sea $f_m(k)$

definida por :

$$f_m(k) = \sum_{j=0}^{m} \frac{(-1)^j\, 3^{m-j}}{(2\,m+2)\,k + 2\,j + 1} \;,\; m \in \mathbb{N} \bigcup \{0\} \tag{60}$$

se tiene :

$$\pi = \frac{2\,q}{3^{m-1}\,p} \sum_{n=0}^{\infty} \left(\frac{s_m}{4}\right)^n \sum_{k=0}^{n} \binom{2\,n-2\,k}{n-k} (-1)^{(m+1)\,k}\, r_m^k\, f_m(k) \tag{61}$$

Los números $p, q,$ se pueden elegir del conjunto generado por la recurrencia :

$$p_{n+1} = 2\,p_n + 3\,q_n \;,\; q_{n+1} = p_n + 2\,q_n \;,\; p_1 = 2 \;,\; q_1 = 1 \tag{62}$$

Ejemplo1 : $m = 0,\ p = 2,\ q = 1,\ s_0 = 1/4,\ r_0 = 16/3$:

$$\pi = 3 \sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n \sum_{k=0}^{n} \binom{2\,n-2\,k}{n-k} \frac{(-1)^k}{2\,k+1} \left(\frac{16}{3}\right)^k \tag{63}$$

Ejemplo2 : $m = 1,\ p = 7,\ q = 4,\ s_1 = 1/49,\ r_1 = 196/27$:

$$\pi = \frac{8}{7} \sum_{n=0}^{\infty} \left(\frac{1}{196}\right)^n \sum_{k=0}^{n} \binom{2\,n-2\,k}{n-k} \left(\frac{3}{4\,k+1} - \frac{1}{4\,k+3}\right) \left(\frac{196}{9}\right)^k \tag{64}$$

Ejemplo3 : $m = 2,\ p = 7,\ q = 4,\ s_2 = 1/49,\ r_2 = 196/27$:

$$\pi = \frac{8}{21} \sum_{n=0}^{\infty} \left(\frac{1}{196}\right)^n \sum_{k=0}^{n} (-1)^k \binom{2\,n-2\,k}{n-k} \left(\frac{9}{6\,k+1} - \frac{3}{6\,k+3} + \frac{1}{6\,k+5}\right) \left(\frac{196}{27}\right)^k \tag{65}$$

Ejemplo4 : $m = 3,\ p = 26,\ q = 15,\ s_3 = 1/676,\ r_3 = 2704/81$:

$$\pi = \frac{5}{39} \sum_{n=0}^{\infty} \left(\frac{1}{2704}\right)^n \sum_{k=0}^{n} \binom{2\,n-2\,k}{n-k} \left(\frac{27}{8\,k+1} - \frac{9}{8\,k+3} + \frac{3}{8\,k+5} - \frac{1}{8\,k+7}\right) \left(\frac{2704}{81}\right)^k \tag{66}$$

Ejemplo5 : $m = 4,\ p = 26,\ q = 15,\ s_4 = 1/676,\ r_4 = 2704/243$:

$$\pi = \frac{5}{117} \sum_{n=0}^{\infty} \left(\frac{1}{2704}\right)^n \sum_{k=0}^{n} (-1)^k \binom{2\,n-2\,k}{n-k} \left(\frac{81}{10\,k+1} - \frac{27}{10\,k+3} + \frac{9}{10\,k+5} - \frac{3}{10\,k+7} + \frac{1}{10\,k+9}\right) \left(\frac{2704}{243}\right)^k \tag{67}$$

Ejemplo6 : $m = 5,\ p = 97,\ q = 56,\ s_5 = 1/9409,\ r_5 = 37\,636/729$:

$$\pi = \frac{112}{7857} \sum_{n=0}^{\infty} \left(\frac{1}{37\,636}\right)^n \sum_{k=0}^{n} \binom{2\,n-2\,k}{n-k} \left(\frac{243}{12\,k+1} - \frac{81}{12\,k+3} + \frac{27}{12\,k+5} - \frac{9}{12\,k+7} + \frac{3}{12\,k+9} - \frac{1}{12\,k+11}\right) \left(\frac{37\,636}{729}\right)^k \tag{68}$$

fórmula 15.

