Lagrangian Analysis of a Class of Quadratic Liénard-Type Oscillator Equations with Exponential-Type Restoring Force function

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Abstract

This research work proposes a Lagrangian and Hamiltonian analysis for the unique class of position-dependent mass oscillator characterized by a harmonic periodic solution and parabolic potential energy and its inverted version admitting a position-dependent mass dynamics.

1. Analysis of the class of quadratic Liénard-type harmonic nonlinear oscillator equations

This section is devoted to the analysis of a class of quadratic Liénard-type nonlinear dissipative oscillator equations that admits exact analytical harmonic periodic solutions. Consider the equation [1, 2]

$$\ddot{x} - \gamma\varphi'(x)\dot{x}^2 + \omega^2 x e^{2\gamma\varphi(x)} = 0 \qquad (1)$$

that represents the class of equations under analysis. γ and ω are arbitrary parameters, and $\varphi(x)$ is an arbitrary function of x. The dot over a symbol means differentiation with respect to time, and prime holds for differentiation with respect to x. By restriction of $\varphi(x) = \ln f(x)$ and $\gamma = -\frac{1}{2}$, the equation (1),yields

$$\ddot{x} + \frac{1}{2}\frac{f'(x)}{f(x)}\dot{x}^2 + \frac{\omega^2 x}{f(x)} = 0 \qquad (2)$$

where $f(x) \neq 0$, is an arbitrary function of x. The equation (1) is of the general form

$$\ddot{x} + F(x)\dot{x}^2 + G(x) = 0 \qquad (3)$$

for which the Lagrangian is given by [3,4]

$$L(\dot{x}, x) = \frac{1}{2}\dot{x}^2 M(x) - V(x) \qquad (4)$$

where

$$M(x) = e^{2\int F(x)dx} \qquad (5)$$

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and

$$V(x)=\int M(x)G(x)dx \tag{6}$$

designate the position dependent mass and the potential function respectively.

The Lagrangian of the equation (1) becomes

$$L(\dot{x},x)=\frac{1}{2}\dot{x}^2e^{-2\gamma\varphi(x)}-\frac{1}{2}\omega^2x^2 \tag{7}$$

Applying the Euler-Lagrange equation formula in [4]

$$\ddot{x}+\frac{1}{2}\frac{M'(x)}{M(x)}\dot{x}^2+\frac{1}{M(x)}\frac{\partial V(x)}{\partial x}=0 \tag{8}$$

to the equation (7), gives the equation (1). By restricting $V(x)$ to the harmonic potential, that is $V(x)=\frac{1}{2}m_0\,\omega^2x^2$, with unit mass, $m_0=1$, the equation (8) becomes identical to the equation (2), with the position-dependent mass function $M(x)=f(x)$. In this regard, the equation (1) represents the unique class of position-dependent mass oscillators exhibiting not only exact harmonic periodic solution but also a harmonic potential function.

Now, using [3]

$$H(p,x)=\frac{p^2}{2M(x)}+V(x) \tag{9}$$

one may deduce from (5) and (6) the Hamiltonian

$$H(p,x)=\frac{p^2}{2}e^{2\gamma\varphi(x)}+\frac{1}{2}\omega^2x^2 \tag{10}$$

Let us now consider, as illustration, some specific examples of (1). Let $\varphi(x)=x$. Then (1) becomes

$$\ddot{x}-\gamma\dot{x}^2+\omega^2xe^{2\gamma x}=0 \tag{11}$$

The equation (10) admits the position dependent mass and the potential

$$M(x)=e^{-2\gamma x}\text{, and } V(x)=\frac{1}{2}\omega^2x^2 \tag{12}$$

respectively, which provides the Lagrangian function

$$L(\dot{x},x)=\frac{1}{2}\dot{x}^2e^{-2\gamma x}-\frac{1}{2}\omega^2x^2 \qquad (13)$$

The application of the Euler-Lagrange equation (8) to (13) gives, as expected, (11). In this regard the Hamiltonian associated to (11) takes the form

$$H(p,x)=\frac{p^2}{2}e^{2\gamma x}+\frac{1}{2}\omega^2x^2 \qquad (14)$$

So, the Hamilton equations

$$\begin{cases}\dot{x}=\dfrac{\partial H}{\partial p}\\ \dot{p}=-\dfrac{\partial H}{\partial x}\end{cases} \qquad (15)$$

yield for (14)

$$\begin{cases}\dot{x}=p\,e^{2\gamma x}\\ \dot{p}=-\gamma\,p^2\,e^{2\gamma x}-\omega^2x\end{cases} \qquad (16)$$

The explicit expression for the conjugate momentum p, as a function of x and $\dot{x}$ takes then the form

$$\dot{p}=-e^{-2\gamma x}\left(\gamma\dot{x}^2+\omega^2xe^{2\gamma x}\right) \qquad (17)$$

Putting now $\varphi(x)=\frac{1}{2}x^2$, into (1), one may obtain as equation

$$\ddot{x}-\gamma x\dot{x}^2+\omega^2xe^{\gamma x^2}=0 \qquad (18)$$

The position dependent mass and the potential of (18) take then the form

$$M(x)=e^{-\gamma x^2}\ \text{and}\ \ V(x)=\frac{1}{2}\omega^2x^2 \qquad (19)$$

respectively.

