ON TRANSFORMATION AND SUMMATION FORMULAS FOR SOME BASIC HYPERGEOMETRIC SERIES

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Abstract

In this paper, we give an alternate and simple proofs for Sear's three term $_3\phi_2$ transformation formula, Jackson's $_3\phi_2$ transformation formula and for a nonterminating form of the q-Saalschütz sum by using q-exponential operator techniques. We also give an alternate proof for a nonterminating form of the q-Vandermonde sum. We also obtain some interesting special cases of all the three identities, some of which are analogous to the identities stated by Ramanujan in his lost notebook.

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1 Introduction

In 1951 Sears [15] has established the following useful three term transformation formula for $_3\phi_2$ series.

Theorem 1.1.

$$\sum_{n=0}^{\infty} \frac{(a,b,c)_n}{(q,e,f)_n} \left(\frac{ef}{abc}\right)^n = \frac{(b,e/a,f/a,ef/bc)_\infty}{(e,f,b/a,ef/abc)_\infty} \sum_{n=0}^{\infty} \frac{(a,e/b,f/b)_n}{(q,aq/b,ef/bc)_n} q^n$$
$$+ \frac{(a,e/b,f/b,ef/ac)_\infty}{(e,f,a/b,ef/abc)_\infty} \sum_{n=0}^{\infty} \frac{(b,e/a,f/a)_n}{(q,bq/a,ef/ac)_n} q^n, \quad (1.1)$$

where $|q| < 1$, $\left|\dfrac{ef}{abc}\right| < 1$ and as usual

$$(a)_\infty := (a; q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n),$$

$$(a)_n := (a; q)_n := \frac{(a)_\infty}{(aq^n)_\infty}, \qquad n \text{ is an integer,}$$

$$(a_1, a_2, a_3, ..., a_m)_n = (a_1)_n(a_2)_n(a_3)_n...(a_m)_n, \qquad n \text{ is an integer or } \infty.$$

Recently, Liu [9] has established (1.1) by parameter augmentation method. This formula was used by Agarwal [1] to deduce an identity of Andrews [2, Thoerem 1] which was instrumental in deriving sixteen partial theta function identities of Ramanujan found in his lost notebook [4][11].

The main objective of this paper is to give an alternate proof for (1.1) and to give proofs for Jackson's $_3\phi_2$ transformation formula and for a nonterminating form of the q-Saalschütz sum found in [5] by using q-exponential operator techniques. And also we give a simple proof for a nonterminating form of the q-Vandermonde sum. Also we obtain a number of interesting applications of these formulas.

We first list some definitions and identities that we use in the remainder of this paper. For any function f, the q-difference operator $D_{q,a}$ is defined by

$$D_{q,a}\{f(a)\} = \frac{f(a) - f(aq)}{a}.$$

The q-shift operator η_a is defined by

$$\eta_a\{f(a)\} = f(aq)$$

and the operator θ_a is given by

$$\theta_a = \eta^{-1}D_{q,a}.$$

The operator identity $T(bD_{q,a})$ [9] is defined by

$$T(bD_{q,a}) = \sum_{n=0}^{\infty} \frac{(bD_{q,a})^n}{(q; q)_n} \tag{1.2}$$

and the basic identity for $T(bD_{q,a})$ operator is

$$T(bD_{q,a})\left\{\frac{1}{(as, at; q)_\infty}\right\} = \frac{(abst; q)_\infty}{(as, at, bs, bt; q)_\infty}. \tag{1.3}$$

The Cauchy operator $T(a, b; D_{q,c})$ [6] is defined by

$$T(a, b; D_{q,c}) := \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n}(bD_{q,c})^n. \tag{1.4}$$

The two basic identities for the cauchy operator (1.4) are

$$T(a, b; D_{q,c})\left\{\frac{1}{(ct; q)_\infty}\right\} = \frac{(abt; q)_\infty}{(bt, ct; q)_\infty}, \qquad |bt| < 1, \tag{1.5}$$

$$T(a, b; D_{q,c})\left\{\frac{(cv; q)_\infty}{(cs, ct; q)_\infty}\right\} = \frac{(abs, cv; q)_\infty}{(bs, cs, ct; q)_\infty}\sum_{n=0}^{\infty} \frac{(a, cs, v/t)_n}{(q, cv, abs)_n}(bt)^n. \tag{1.6}$$

The q-exponential operator $R(bD_{q,a})$ [7] is defined by

$$R(bD_{q,a}) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} b^n}{(q; q)_n} D_{q,a}^n. \tag{1.7}$$