$$\frac{1}{\pi} = \frac{5}{741} \sum_{n=0}^{\infty} \left(\frac{1}{6084}\right)^n \sum_{k=0}^{n} \binom{2\,n-2\,k}{n-k} (408\,k+47) \left(\frac{1521}{1444}\right)^k \sum_{m=0}^{k} \binom{k}{m}^4 \tag{69}$$

fórmula 16.

$$\frac{1}{\pi} = \frac{2}{9} \sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^{2n} \sum_{k=0}^{n} \binom{2n-2k}{n-k} \binom{2k}{k} (5k+1) \left(\frac{3}{8}\right)^{k} \sum_{m=0}^{k} \binom{k}{m}^{3} \tag{70}$$

fórmula 17.

$$\frac{1}{\pi} = \frac{9}{125} \sum_{n=0}^{\infty} \left(-\frac{1}{50}\right)^{n} \sum_{k=0}^{n} \binom{2n-2k}{n-k} \binom{2k}{k} (16k+3) \left(-\frac{1}{2}\right)^{k} \sum_{m=0}^{k} \binom{k}{m}^{2} \binom{2m}{m} \tag{71}$$

fórmula 18.

$$\frac{1}{\pi} = \frac{13}{33\,775} \sum_{n=0}^{\infty} \left(\frac{1}{2 \cdot 7 \cdot 193}\right)^{2n} \sum_{k=0}^{n} \binom{2n-2k}{n-k} \binom{2k}{k}^{2} \binom{3k}{k} (14\,151\,k + 827) \left(-\frac{7^2 \cdot 193^2}{2^4 \cdot 3^3 \cdot 5^6}\right)^{k} \tag{72}$$

fórmula 19.

$$\frac{1}{\pi} = \frac{560}{1\,940\,499} \sum_{n=0}^{\infty} \left(\frac{1}{2 \cdot 17 \cdot 1153}\right)^{2n} \sum_{k=0}^{n} \binom{2n-2k}{n-k} \frac{(4k)!}{(k!)^4} (26\,390\,k + 1103) \left(\frac{17 \cdot 1153}{2^3 \cdot 3^4 \cdot 11^2}\right)^{2k} \tag{73}$$

fórmula 20.

$$\frac{1}{\pi} = A \sum_{n=0}^{\infty} \frac{(-1)^n}{B^n} \sum_{k=0}^{n} \binom{2n-2k}{n-k} \frac{(6k)!}{(k!)^3\,(3k)!} (a + b\,k) \frac{(5 \cdot 131 \cdot 97\,771)^{2k}}{2^{12k}\,(3 \cdot 23 \cdot 29)^{3k}} \tag{74}$$

$$A = \frac{41 \cdot 101 \cdot 15\,461}{2^9 \cdot 5^4 \cdot 23 \cdot 29 \cdot 131 \cdot 97\,771}, \quad B = 2^6 \cdot 5^5 \cdot 131^2 \cdot 97\,771^2 \tag{75}$$

$$a = 13\,591\,409, \quad b = 545\,140\,134 \tag{76}$$

fórmula 21.

$$\pi = \frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^{2n} \sum_{k=0}^{n} (-1)^k \binom{2n-2k}{n-k} \left(\frac{4}{6k+1} + \frac{1}{6k+3} + \frac{1}{6k+5}\right) \left(-\frac{9}{2}\right)^{k} \tag{77}$$

fórmula 22.

$$\frac{1}{\pi} = \frac{4}{135} \sum_{n=0}^{\infty} 3^{-4n} \sum_{k=0}^{n} \frac{(1/6)_k\,(5/6)_k\,(1/2)_k\,(1/2)_{n-k}}{(k!)^3\,(n-k)!} (84k + 9) \left(\frac{3^7}{5^3}\right)^{k} \tag{78}$$

fórmula 23.

$$\frac{1}{\pi} = \frac{48}{155} \sum_{n=0}^{\infty} 31^{-2n} \sum_{k=0}^{n} \frac{(1/6)_k\,(5/6)_k\,(1/2)_k\,(1/2)_{n-k}}{(k!)^3\,(n-k)!} (11k + 1) \left(\frac{3844}{125}\right)^{k} \tag{79}$$

fórmula 24.