The associated Lagrangian becomes

$$L(\dot{x},x)=\frac{1}{2}\dot{x}^2e^{-\gamma x^2}-\frac{1}{2}\omega^2x^2 \qquad (20)$$

The application of the Euler-Lagrange equation (8) to (20) gives with satisfaction (18). So, the associated Hamiltonian may be written as

$$H(p,x)=\frac{p^2}{2}e^{\gamma x^2}+\frac{1}{2}\omega^2 x^2 \tag{21}$$

such that the Hamilton equations take the form

$$\begin{cases}\dot{x}=pe^{\gamma x^2}\\ \dot{p}=-\gamma\, p^2 x e^{\gamma x^2}-\omega^2 x\end{cases} \tag{22}$$

The relation between $\dot{x}$ and $\dot{p}$ reads in this perspective

$$\dot{p}=-xe^{-\gamma x^2}(\gamma\dot{x}^2+\omega^2 e^{\gamma x^2}) \tag{23}$$

2. Analysis of inverted versions

Consider now the inverted version of (1)

$$\ddot{x}+\gamma\varphi'(x)\dot{x}^2+\omega^2 xe^{2\gamma\varphi(x)}=0 \tag{24}$$

which gives for $\varphi(x)=x$, the following equation

$$\ddot{x}+\gamma\dot{x}^2+\omega^2 xe^{2\gamma x}=0 \tag{25}$$

The position dependent mass and potential function of (25) may be then deduced from (4) as

$$M(x)=e^{2\gamma x}\text{ and }\ V(x)=\frac{\omega^2}{4\gamma}xe^{4\gamma x}-\frac{\omega^2}{16\gamma^2}e^{4\gamma x} \tag{26}$$

respectively.

Therefore, the Lagrangian for (25) may be written in the form

$$L(\dot{x},x)=\frac{1}{2}\dot{x}^2e^{2\gamma x}+\frac{\omega^2}{16\gamma^2}e^{4\gamma x}-\frac{\omega^2}{4\gamma}xe^{4\gamma x} \tag{27}$$

In this perspective, it may be verified that the application of the Euler-Lagrange equation (8) to (27) yields, as expected, (25). The Hamiltonian for (25) may also be computed as

$$H(p,x)=\frac{p^2}{2}e^{-2\gamma x}+\frac{\omega^2}{4\gamma}xe^{4\gamma x}-\frac{\omega^2}{16\gamma^2}e^{4\gamma x} \tag{28}$$

which gives the Hamiltonian equations

$$\begin{cases}\dot{x}=p\,e^{-2\gamma x}\\ \dot{p}=\gamma\, p^2\, e^{-2\gamma x}-\omega^2 xe^{4\gamma x}\end{cases} \tag{29}$$

from which the conjugate momentum becomes

$$\dot{p} = e^{2\gamma x}\left(\gamma\dot{x}^2 - \omega^2 x e^{2\gamma x}\right) \quad (30)$$

By analysis, other forms of equations are also suggested by the previous studied equations. So, the following equations may also be considered in the perspective of this study, that is

$$\ddot{x} + \gamma x\dot{x}^2 + \omega^2 x e^{\gamma x^2} = 0 \quad (31)$$

or in general

$$\ddot{x} + \gamma\varphi'(x)\dot{x}^2 + \omega^2 x e^{\gamma\varphi(x)} = 0 \quad (32)$$

$$\ddot{x} - \gamma\varphi'(x)\dot{x}^2 + \omega^2 x e^{\gamma\varphi(x)} = 0 \quad (33)$$

Finally one may consider the following more generalizations

$$\ddot{x} + \gamma\varphi'(x)\dot{x}^2 + \omega^2 h(x) e^{\gamma\varphi(x)} = 0 \quad (34)$$

$$\ddot{x} - \gamma\varphi'(x)\dot{x}^2 + \omega^2 h(x) e^{\gamma\varphi(x)} = 0 \quad (35)$$

$$\ddot{x} + \gamma\varphi'(x)\dot{x}^2 + \omega^2 h(x) e^{2\gamma\varphi(x)} = 0 \quad (36)$$

$$\ddot{x} - \gamma\varphi'(x)\dot{x}^2 + \omega^2 h(x) e^{2\gamma\varphi(x)} = 0 \quad (37)$$

$$\ddot{x} + \gamma\varphi'(x)\dot{x}^2 + \omega^2 h(x) e^{-\gamma\varphi(x)} = 0 \quad (38)$$

$$\ddot{x} + \gamma\varphi'(x)\dot{x}^2 + \omega^2 h(x) e^{-\gamma\varphi(x)} = 0 \quad (39)$$

$$\ddot{x} + \gamma\varphi'(x)\dot{x}^2 + \omega^2 h(x) e^{-2\gamma\varphi(x)} = 0 \quad (40)$$

These equations will be investigated in a subsequent work.

References

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