The two basic identities for $R(bD_{q,a})$ are

$$R(bD_{q,a})\left\{\frac{1}{(at; q)_\infty}\right\} = \frac{(bt; q)_\infty}{(at; q)_\infty}, \tag{1.8}$$

and

$$R(bD_{q,a})\left\{\frac{(av; q)_\infty}{(at, as; q)_\infty}\right\} = \frac{(bs; q)_\infty}{(as; q)_\infty}\sum_{n=0}^{\infty} \frac{(v/t, b/a)_n}{(q, bs)_n}(at)^n. \tag{1.9}$$

q-binomial theorem [5, equation(II.3), p.354] is given by

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n}z^n = \frac{(az)_\infty}{(z)_\infty}. \tag{1.10}$$

Heine's transformations for $_2\phi_1$-series [5, equation(III.1), (III.2), p.359] is given by

$$\sum_{n=0}^{\infty} \frac{(\alpha, \ \beta)_n}{(q, \ \gamma)_n}z^n = \frac{(\beta, \ \alpha z)_\infty}{(\gamma, \ z)_\infty}\sum_{n=0}^{\infty} \frac{(\gamma/\beta, z)_n}{(q, \ \alpha z)_n}\beta^n. \tag{1.11}$$

The Rogers-Fine identity [12, equation(12), p.576] is given by

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n}z^n = \sum_{n=0}^{\infty} \frac{(\alpha, \alpha zq/\beta)_n \beta^n z^n q^{n^2-n}(1 - \alpha zq^{2n})}{(\beta)_n(z)_{n+1}}. \tag{1.12}$$

Sears' transformation for $_3\phi_2$-series [5, equation (III.9), p.359] is given by

$$\sum_{n=0}^{\infty} \frac{(\alpha,\beta,\gamma)_n}{(q,\delta,\epsilon)_n}\left(\frac{\delta\epsilon}{\alpha\beta\gamma}\right)^n = \frac{(\epsilon/\alpha,\delta\epsilon/\beta\gamma)_\infty}{(\epsilon,\delta\epsilon/\alpha\beta\gamma)_\infty}\sum_{n=0}^{\infty}\frac{(\alpha,\delta/\beta,\delta/\gamma)_n}{(q,\delta,\delta\epsilon/\beta\gamma)_n}\left(\frac{\epsilon}{\alpha}\right)^n. \quad (1.13)$$

Three-term $_2\phi_1$ transformation formula [5, equation (III.31), p.363] is given by

$$\sum_{n=0}^{\infty}\frac{(\alpha,\beta)_n}{(q,\gamma)_n}z^n = \frac{(\alpha\beta z/\gamma,\ q/\gamma)_\infty}{(\alpha z/\gamma,\ q/\alpha)_\infty}\sum_{n=0}^{\infty}\frac{(\gamma/\alpha,\gamma q/\alpha\beta z)_n}{(q,\gamma q/\alpha z)_n}\left(\frac{\beta q}{\gamma}\right)^n$$
$$-\frac{(\beta,q/\gamma,\gamma/\alpha,\alpha z/q,q^2/\alpha z)_\infty}{(\gamma/q,\beta q/\gamma,q/\alpha,\alpha z/\gamma,\gamma q/\alpha z)_\infty}\sum_{n=0}^{\infty}\frac{(\alpha q/\gamma,\beta q/\gamma)_n}{(q,q^2/\gamma)_n}z^n. \quad (1.14)$$

Jackson's transformation [3, p. 526] is given by

$$\sum_{n=0}^{\infty}\frac{(\alpha,\beta)_n}{(q,\gamma)_n}z^n = \frac{(\alpha z)_\infty}{(z)_\infty}\sum_{n=0}^{\infty}\frac{(\alpha,\gamma/\beta)_n(-\beta z)^n}{(\gamma,\alpha z,q)_n}q^{n(n-1)/2}. \quad (1.15)$$

Ramanujan's [10, Ch. 16] definition of the theta function is

$$f(a,b) = \sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2}, \qquad |ab| < 1. \quad (1.16)$$

The Jacobi's triple product identity [8] is given by

$$\sum_{n=-\infty}^{\infty}q^{n^2}z^n = (-qz,-q/z,q^2;q^2)_\infty, \quad z \neq 0. \quad (1.17)$$

If we set $qz = a, q/z = b$ in (1.17), we obtain

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \quad (1.18)$$

which is the Jacobi's triple product identity in Ramanujan's notation [10, Ch. 16, entry 19]. It follows from (1.16) and (1.18) that [10, Ch. 16, entry 22]

$$\varphi(q) := f(q,q) = 1 + 2\sum_{n=1}^{\infty}q^{n^2} = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, \quad (1.19)$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty}q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \quad (1.20)$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} = (q;q)_\infty \quad (1.21)$$

and

$$\chi(q) := (-q; q^2). \tag{1.22}$$

Ramanujan's functions are given by [4][11]

$$G_6(q) := (q^3; q^6)_\infty^2 (q^6; q^6)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2} = \varphi(-q^3), \tag{1.23}$$

$$H_6(q) := (q; q^6)_\infty (q^5; q^6)_\infty (q^6; q^6)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2-2n} = f(-q, -q^5) \tag{1.24}$$

and

$$J_6(q) := (-q; q^3)_\infty (-q^2; q^3)_\infty (q^3; q^3)_\infty) = \sum_{n=0}^{\infty} q^{n(3n+1)/2} = f(q, q^2). \tag{1.25}$$

2 Main Theorems

In this section, we prove the main results.