$$\frac{1}{\pi} = \frac{1728}{43\,435} \sum_{n=0}^{\infty} 511^{-2\,n} \sum_{k=0}^{n} \frac{(1/6)_k\,(5/6)_k\,(1/2)_k\,(1/2)_{n-k}}{(k\,!)^3\,(n-k)\,!}\,(133\,k+8)\left(\frac{2^6\cdot 7^2\cdot 73^2}{85^3}\right)^k \tag{80}$$

fórmula 25.

$$\frac{1}{\pi} = \frac{107}{336} - \frac{1}{336}\sum_{n=1}^{\infty}\left(\frac{47-21\sqrt{5}}{2^{13}}\right)^n c_n \tag{81}$$

donde $c_n \in \mathbb{N}$, se define por :

$$c_n = 2^{12}\binom{2\,n-2}{n-1}^3 (42\,n-37) - \binom{2\,n}{n}^3 (1302\,n+107)\,,\ n\in\mathbb{N} \tag{82}$$

$$c_n = \binom{2\,n}{n}^3 \frac{a_n}{(2\,n-1)^3} = 8\binom{2\,n-2}{n-1}^3 \frac{a_n}{n^3}\,,\ n\in\mathbb{N} \tag{83}$$

$$a_n = 107 + 660\,n - 6528\,n^2 - 4176\,n^3 + 11\,088\,n^4\,,\ n\in\mathbb{N} \tag{84}$$

$$a_{n+5} = 5\,a_{n+4} - 10\,a_{n+3} + 10\,a_{n+2} - 5\,a_{n+1} + a_n\,,\ n\in\mathbb{N} \tag{85}$$

$$a_1 = 1151,\ a_2 = 119\,315\,,\ a_3 = 728\,711\,,\ a_4 = 2\,469\,563\,,\ a_5 = 6\,248\,207 \tag{86}$$

fórmula 26.

$$\frac{1}{\pi} = \frac{1}{3} - \frac{1}{6}\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{17-12\sqrt{2}}{64}\right)^n c_n \tag{87}$$

donde $c_n \in \mathbb{N}$, se define por :

$$c_n = \binom{2\,n}{n}^3 (15\,n+2) + 64\binom{2\,n-2}{n-1}^3 (3\,n-2)\,,\ n\in\mathbb{N} \tag{88}$$

$$c_n = \binom{2\,n}{n}^3 \frac{a_n}{(2\,n-1)^3} = 8\binom{2\,n-2}{n-1}^3 \frac{a_n}{n^3}\,,\ n\in\mathbb{N} \tag{89}$$

$$a_n = -2 - 3\,n + 66\,n^2 - 180\,n^3 + 144\,n^4\,,\ n\in\mathbb{N} \tag{90}$$

$$a_{n+5} = 5\,a_{n+4} - 10\,a_{n+3} + 10\,a_{n+2} - 5\,a_{n+1} + a_n\,,\ n\in\mathbb{N} \tag{91}$$

$$a_1 = 25,\ a_2 = 1120\,,\ a_3 = 7387\,,\ a_4 = 26\,386\,,\ a_5 = 69\,133 \tag{92}$$

fórmula 27.

$$\frac{\sqrt[4]{12}}{\pi} = \frac{33}{56} + \frac{1}{56}\sum_{n=1}^{\infty}\left(\frac{97-56\sqrt{3}}{64}\right)^n c_n \tag{93}$$

donde $c_n \in \mathbb{N}$, se define por :

$$c_n = \binom{2\,n}{n}^3 (312\,n+33) - \binom{2\,n-2}{n-1}^3 (1536\,n-960)\,,\ n\in\mathbb{N} \tag{94}$$

$$c_n = 3\binom{2\,n}{n}^3 \frac{a_n}{(2\,n-1)^3} = 24\binom{2\,n-2}{n-1}^3 \frac{a_n}{n^3}\,,\ n\in\mathbb{N} \tag{95}$$

$$a_n = -11 - 38\,n + 492\,n^2 - 1120\,n^3 + 768\,n^4\,,\ \ n \in \mathbb{N} \tag{96}$$

$$a_{n+5} = 5\,a_{n+4} - 10\,a_{n+3} + 10\,a_{n+2} - 5\,a_{n+1} + a_n\,,\ \ n \in \mathbb{N} \tag{97}$$

$$a_1 = 91,\ \ a_2 = 5209\,,\ \ a_3 = 36\,271\,,\ \ a_4 = 132\,637\,,\ \ a_5 = 352\,099 \tag{98}$$

fórmula 28.

