Exact Harmonic Periodic Solution of a Class of Liénard-Type Position-Dependent Mass Oscillator Equations and Exact Periodic Solution of a Class of Duffing Equations

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Abstract

The main objective of this paper is to propose two simple analytical linearizing transformation for determining the exact analytical harmonic periodic solution of a class of Liénard-type position-dependent mass oscillator equations. The performed exact analysis shows that both employed analytical methods are efficient to solve this class of equations.

Keywords: Position-Dependent Mass Oscillator, Quadratic Liénard Equation, Duffing Equation, Exact Harmonic Periodic Solution,

Introduction

Consider the following class of quadratic Liénard-type position-dependent mass oscillator equations

$$\ddot{x}-\gamma\frac{\dot{x}^2}{x}+\omega^2 x^{2\gamma+1}=0 \tag{1}$$

where γ and ω are arbitrary parameters and overdot denotes the differentiation with respect to time.

At present, we propose to determine the exact analytical trigonometric periodic solution of this equation via two analytical methods

1. First method of linearizing transformation

In order to solve the equation (1), we carry out the following suitable substitution

$$u=x^{1-\gamma} \tag{2}$$

Introducing the equation (2) into the equation (1), we obtain after some algebraic manipulations

$$\ddot{u}+(1-\gamma)\omega^2 u^{\frac{1+\gamma}{1-\gamma}}=0 \tag{3}$$

avec $\gamma\neq 1$

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The so obtained equation (3) represents an anharmonic generalized Duffing oscillator equation. It is interesting to show that when $(1-\gamma)\omega^2 = 1$, and $\frac{1+\gamma}{1-\gamma} = m$, where m is an arbitrary positive parameter, the equation (3) becomes

$$\ddot{u} + u^m = 0 \tag{4}$$

Under this form, it is now important to signal that a complex saw-tooth time transformation has been employed by [1] for determining an analytical solution of the equation (4).

On the other hand, various approaches have been developed in open literature for, and used by several investigators for finding an approximate analytical solution of the equation (4) for any arbitrary exponent m [2-3].

Recently Monsia et al. [4] have proposed a method based on the linearizing transformation for determining the exact analytical periodic solution of a class of quadratic Liénard type nonlinear dissipative equation. Thus, by applying then the following nonlocal transformation

$$y(\tau) = u(t), \text{ and } d\,\tau = u^{\frac{\gamma}{1-\gamma}} dt \tag{5}$$

the equation (3) becomes

$$y'' + \left(\frac{1}{1-\gamma} - 1\right)\frac{y'^2}{y} + \frac{\omega^2}{\frac{1}{1-\gamma}} y = 0 \tag{6}$$

where the prime denotes the differentiation with respect to τ. The equation (6) is well-known as a Painlevé type equation where the exact analytical periodic solution can be written as follows [2,5]

$$y = [A_0 \sin(\omega\tau + \phi_0)]^{1-\gamma} \tag{7}$$

where A_0 and ϕ_0 are arbitrary constants. Then, the exact analytical periodic solution of the equation (3) takes the following form

$$u(t) = (A_0 \sin\phi(t))^{1-\gamma} \tag{8}$$

where the function $\phi(t) = \omega\tau + \phi_0$ verifies the relation

$$\omega A_0^{\gamma} t = \int_{\phi_0}^{\phi} \frac{d\,\phi}{\sin^{\gamma}\phi} \tag{9}$$

Now, taking into account the equation (2), we obtain the desired exact analytical harmonic periodic solution of the equation (1)

$$x(t) = A_0 \sin\phi(t) \tag{10}$$

where the function $\phi(t) = \omega\tau + \phi_0$, satisfies (9)

2- Second method

Here, we use directly the following nonlocal transformation proposed by Monsia et al [4]

$$y(\tau) = x(t), \text{ and } d\tau = x^{\gamma} dt \tag{11}$$

Applying this transformation, the equation (1) becomes

$$y'' + \omega^2 y = 0 \tag{12}$$

where the prime designates the differentiation with respect to τ. The equation (12) is a linear harmonic oscillator equation. This equation admits a well-known analytical harmonic periodic solution that can be written as

$$y(\tau) = A_0 \sin(\omega \tau + \phi_0) \tag{13}$$

where A_0 and ϕ_0 are arbitrary constants. Therefore, by taking in consideration the equation (11), we obtain finally the exact analytical harmonic periodic solution of the equation (1)

$$x(t) = A_0 \sin \phi(t) \tag{14}$$

where the function $\phi(t) = \omega \tau + \phi_0$, is solution of the equation

$$\omega A_0^{\gamma} t = \int_{\phi_0}^{\phi} \frac{d\phi}{\sin^{\gamma} \phi} \tag{15}$$

When $\gamma = 2$, for example, the exact analytical harmonic periodic solution of quadratic Liénard type nonlinear dissipative equation (1) via both linearizing methods can be written as

$$x(t) = A_0 \sin\left[\arctan\left[\frac{1}{\cot an(\phi_0) - \omega A_0^2 t} \right] \right] \tag{16}$$

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