Proof of theorem 1.1. Setting $\alpha = b, \beta = a/c, \gamma = qb/c$ and $z = q$ in (1.14), we obtain

$$\sum_{n=0}^{\infty} \frac{(b, a/c)_n}{(q, qb/c)_n} q^n = \frac{(a, c/b)_\infty}{(c, q/b)_\infty} \sum_{n=0}^{\infty} \frac{(q/a)_n}{(q)_n} \left(\frac{a}{b}\right)^n$$
$$- \frac{(a/c, c/b, b)_\infty}{(b/c, a/b, c)_\infty} \sum_{n=0}^{\infty} \frac{(c, a/b)_n}{(q, qc/b)_n} q^n. \tag{2.1}$$

On using q-binomial theorem for the first series on the right side of (2.1), we obtain

$$\sum_{n=0}^{\infty} \frac{(b, a/c)_n}{(q, qb/c)_n} q^n + \frac{(a/c, c/b, b)_\infty}{(b/c, a/b, c)_\infty} \sum_{n=0}^{\infty} \frac{(c, a/b)_n}{(q, qc/b)_n} q^n = \frac{(a, c/b)_\infty}{(c, a/b)_\infty}. \tag{2.2}$$

divide the identity (2.2) throughout by $(a/c, c/b, b)_\infty$ to obtain

$$\frac{(a)_\infty}{(b, c, a/b, a/c)_\infty} = \frac{1}{(b, c/b)_\infty} \sum_{n=0}^{\infty} \frac{(b)_n q^n}{(q, qb/c)_n (aq^n/c)_\infty}$$
$$+ \frac{1}{(c, b/c)_\infty} \sum_{n=0}^{\infty} \frac{(c)_n q^n}{(q, qc/b)_n (aq^n/b)_\infty}. \tag{2.3}$$

Applying $T(d, e; D_{q,a})$ to both the sides of the identity (2.3) and using (1.5) and (1.6), we obtain

$$\frac{(a, de/b)_\infty}{(b, c, a/b, a/c, e/b)_\infty} \sum_{n=0}^\infty \frac{(d, a/b, c)_n}{(q, de/b, a)_n} \left(\frac{e}{c}\right)^n$$
$$= \frac{1}{(b, c/b)_\infty} \sum_{n=0}^\infty \frac{(b)_n (deq^n/c)_\infty q^n}{(q, qb/c)_n (aq^n/c, eq^n/c)_\infty}$$
$$+ \frac{1}{(c, b/c)_\infty} \sum_{n=0}^\infty \frac{(c)_n (deq^n/b)_\infty q^n}{(q, qc/b)_n (aq^n/b, eq^n/b)_\infty}. \quad (2.4)$$

Multiply the identity (2.4) throughout by $(b, c, a/b, a/c, e/b)_\infty/(a, de/b)_\infty$ to obtain

$$\sum_{n=0}^\infty \frac{(d, a/b, c)_n}{(q, de/b, a)_n} \left(\frac{e}{c}\right)^n = \frac{(c, a/b, e/b, de/c)_\infty}{(a, c/b, e/c, de/b)_\infty} \sum_{n=0}^\infty \frac{(b, a/c, e/c)_n}{(q, qb/c, de/c)_n} q^n$$
$$+ \frac{(b, a/c)_\infty}{(a, b/c)_\infty} \sum_{n=0}^\infty \frac{(c, a/b, e/b)_n}{(q, qc/b, de/b)_n} q^n. \quad (2.5)$$

Change a to A, b to C, c to B, d to A/D and e to E in (2.5) to obtain

$$\sum_{n=0}^\infty \frac{(B, A/D, A/C)_n}{(q, A, AE/CD)_n} \left(\frac{E}{B}\right)^n$$
$$= \frac{(B, A/C, E/C, AE/BD)_\infty}{(A, B/C, E/B, AE/CD)_\infty} \sum_{n=0}^\infty \frac{(C, A/B, E/B)_n}{(q, Cq/B, AE/BD)_n} q^n$$
$$+ \frac{(C, A/B)_\infty}{(A, C/B)_\infty} \sum_{n=0}^\infty \frac{(B, A/C, E/C)_n}{(q, Bq/C, AE/CD)_n} q^n. \quad (2.6)$$

Setting $\alpha = B$, $\beta = A/D$, $\gamma = A/C$, $\delta = A$ and $\epsilon = AE/CD$ in (1.13), using the resulting identity on the left side of (2.6) and then multiplying the resulting identity throughout by $(E/B, AE/CD)_\infty/(E, AE/BCD)_\infty$; change A to e, B to b, C to a, D to c and E to f in the resulting identity, we obtain (1.1).