$$\frac{1}{\pi} = \frac{3}{2} - \frac{3}{2}\sum_{n=1}^{\infty}\left(\frac{2-\sqrt{3}}{8}\right)^n c_n \tag{99}$$

$$c_n = 8\sum_{k=0}^{[(n-1)/4]} 2^{6\,k}\,(8\,k+3)\binom{2\,n-8\,k-2}{n-4\,k-1}\binom{2\,k}{k}^3 - \sum_{k=0}^{[n/4]} 2^{6\,k}\,(4\,k+1)\binom{2\,n-8\,k}{n-4\,k}\binom{2\,k}{k}^3 \tag{100}$$

fórmula 29.

$$\pi = 4\sum_{n=0}^{\infty} 2^{-2\,n}\sum_{k=0}^{n}\frac{1}{2\,k+1}\binom{2\,n-2\,k}{n-k}\binom{2\,k}{k}\left(\frac{2}{5}\left(\frac{1}{5}\right)^n + \frac{5}{16}\left(\frac{3}{128}\right)^{n-k}\left(\frac{1}{10}\right)^k\right) \tag{101}$$

fórmula 30.

$$\pi = 4\sum_{n=0}^{\infty} 2^{-2\,n}\sum_{k=0}^{n}\frac{1}{2\,k+1}\binom{2\,n-2\,k}{n-k}\binom{2\,k}{k}\left(\frac{4}{11}\left(\frac{5}{121}\right)^{n-k}\left(\frac{4}{29}\right)^k + \frac{3}{8}\left(\frac{3}{32}\right)^{n-k}\left(\frac{9}{58}\right)^k\right) \tag{102}$$

fórmula 31.

$$\pi = 8\sum_{n=0}^{\infty} 2^{-2\,n}\sum_{k=0}^{n}\frac{1}{2\,k+1}\binom{2\,n-2\,k}{n-k}\binom{2\,k}{k}\left(\frac{6}{19}\left(\frac{1}{361}\right)^{n-k}\left(\frac{1}{10}\right)^k + \frac{7}{99}\left(\frac{1}{9801}\right)^{n-k}\left(\frac{1}{50}\right)^k\right) \tag{103}$$

fórmula 32.

$$\pi = 8\sum_{n=0}^{\infty} 2^{-2\,n}\sum_{k=0}^{n}\frac{1}{2\,k+1}\binom{2\,n-2\,k}{n-k}\binom{2\,k}{k}\left(\frac{20}{51}\left(\frac{1}{A}\right)^{n-k}\left(\frac{1}{B}\right)^k - \frac{239}{114\,243}\left(\frac{1}{C}\right)^{n-k}\left(\frac{1}{D}\right)^k\right) \tag{104}$$

$$A = 2601\,,\ \ B = 26\,,\ \ C = 13\,051\,463\,049\,,\ \ D = 57\,122 \tag{105}$$

fórmula 33.

$$\pi = 2\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n\sum_{k=0}^{n}\binom{2\,k}{k}\frac{(-3/16)^k}{2\,n-2\,k+1} + 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2\,n+1}\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^{4\,n+2} \tag{106}$$

fórmula 34.

$$\pi = \frac{16}{9}\sum_{n=0}^{\infty}\left(-\frac{1}{5}\right)^n\sum_{k=0}^{n}\binom{2\,k}{k}\frac{(-5/324)^k}{2\,n-2\,k+1} + 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2\,n+1}\left(\frac{\sqrt{5}-1}{2}\right)^{4\,n+2} \tag{107}$$

fórmula 35.

$$\pi = 4 - 16 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, n \left(\sqrt{2} - 1\right)^{2n}}{4n^2 - 1} \tag{108}$$

$$\pi = 3 + 6 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \left(2 - \sqrt{3}\right)^{2n}}{4n^2 - 1} \tag{109}$$

fórmula 36.

