Generalized Duffing-van der Pol and Modified Emden Type Equations as Limiting Cases of the Monsia et al. [2] Nonlinear Oscillator Equation

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Abstract

The objective in this paper is to show that the generalized Duffing-van der Pol and modified Emden type equations consist of limiting cases of the exactly integrable Monsia et al.[2] nonlinear oscillator equation by expanding the exponential-type damping and restoring forces in a Taylor series.

1. Consider the exactly solvable mixed Liénard-type nonlinear dissipative equation [1-5]

$$\ddot{x} - \gamma\varphi'(x)\dot{x}^2 + \mu\dot{x}\exp(\gamma\varphi(x)) + \omega^2 x\exp(2\gamma\varphi(x)) = 0 \tag{1}$$

where prime denotes differentiation with respect to x, and dot over a symbol means differentiation with respect to time. γ, μ and ω are arbitrary parameters. $\varphi(x)$ is an arbitrary function of x.

By expanding the exponential functions

$$\exp(\gamma\varphi(x)) = 1 + \gamma\varphi(x) + \frac{\gamma^2\varphi(x)^2}{2} + \frac{\gamma^3\varphi(x)^3}{3!} + \ldots \tag{2}$$

and

$$\exp(2\gamma\varphi(x)) = 1 + 2\gamma\varphi(x) + \frac{4\gamma^2\varphi(x)^2}{2} + \frac{8\gamma^3\varphi(x)^3}{3!} + \ldots \tag{3}$$

as $\gamma \to 0$, the equation (1) becomes

$$\ddot{x} - \gamma\varphi'(x)\dot{x}^2 + \mu\dot{x}(1 + \gamma\varphi(x) + \frac{\gamma^2\varphi(x)^2}{2} + \frac{\gamma^3\varphi(x)^3}{3!} + \ldots) + \omega^2 x(1 + 2\gamma\varphi(x) + \frac{4\gamma^2\varphi(x)^2}{2} + \frac{8\gamma^3\varphi(x)^3}{3!} + \ldots) = 0 \tag{4}$$

By neglecting the term $\gamma\varphi'(x)\dot{x}^2$, as $\gamma \to 0$, the equation (4) takes the form

$$\ddot{x} + \mu\dot{x}(1 + \gamma\varphi(x) + \frac{\gamma^2\varphi(x)^2}{2} + \frac{\gamma^3\varphi(x)^3}{3!} + \ldots) + \omega^2 x(1 + 2\gamma\varphi(x) + \frac{4\gamma^2\varphi(x)^2}{2} + \frac{8\gamma^3\varphi(x)^3}{3!} + \ldots) = 0 \tag{5}$$

The equation (5) may give, following the number of terms to be kept in the series expansion, the desired generalized Duffing-van der Pol and modified Emden type equations.

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1.1 Generalized Duffing-van der Pol type equation

Let $\varphi(x) = x^2$. Then, keeping the first two terms in the series expansion of the exponential-type damping force and the first three terms in the series expansion of the exponential-type restoring force, the equation (5) immediately yields the generalized Duffing-van der Pol type equation under question

$$\ddot{x} + \mu\dot{x}(1+\gamma x^2) + \omega^2 x(1+2\gamma x^2 + 2\gamma^2 x^4) = 0 \tag{6}$$

The general exact solution of (6) is, in principle, secured by the exact analytical solution of (1) under $\varphi(x) = x^2$, as $\gamma \to 0$.

1.2 Generalized modified Emden type equation

By keeping the first two terms in the series expansion of the exponential-type damping force and the first three terms in the series expansion of the exponential-type restoring force, the equation (5) gives with $\varphi(x) = x$, the following form of the modified Emden type equation

$$\ddot{x} + \mu\dot{x}(1+\gamma x) + \omega^2 x(1+2\gamma x + 2\gamma^2 x^2) = 0 \tag{7}$$

The general exact solution of the equation (7) is also secured, in principle, by the exact solution of the equation (1) under $\varphi(x) = x$. Other forms of generalized Duffing-van der Pol and modified Emden type differential equations may be obtained by keeping the subsequent number of terms in the Taylor expansion.

2. In [2], it was shown that the exactly integrable mixed Liénard-type nonlinear oscillator equation [1]

$$\ddot{x} + (l\frac{g'(x)}{g(x)} - \gamma\varphi'(x))\dot{x}^2 + \mu\dot{x}\exp(\gamma\varphi(x)) + \frac{\omega^2 \exp(2\gamma\varphi(x))\int g(x)^l dx}{g(x)^l} = 0 \tag{8}$$

may give, under the choice $\varphi(x) = \ln(f(x))$, $f(x) = g(x) = x$, and $l = \gamma$, the generalized modified Emden type nonlinear oscillator equation

$$\ddot{x} + \mu x^l \dot{x} + \frac{\omega^2}{1+l} x^{2l+1} = 0 \tag{9}$$

which admits a general exact analytical solution following the general linearizing transformation [1]

$$y(\tau) = \frac{1}{l+1} x^{l+1}\ ,\ d\tau = x^l dt \tag{10}$$

where l is an arbitrary parameter, and $y(\tau)$ satisfies the damped linear harmonic oscillator equation

$$y''(\tau) + \mu y'(\tau) + \omega^2 y(\tau) = 0 \tag{11}$$

and prime designates here differentiation with respect to variable τ . So, the general exact solution of (9) may be written

$$x(t)=[(l+1)y(\phi(t))]^{\frac{1}{l+1}} \tag{12}$$

where the function $\tau=\phi(t)$ satisfies

$$\frac{d\tau}{dt}=[(l+1)y(\tau)]^{\frac{l}{l+1}} \tag{13}$$

The general exact analytical solution of (9) may be then written for all the three distinct damped dynamical regimes of (11) , that is for the over-damped, critically damped and under-damped oscillations.

References

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