Remark 1. The identity (2.3) can be used to prove Lemma 2.1 of Somashekara, Narasimha Murthy and Shalini [13], which played a key role in giving a unified approach to the proofs of the reciprocity theorem of Ramanujan and its generalizations.

Remark 2. The identity (2.3) can also be used to prove Theorem 2.2 of Somashekara, Kim, Kwon and Shalini [14], which played a key role in giving proofs for ten identities of Ramanujan found in his lost notebook [4].

Theorem 2.1. [5, equation III.5, p. 359] *We have*

$$\sum_{n=0}^{\infty} \frac{(a,b)_n}{(q,c)_n} z^n = \frac{(abz/c)_\infty}{(bz/c)_\infty} \sum_{n=0}^{\infty} \frac{(a,c/b,0)_n}{(q,c,cq/bz)_n} q^n$$
$$+ \frac{(a,bz,c/b)_\infty}{(c,z,c/bz)_\infty} \sum_{n=0}^{\infty} \frac{(z,abz/c,0)_n}{(q,bz,bzq/c)_n} q^n. \quad (2.7)$$

Proof. Applying $R(dD_{q,a})$ to both the sides of the identity (2.3) and using (1.8) and (1.9), we obtain

$$\frac{(d/c)_\infty}{(b,c,a/c)_\infty} \sum_{n=0}^{\infty} \frac{(b,d/a)_n}{(q,d/c)_n} \left(\frac{a}{b}\right)^n = \frac{1}{(b,c/b)_\infty} \sum_{n=0}^{\infty} \frac{(b)_n(dq^n/c)_\infty}{(q,bq/c)_n(aq^n/c)_\infty} q^n$$
$$+ \frac{1}{(c,b/c)_\infty} \sum_{n=0}^{\infty} \frac{(c)_n(dq^n/b)_\infty}{(q,cq/b)_n(aq^n/b)_\infty} q^n. \quad (2.8)$$

Multiply the identity (2.8) throughout by $(b,c,a/c)_\infty/(d/c)_\infty$ to obtain

$$\sum_{n=0}^{\infty} \frac{(b,d/a)_n}{(q,d/c)_n} \left(\frac{a}{b}\right)^n = \frac{(c)_\infty}{(c/b)_\infty} \sum_{n=0}^{\infty} \frac{(b,a/c,0)_n}{(q,bq/c,d/c)_n} q^n$$
$$+ \frac{(b,a/c,d/b)_\infty}{(a/b,b/c,d/c)_\infty} \sum_{n=0}^{\infty} \frac{(c,a/b,0)_n}{(q,cq/b,d/b)_n} q^n. \quad (2.9)$$

Change a to az, b to a, c to abz/c and d to abz in (2.9) to obtain (2.7). $\square$

Theorem 2.2. [5, equation II.23, p. 356] *We have*

$$\sum_{n=0}^{\infty} \frac{(a,b)_n}{(q,c)_n} q^n + \frac{(q/c,a,b)_\infty}{(c/q,aq/c,bq/c)_\infty} \sum_{n=0}^{\infty} \frac{(aq/c,bq/c)_n}{(q,q^2/c)_n} q^n = \frac{(q/c,abq/c)_\infty}{(aq/c,bq/c)_\infty}. \quad (2.10)$$

Proof. Change lower case letters to upper case letters in (2.2) and then change B to a, A/C to b and Bq/C to c to obtain (2.10). $\square$

Theorem 2.3. [5, equation II.24, p. 356] *We have*

$$\sum_{n=0}^{\infty} \frac{(a,b,c)_n}{(q,e,f)_n} q^n + \frac{(q/e,a,b,c,qf/e)_\infty}{(e/q,aq/e,bq/e,cq/e,f)_\infty} \sum_{n=0}^{\infty} \frac{(aq/e,bq/e,cq/e)_n}{(q,q^2/e,qf/e)_n} q^n$$
$$= \frac{(q/e,f/a,f/b,f/c)_\infty}{(aq/e,bq/e,cq/e,f)_\infty}, \quad (2.11)$$

where ef=abcq.