$$\pi = \frac{28}{9} + \frac{2}{3} \sum_{n=1}^{\infty} \left(\sqrt{2} - 1\right)^{4n} \left(\frac{5}{4n-1} + \frac{5}{4n+1} - \frac{1}{4n-3} - \frac{1}{4n+3}\right) \tag{110}$$

$$\pi = \frac{22}{7} - \frac{3}{14} \sum_{n=1}^{\infty} \left(2 - \sqrt{3}\right)^{4n} \left(\frac{15}{4n-1} - \frac{15}{4n+1} - \frac{1}{4n-3} + \frac{1}{4n+3}\right) \tag{111}$$

fórmula 37.

$$\pi = \frac{2}{7} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{2n-2k}{n-k} \frac{(-1)^k}{2k+1} \left(\frac{1}{14}\right)^{2n-2k} \left(8 \cdot 3^{-3k} + 2^{-4k} \cdot 3^{-k+1}\right) \tag{112}$$

$$\pi = \frac{2}{7} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{2k}{k} \frac{(-1)^{n-k}}{2n-2k+1} \left(\frac{1}{14}\right)^{2k} \left(8 \cdot 3^{-3n+3k} + 2^{-4n+4k} \cdot 3^{-n+k+1}\right) \tag{113}$$

fórmula 38.

$$\frac{1}{\pi} = \frac{3}{544} \sum_{n=0}^{\infty} \left(\frac{1}{34}\right)^{2n} \sum_{k=0}^{n} \binom{2n-2k}{n-k} \binom{4k}{2k} \binom{2k}{k}^2 \left(\frac{17}{72}\right)^{2k} \left(-\frac{2088}{2k-1} + \frac{333}{4k-1} + \frac{5103}{4k-3}\right) \tag{114}$$

fórmula 39.

$$\frac{20\sqrt{3} + 9\sqrt{15}}{\pi} = \frac{79}{4} + \frac{1}{4} \sum_{n=1}^{\infty} \left(161 - 72\sqrt{5}\right)^n c_n \tag{115}$$

$$c_n = (480n + 79) \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 + \sum_{k=0}^{n-1} \binom{n-1}{k}^2 \binom{n+k-1}{k}^2, \quad n \in \mathbb{N} \tag{116}$$

fórmula 40.

$$\frac{\sqrt{3}}{\pi} = \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \left(161 - 72\sqrt{5}\right)^n c_n \tag{117}$$

$$c_n = (6n + 1) \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 - (6n - 1) \sum_{k=0}^{n-1} \binom{n-1}{k}^2 \binom{n+k-1}{k}^2, \quad n \in \mathbb{N} \tag{118}$$

fórmula 41.

$$\frac{884\sqrt{21650+5967\sqrt{5}}}{\pi} = 52587 + \sum_{n=1}^{\infty}\left(\frac{884\sqrt{5}-1975}{2129600}\right)^{n} c_n \tag{119}$$

$$c_n = (739024\,n + 52587)\binom{6n}{3n}\binom{3n}{2n}\binom{2n}{n} - 31944000\binom{6n-6}{3n-3}\binom{3n-3}{2n-2}\binom{2n-2}{n-1} \tag{120}$$

fórmula 42.

$$\pi = 2\sqrt{2}\int_0^{\infty}\left(\sqrt{2} - x\sqrt{\sqrt{1+4x^{-2}}-1}\right)dx \tag{121}$$

$$\pi = \frac{8}{3} + \frac{2\sqrt{2}}{3}\int_2^{\infty}\left(\sqrt{2} - \sqrt{4-x^2+\sqrt{x^4-4x^2}}\right)dx \tag{122}$$

fórmula 43.

$$\pi = 8\sum_{n=0}^{\infty}\frac{\operatorname{Im}(z^{2n+1})}{2n+1} + 16\sum_{n=1}^{\infty}\binom{4n-2}{2n-1}\frac{(-1)^{n-1}\,2^{-6n}}{2n-1} \tag{123}$$

$$z = \frac{\sqrt{2+\sqrt{5}}-1}{2} + i\,\frac{1-\sqrt{\sqrt{5}-2}}{2} \tag{124}$$

fórmula 44.

$$\pi = \int_0^1\int_0^1 \frac{4\left(8\sqrt{7}-16-\left(8\sqrt{7}-21\right)y^2\right)}{\sqrt{(1-x^2)(1-y^2)\left(16-\left(8-3\sqrt{7}\right)x^2\right)\left(16-\left(8-3\sqrt{7}\right)y^2\right)}}\,dx\,dy \tag{125}$$

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