CYCLE AND THE COLLATZ CONJECTURE

W.Wongcharoenbhorn

30 August 2016

Abstract

We study on cycle in the Collatz conjecture and there is something surprise us. Our goal is to show that there is no collatz cycle.

1 Introduction

In 1937, Lothar Collatz had proposed the conjecture in number theory called *Collatz conjecture* or also known as $3n+1$ *conjecture.* It is still a conjecture until now. We define collatz funtion T as $T(n) = n/2$ if n is even and $T(n) = \dfrac{3n+1}{2}$ if n is odd and we let $T^k(n) = T(T(T(T...T(n))))$ for k times mapping

Conjecture (*Collatz*,1937) There exist $k \in N$ such that for any positive integer n, $T^k(n) = 1$.

It means that for any natural number n, we can find a positive integer k which we apply collatz function k times on n to reach 1 ,for example, the trajectory of thirteen for seven times is

$$13 \to 20 \to 10 \to 5 \to 8 \to 4 \to 2 \to 1$$

which satisfies the conjecture ($T^7(13) = 1$).
The parity vector of a number is its trajectory considered modulo two.
Hence, the parity vector of 13 to reach one is $\langle\ 1, 0, 0, 1, 0, 0, 0\rangle$

A lot of mathematicians had studied the Collatz conjecture and concluded that every positive integers less than 10^{10} have the behaviors like Collatz proposed.

Definition I $T_{<0>}(n) = n/2$ and $T_{<1>}(n) = (3n + 1)/2$

Definition II Let $< v_1, v_2, ...v_k >$ be the parity vector of n then we write

$$T_{<v_1,v_2,...,v_k>}(n) = T_{<v_k>}(T_{<v_{k-1}>}(...(T_{<v_1>}(n))))...)$$

The above two definitions say that v_1 can classify n. If $v_1 = 0$ it means that n is even and else, i.e. $v_1 = 1$, n is odd.

Definition III (*Collatz cycle*)

Collatz cycle means a loop containing of k elements that every element is greater than 1.If x is one of those elements then it must satisfy the equation $T^k(x) = x$. Therefore, after mapping T on x for k times it reach itself.

Theorem (*Latourette*) Let $< v_1, v_2, ...v_k >$ in which $\sum_{i=1}^{k} v_i = n$ be the parity vector of x.Then $T_{<v_1,v_2,..v_k>}(x) = T^k(x) = \frac{3^n}{2^k}(x) + \sum_{i=0}^{k-1} \frac{a_i}{2^{k-i}}$.
Where $a_i = 3^{(v_k+v_{k-1}+...+vi+2)}v_{i+1}$

2 Result

Proposition There is no cycle in the collatz problem.
Proof

We assume that there is a Collatz cycle with the number of the elements is k (not a trivial loop as $4, 2, 1$). therefore, $T^k(a) = a$ for any integer $a > 1$ in that cycle.

Considering in odd x of the loop (The cycle must have odd number . if it has only even, all of the elements must be in the form of 2^s for some natural s, which will reach one rapidly so it will not be a cycle.) and we see at the following trajectory compared to $x \to \frac{3x+1}{2}$

$$4x + 1 \to 6x + 2 \to 3x + 1 \to \frac{3x + 1}{2}$$

We have that $T^k(x) = x, T^{k-1}(\frac{3x + 1}{2}) = x$ and $T^3(4x + 1) = \frac{3x + 1}{2}$
i.e. $T^{k+2}(4x + 1) = x$.

Let $< v_1, v_2, ..., v_k >$ be the parity vector of x with the number of ones is n or $\sum_{i=1}^{k} v_i = n$. Hence, $< v_2, v_3, ...v_k >$ must be the parity vector starting at $\frac{3x + 1}{2}$ and its value after applying Collatz function is x .

The trajectory of $4x+1$ above shows that $4x+1$ have the parity vector as $< 1,0,0 >$ before reaching $(3x+1)/2$. After combining $\left(\frac{3x+1}{2}\right)$- parity vector with $< 1,0,0 >$ we get a whole parity vector of $4x+1$ for reaching x as $< 1,0,0,v_2,v_3,...,v_k >$. We clearly see that $v_1 = 1$ (x is odd) so $n = \sum_{i=1}^{k} v_i = 1+0+0+\sum_{i=2}^{k} v_i$

From Latourrette, let a_i, a'_i be the coefficient of $x, 4x+1$ in the equation respectively. Hence,We have $x = T^k(x) = \frac{3^n}{2^k}(x) + \sum_{i=0}^{k-1} \frac{a_i}{2^{k-i}}...(i)$

and $x = T^{k+2}(4x+1) = \frac{3^n}{2^{k+2}}(4x+1) + \frac{a'_0}{2^{k+2}} + \frac{a'_1}{2^{k+1}} + \frac{a'_2}{2^k} + \sum_{i=3}^{k+1} \frac{a'_i}{2^{k+2-i}}...(ii)$

Note $a_0 = 3^{n-1}, a'_0 = 3^{n-2}$ and $a'_1 = a'_2 = 0$.

Observation I For any $3 \leq i \leq k+2$. We have $a'_i = a_{i-2}$

Proof at $i=3$. It is clear that $a'_3 = 3^{(v_k+v_{k-1}+..+v_3)}v_2 = a_1$.

Suppose that at $i = s$ the Observation is also true. So $a'_s = a_{s-2} = 3^{(v_k+v_{k-1}+...+v_s)}v_{s-1}$ and the next step of i would be $a'_{s+1} = 3^{(v_k+v_{k-1}+...+v_{s+1})}v_s$ which is still equal to a_{s-1}. The proof is complete.

By using **Note** and **Observation I** the equation (ii) becomes

$$x = \frac{3^n}{2^{k+2}}(4x+1) + \frac{3^{n-2}}{2^{k+2}} + \sum_{i=3}^{k+1} \frac{a_{i-2}}{2^{k-(i-2)}} = \frac{3^n}{2^{k+2}}(4x+1) + \frac{3^{n-2}}{2^{k+2}} + \sum_{i=1}^{k-1} \frac{a_i}{2^{k-i}}...(iii)$$

We conclude from $(i) = (iii)$ that

$$\frac{3^n}{2^k}(x) + \frac{3^{n-1}}{2^k} + \sum_{i=1}^{k-1} \frac{a_i}{2^{k-i}} = \frac{3^n}{2^{k+2}}(4x+1) + \frac{3^{n-2}}{2^{k+2}} + \sum_{i=1}^{k-1} \frac{a_i}{2^{k-i}}$$

which is equivalent to $10 = 12$. Contradiction, hence, the proof is complete.

Reference

[1] Kelly S.Latourette,*Explorations of the collatz conjecture* (2007)