Proof. Divide (2.3) throughout by $(a)_\infty$ to obtain

$$\frac{1}{(b,c,a/b,a/c)_\infty} = \frac{1}{(b,c/b)_\infty} \sum_{n=0}^{\infty} \frac{(b)_n q^n}{(q,qb/c)_n(aq^n/c,a)_\infty}$$
$$+ \frac{1}{(c,b/c)_\infty} \sum_{n=0}^{\infty} \frac{(c)_n q^n}{(q,qc/b)_n(aq^n/b,a)_\infty}. \quad (2.12)$$

Applying $T(dD_{q,a})$ to both the sides of the identity (2.12) and using (1.3), we obtain

$$\frac{(ad/bc)_\infty}{(b,c,a/b,a/c,d/b,d/c)_\infty} = \frac{1}{(b,c/b)_\infty} \sum_{n=0}^{\infty} \frac{(b)_n(adq^n/c)_\infty}{(q,bq/c)_n(aq^n/c,a,dq^n/c,d)_\infty} q^n$$
$$+ \frac{1}{(c,b/c)_\infty} \sum_{n=0}^{\infty} \frac{(c)_n(adq^n/b)_\infty}{(q,qc/b)_n(aq^n/b,a,dq^n/b,d)_\infty} q^n. \quad (2.13)$$

Multiply the identity (2.13) throughout by $(a,b,d,a/c,c/b,d/c)_\infty/(ad/c)_\infty$ to obtain

$$\sum_{n=0}^{\infty} \frac{(b,a/c,d/c)_n q^n}{(q,bq/c,ad/c)_n} + \frac{(c/b,b,a/c,d/c,ad/b)_\infty}{(b/c,c,a/b,d/b,ad/c)_\infty} \sum_{n=0}^{\infty} \frac{(c,a/b,d/b)_n q^n}{(q,qc/b,ad/b)_n}$$
$$= \frac{(c/b,ad/bc,d,a)_\infty}{(c,a/b,d/b,ad/c)_\infty}. \quad (2.14)$$

Change lower case letters to upper case letters in (2.14) and then change B to a, A/C to b, D/C to c, Bq/C to e and AD/C to f to obtain (2.11). $\square$

3   Some Applications of Main Results

In this section, we derive some interesting special cases of the main identities. These special cases are found to be analogues to some identities of Ramanujan found in his lost notebook [4][11].

Setting $a = C, b = B/A, c = D$ and $z = A$ in (2.7), we obtain

$$\sum_{n=0}^{\infty} \frac{(C,B/A)_n}{(q,D)_n} A^n = \frac{(BC/D)_\infty}{(B/D)_\infty} \sum_{n=0}^{\infty} \frac{(C,AD/B)_n}{(q,D,qD/B)_n} q^n$$
$$+ \frac{(B,C,AD/B)_\infty}{(A,D,D/B)_\infty} \sum_{n=0}^{\infty} \frac{(A,BC/D)_n}{(q,B,qB/D)_n} q^n. \quad (3.1)$$

Change B to β, C to τ, D to τq and then let $A \to 0$ in (3.1) to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n \beta^n q^{n(n-1)/2}}{(q;q)_n(1-\tau q^n)} = \frac{(\beta/q)_\infty}{(\beta/\tau q)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n(\tau q^2/\beta)_n(1-\tau q^n)}$$
$$+ \frac{(1-\beta/q)(\beta)_\infty}{(\tau q/\beta)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n(\beta/\tau)_n(1-\beta q^{n-1})}. \quad (3.2)$$

Change q to q^2 and set $\tau = -1$ and $\beta = -q^3$ in (3.2) to obtain

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^2;q^2)_n(1+q^{2n})} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q;q)_{2n-1}(1-q^{4n})}$$
$$- \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q;q)_{2n}(1-q^{4n+2})}. \quad (3.3)$$

Use (1.22) to obtain

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^2;q^2)_{n-1}(1-q^{4n})} = \frac{\chi(q)}{\chi(-q)} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q;q)_{2n-1}(1-q^{4n})}$$
$$- \frac{\chi(q)}{\chi(-q)} \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q;q)_{2n}(1-q^{4n+2})}.$$

Setting $\alpha = B/A, \beta = C, \gamma = D$ and $z = A$ in (1.11), we obtain

$$\sum_{n=0}^{\infty} \frac{(B/A,C)_n}{(q,D)_n} A^n = \frac{(B,C)_\infty}{(A,D)_\infty} \sum_{n=0}^{\infty} \frac{(A,D/C)_n}{(q,B)_n} C^n. \quad (3.4)$$

Using (3.4) in (3.1) and then multiplying the resulting identity throughout by $(A,D)_\infty/(B,C)_\infty$, we obtain

$$\sum_{n=0}^{\infty} \frac{(D/C,A)_n}{(q,B)_n} C^n = \frac{(A,D,BC/D)_\infty}{(B,C,B/D)_\infty} \sum_{n=0}^{\infty} \frac{(C,AD/B)_n}{(q,D,Dq/B)_n} q^n$$
$$+ \frac{(AD/B)_\infty}{(B/D)_\infty} \sum_{n=0}^{\infty} \frac{(A,BC/D)_n}{(q,B,qB/D)_n} q^n. \quad (3.5)$$

Change q to q^2 and set $A = t, B = -aq^3, C = -a$ and $D = -aq^2$ in (3.5) and then let $t \to 0$; divide the resulting identity throughout by $(1+aq)$, we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^n}{(-aq;q^2)_{n+1}} = \frac{1}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q;q)_{2n}(1+aq^{2n})}$$
$$- \frac{1}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q;q)_{2n+1}(1+aq^{2n+1})}. \quad (3.6)$$

In Rogers-Fine identity, change q to q^2, set $\alpha = 0, \beta = -aq^3$ and $z = -a$; multiply the resulting identity throughout by $1/(1 + aq)$ to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^n}{(-aq; q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{a^{2n} q^{2n^2+n}}{(-a; q^2)_{n+1}(-aq; q^2)_{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{a^{2n} q^{2n^2+n}}{(-a; q)_{2n+2}}$$

$$= \sum_{n=0}^{\infty} \frac{a^{2n} q^{2n^2+n}((1 + aq^{2n+1}) - aq^{2n+1})}{(-a; q)_{2n+2}}$$

$$= \sum_{n=0}^{\infty} \frac{a^{2n} q^{2n^2+n}}{(-a; q)_{2n+1}} - \sum_{n=0}^{\infty} \frac{a^{2n+1} q^{2n^2+3n+1}}{(-a; q)_{2n+2}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{n(n+1)/2}}{(-a; q)_{n+1}}. \tag{3.7}$$

Use (3.7) in (3.6) and also use (1.21) to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{n(n+1)/2}}{(-a; q)_{n+1}} = \frac{f(-q^2)}{f(-q)} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q; q)_{2n}(1 + aq^{2n})}$$

$$- \frac{f(-q^2)}{f(-q)} \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q; q)_{2n+1}(1 + aq^{2n+1})}. \tag{3.8}$$

Change q to q^2, set $A = t, B = aq^3, C = -aq$ and $D = -aq^3$ in (3.5); let $t \to 0$ in the resulting identity; multiply the resulting identity throughout by $1/(1 - aq)$ and also use (1.21) to obtain on some simplifications

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^n q^n}{(aq; q^2)_{n+1}} = \frac{f(-q^2)}{f(-q^4)} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^4; q^4)_n (1 - a^2 q^{4n+2})}. \tag{3.9}$$

In Rogers-Fine identity, replace q by q^2, set $\alpha = 0, \beta = aq^3$ and $z = -aq$ and then multiply the resulting identity throughout by $1/(1 - aq)$ to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^n q^n}{(aq; q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{2n^2+2n}}{(a^2 q^2; q^4)_{n+1}}. \tag{3.10}$$

Use (3.10) in (3.9) to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{2n^2+2n}}{(a^2 q^2; q^4)_{n+1}} = \frac{f(-q^2)}{f(-q^4)} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^4; q^4)_n (1 - a^2 q^{4n+2})}. \tag{3.11}$$

Change q to q^2, set $A = t, B = aq^3, C = -aq$ and $D = -aq^3$ in (3.5); multiply the resulting identity throughout by $1/(1-aq)$ to obtain

$$\sum_{n=0}^{\infty} \frac{(t;q^2)_n(-aq)^n}{(aq;q^2)_{n+1}} = \frac{(t;q^2)_\infty}{2(-q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-t;q^2)_n q^{2n}}{(q^4;q^4)_n(1+aq^{2n+1})}$$
$$+ \frac{(-t;q^2)_\infty}{2(-q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(t;q^2)_n q^{2n}}{(q^4;q^4)_n(1-aq^{2n+1})}. \quad (3.12)$$

Set $a = -1$ and $t = q$ in (3.12) to obtain

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^n}{(-q;q^2)_{n+1}} = \frac{(q;q^2)_\infty}{2(-q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{2n}}{(q^4;q^4)_n(1-q^{2n+1})}$$
$$+ \frac{(-q;q^2)_\infty}{2(-q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n}}{(q^4;q^4)_n(1+q^{2n+1})}. \quad (3.13)$$

In Rogers-Fine identity, replace q by q^2, set $\alpha = z = q$ and $\beta = -q^3$; multiply the resulting identity throughout by $1/(1+q)$ to obtain

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^n}{(-q;q^2)_{n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{2n(n+1)}. \quad (3.14)$$

Use (3.14) in (3.13) and also use (1.19), (1.20) and (1.21) to obtain

$$2\sum_{n=0}^{\infty} (-1)^n q^{2n(n+1)} = \frac{f(-q)}{f(-q^4)} \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{2n}}{(q^4;q^4)_n(1-q^{2n+1})}$$
$$+ \frac{\varphi(q)}{\psi(q)} \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n}}{(q^4;q^4)_n(1+q^{2n+1})}. \quad (3.15)$$

In (3.5), set $A = q, B = -aq, C = \tau$ and $D = a^2q$ to obtain

$$\sum_{n=0}^{\infty} \frac{(a^2q/\tau)_n}{(-aq)_n} \tau^n = \frac{(-\tau/a, q, a^2q)_\infty}{(-1/a, -aq, \tau)_\infty} \sum_{n=0}^{\infty} \frac{(\tau)_n}{(q, a^2q)_n} q^n$$
$$+ \frac{(-aq)_\infty}{(-1/a)_\infty} \sum_{n=0}^{\infty} \frac{(-\tau/a)_n}{(-aq, -q/a)_n} q^n. \quad (3.16)$$

In Rogers-Fine identity, set $\alpha = a^2q/\tau, \beta = -aq$ and $z = \tau$ to obtain

$$\sum_{n=0}^{\infty} \frac{(a^2q/\tau)_n}{(-aq)_n} \tau^n = \sum_{n=0}^{\infty} \frac{(-1)^n (a^2q/\tau)_n a^n q^{n^2}(1-a^2q^{2n+1})}{(\tau)_{n+1}}. \quad (3.17)$$

Use (3.17) in (3.16) and then let $\tau \to 0$ in the resulting identity to obtain

$$\sum_{n=0}^{\infty} a^{3n} q^{n(3n+1)/2} (1 - a^2 q^{2n+1}) = \frac{(a^2 q)_\infty f^2(-q)}{f(aq, 1/a)} \sum_{n=0}^{\infty} \frac{q^n}{(q, a^2 q)_n}$$
$$+ \frac{(-aq)_\infty}{(-1/a)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(-aq, -q/a)_n}. \quad (3.18)$$

Set $a = 1$ in (3.18) to obtain

$$\sum_{n=0}^{\infty} q^{n(3n+1)/2} (1 - q^{2n+1}) = \frac{f^3(-q)}{f(q, 1)} \sum_{n=0}^{\infty} \frac{q^n}{(q)_n^2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{q^n}{(-q)_n^2}. \quad (3.19)$$

In (1.11), set $\gamma = z = q$ and then $\alpha = 0, \beta = 0$ to obtain

$$\sum_{n=0}^{\infty} \frac{q^n}{(q)_n^2} = \frac{1}{(q)_n^2} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}. \quad (3.20)$$

Use (3.20) in (3.19) and also use (1.20) to obtain

$$\sum_{n=0}^{\infty} q^{n(3n+1)/2} (1 - q^{2n+1}) = \frac{f(-q)\psi(-q)}{f(q, 1)} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{q^n}{(-q)_n^2}.$$

In (3.16), let $\tau \to 0$ to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n(n+1)/2}}{(-aq)_n} = \frac{(q, a^2 q)_\infty}{(-1/a, -aq)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(q, a^2 q)_n}$$
$$+ \frac{(-aq)_\infty}{(-1/a)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(-aq, -q/a)_n}. \quad (3.21)$$

Set $a = 1$ in (3.21) to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-q)_n} = \frac{f^3(-q)}{f(1, q)} \sum_{n=0}^{\infty} \frac{q^n}{(q)^2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{q^n}{(-q)_n^2}. \quad (3.22)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-q)_n} = \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(-q; q)_{2n}} - \sum_{n=0}^{\infty} \frac{q^{(n+1)(2n+1)}}{(-q; q)_{2n+1}}$$
$$= \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}((1 + q^{2n+1}) - q^{2n+1})}{(-q; q)_{2n+1}}$$
$$= \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(-q; q)_{2n+1}}. \quad (3.23)$$

Use (3.23) in (3.22) to obtain

$$\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(-q;q)_{2n+1}} = \frac{f^3(-q)}{f(1,q)} \sum_{n=0}^{\infty} \frac{q^n}{(q)^2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{q^n}{(-q)_n^2}. \qquad (3.24)$$

Use (3.20) in (3.24) to obtain

$$\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(-q;q)_{2n+1}} = \frac{f^3(-q)}{f(1,q)} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{q^n}{(-q)_n^2}. \qquad (3.25)$$

Use the definition of ψ to obtain

$$\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(-q;q)_{2n+1}} = \frac{f(-q)\psi(-q)}{f(1,q)} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{q^n}{(-q)_n^2}.$$

In (3.5), replace q by q^2, set $A = q^2, B = -aq^3, C = \tau$ and $D = a^2 q^2$; multiply the resulting identity throughout by $1/(1 + aq)$ to obtain to obtain

$$\sum_{n=0}^{\infty} \frac{(a^2 q^2/\tau; q^2)_n}{(-aq; q^2)_n} \tau^n = \frac{(q^2, a^2 q^2, -q\tau/a; q^2)_\infty}{(-aq, \tau, -q/a; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(\tau; q^2)_n}{(q^2, a^2 q^2; q^2)_n} q^{2n}$$

$$+ \frac{(-aq; q^2)_\infty}{(-q/a; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-q\tau/a; q^2)_n}{(-q^3/a; q^2)_n(-aq; q^2)_{n+1}} q^{2n}. \qquad (3.26)$$

In Rogers-Fine identity, replace q by q^2, set $\alpha = a^2 q^2/\tau, \beta = -aq^3, z = \tau$ and then multiply the resulting identity throughout by $1/(1 + aq)$ to obtain

$$\sum_{n=0}^{\infty} \frac{(a^2 q^2/\tau; q^2)_n}{(-aq; q^2)_n} \tau^n = \sum_{n=0}^{\infty} \frac{(-1)^n (a^2 q^2/\tau; q^2)_n \tau^n a^n q^{2n(n+1)}(1 - a^2 q^{4n+2})}{(1 + aq^{2n+1})(\tau; q^2)_{n+1}}. \qquad (3.27)$$

Use (3.27) in (3.26) and then let $\tau \to 0$ to obtain

$$\sum_{n=0}^{\infty} a^{3n} q^{3n^2+2n}(1 - aq^{2n+1}) = \frac{(q^2, a^2 q^2; q^2)_\infty}{(-q/a, -aq; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2, a^2 q^2; q^2)_n}$$

$$+ \frac{(-aq; q^2)_\infty}{(-q/a; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n}}{(-aq; q^2)_{n+1}(-q^3/a; q^2)_n}. \qquad (3.28)$$

In (3.5), replace q to q^2, set $A = q^2, B = -q^3, D = q^2$ and then let $C \to 0$; multiply the resulting identity throughout by $1/(1 + q)$ to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+n}}{(-q; q^2)_{n+1}} = \frac{(q^2; q^2)_\infty^2}{(-q; q^2)_\infty^2} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2; q^2)_n^2} + (1 + q) \sum_{n=0}^{\infty} \frac{q^{2n}}{(-q; q^2)_{n+1}^2}. \qquad (3.29)$$

In (2.11), replace q by q^6, set $a = q, b = q^4, c = q^2, e = q^3$ and $f = q^7$ to obtain

$$\sum_{n=0}^{\infty} \frac{(q^2, q^4; q^6)_n q^{6n}}{(q^3, q^6; q^6)_n (1 - q^{6n+1})} - \frac{(q^2, q^4; q^6)_\infty}{(q, q^5; q^6)_\infty} \sum_{n=0}^{\infty} \frac{(q; q^6)_{n+1} (q^5; q^6)_n q^{6n+3}}{(q^3; q^6)_{n+1} (q^6; q^6)_n (1 - q^{6n+4})}$$
$$= (1 - q) \frac{(q^3; q^6)_\infty^2 (q^6; q^6)_\infty}{(q; q^6)_\infty^2 (q^4; q^6)_\infty}. \quad (3.30)$$

Use (1.21), (1.23) and (1.24) to obtain on some simplifications

$$\sum_{n=0}^{\infty} \frac{(q^2, q^4; q^6)_n q^{6n}}{(q^3, q^6; q^6)_n (1 - q^{6n+1})} - \frac{f(-q^2)}{H_6(q)} \sum_{n=0}^{\infty} \frac{(q; q^6)_{n+1} (q^5; q^6)_n q^{6n+3}}{(q^3; q^6)_{n+1} (q^6; q^6)_n (1 - q^{6n+4})}$$
$$= (1 - q) \frac{G_6^2(q) H_6^2(q) f(-q^2)}{(q; q^6)_\infty^2 f(-q) f^2(-q^6)} \quad (3.31)$$

In (2.11), replace q by q^3, set $a = c = -q, b = e = -q^2$ and $f = q^3$ to obtain

$$\sum_{n=0}^{\infty} \frac{(-q; q^3)_n^2}{(q^3; q^3)_n^2} q^{3n} + \frac{(-q; q^3)_\infty^2 (q^2; q^6)_\infty^2}{(q; q)_\infty^2} \sum_{n=0}^{\infty} \frac{(q^2; q^3)_n^2 q^{3n+1}}{(-q; q^3)_{n+1}^2}$$
$$= \frac{(-q; q^3)_\infty^2 (-q^2; q^3)_\infty^2 (q^3; q^3)_\infty^2}{(q^2; q^3)_\infty^2 (q^3; q^3)_\infty^4}. \quad (3.32)$$

Use (1.25) to obtain

$$\sum_{n=0}^{\infty} \frac{(-q; q^3)_n^2}{(q^3; q^3)_n^2} q^{3n} + \frac{(-q; q^3)_\infty^2 (q^2; q^6)_\infty^2}{(q; q)_\infty^2} \sum_{n=0}^{\infty} \frac{(q^2; q^3)_n^2 q^{3n+1}}{(-q; q^3)_{n+1}^2} = \frac{J_6^2(q) (q; q^3)_\infty^2}{(q; q)_\infty^2 (q^3; q^3)_\infty^4}.$$

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