EXPANSION OF THE EULER ZIGZAG NUMBERS

GYEONGMIN YANG

Abstract. This article is based on how to look for a closed-form expression related to the value of the Riemann zeta funtion at odd positive integers and explained what meaning of the expansion of the Euler zigzag numbers is.

1. Introduction

For all s $\in \mathbb{C}$, the Riemann zeta funtion and the Dirichlet lambda, eta, beta function are defined as

$$\zeta(s) = \sum_{m=0}^{\infty} \frac{1}{(m+1)^s} \qquad \Re(s) > 1, \tag{1}$$

$$\lambda(s) = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^s} = \left(1 - \frac{1}{2^s}\right)\zeta(s) \qquad \Re(s) > 1, \tag{2}$$

$$\eta(s) = (1 - 2^{1-s})\zeta(s) \tag{3}$$

and

$$\beta(s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^s} \qquad \Re(s) > 0. \tag{4}$$

For every m $\in \mathbb{Z}^*$, the Euler zigzag number A_m can be expressed as

$$\sec x + \tan x = \sum_{m=0}^{\infty} \frac{A_m}{m!} x^m \qquad |x| < \frac{\pi}{2} \tag{5}$$

which the power series of $\sec x + \tan x$ involves. Furthermore, we will use

$$\sec x = \sum_{m=0}^{\infty} \frac{A_{2m}}{(2m)!} x^{2m} \qquad |x| < \frac{\pi}{2}, \tag{6}$$

$$\tan x = \sum_{m=0}^{\infty} \frac{A_{2m+1}}{(2m+1)!} x^{2m+1} \qquad |x| < \frac{\pi}{2} \tag{7}$$

in order to seperate the even and odd parts of A_m. Then, we will be following A'_{2m} rather than A_{2m} in this article. According to this suggestion, the $\sec x$ is represented by

$$\sec x = \sum_{m=0}^{\infty} \frac{A'_{2m}}{(2m)!} x^{2m} \qquad |x| < \frac{\pi}{2}. \tag{8}$$

2. Basic properties

Lemma 1. *For every $n \in \mathbb{N}$,*

$$\lambda(2n) = \beta(1)\frac{A_{2n-1}}{(2n-1)!}\left(\frac{\pi}{2}\right)^{2n-1}, \tag{9}$$

$$\beta(2n-1) = \beta(1)\frac{A'_{2n-2}}{(2n-2)!}\left(\frac{\pi}{2}\right)^{2n-2}. \tag{10}$$

Lemma 2. *For every $n \in \mathbb{N}$ and $0 < x < \pi/2$,*

$$\ln(\cot x) = 2\sum_{m=1}^{\infty}\frac{\cos((4m-2)x)}{2m-1}. \tag{11}$$

Proof. We consider [2]

$$\ln(\sin x) = -\ln 2 - \sum_{m=1}^{\infty}\frac{\cos(2mx)}{m} \tag{12}$$

which was studied by Euler. (12) is replaced by

$$\ln(\cos x) = -\ln 2 - \sum_{m=1}^{\infty}\frac{(-1)^m}{m}\cos(2mx). \tag{13}$$

To subtract (12) from (13) is

$$\ln(\cot x) = 2\sum_{m=1}^{\infty}\frac{\cos((4m-2)x)}{2m-1}.$$

□

Lemma 3. *For every $n \in \mathbb{N}$,*

$$\int_0^{\frac{\pi}{2}}\frac{x^n}{\sin x}dx = \sum_{m=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l A'_{2m}}{(2m+l+1)(2m)!}\left(\frac{\pi}{2}\right)^{2m+n+1} \tag{14}$$

the binomial coefficient is defined by the next expression:

$$\binom{n}{l} = \frac{n!}{(l)!(n-l)!}.$$

Lemma 4. *For every $n \in \mathbb{N}$,* [1]

$$\frac{1}{n!}\int_0^{\frac{\pi}{2}}\frac{x^n}{\sin x}dx = \sum_{m=0}^{\infty}\frac{A'_{2m}}{(2m+n+1)!}\left(\frac{\pi}{2}\right)^{2m+n+1}. \tag{15}$$

Lemma 5. *For every $n \in \mathbb{N}$,*

$$\frac{1}{n!}\int_0^{\frac{\pi}{2}}x^n\cot x\,dx = \sum_{m=0}^{\infty}\frac{A_{2m+1}}{(2m+n+2)!}\left(\frac{\pi}{2}\right)^{2m+n+2}. \tag{16}$$

3. Proof of Theorem 1

Theorem 1. *For every $n \in \mathbb{N}$,*

$$\sum_{m=0}^{\infty}\frac{A'_{2m}}{(2m+n+1)!}\left(\frac{\pi}{2}\right)^{2m} = \frac{A_n}{n!}\cos\left(\frac{n}{2}\pi\right) + \sum_{l=1}^{n}\frac{A'_l}{(l)!(n-l)!}\sin\left(\frac{l}{2}\pi\right). \tag{17}$$

Proof. Multiplying x^{n-1} and integrating 0 to $\pi/4$ for the both terms of Lemma 2 is

$$\int_0^{\frac{\pi}{4}} x^{n-1} \ln(\cot x) dx = 2 \int_0^{\frac{\pi}{4}} \sum_{m=1}^{\infty} \frac{x^{n-1}}{2m-1} \cos((4m-2)x) dx. \tag{18}$$

Carrying out partial integral the left side of (18), equation (18) will be

$$\int_0^{\frac{\pi}{4}} \frac{x^n}{\sin 2x} dx = \int_0^{\frac{\pi}{4}} \sum_{m=1}^{\infty} \frac{nx^{n-1}}{2m-1} \cos((4m-2)x) dx. \tag{19}$$

Calculating the integral terms in (19), we find the following expressions:

$$\beta(2) = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{x}{\sin x} dx, \tag{20}$$

$$\lambda(2n+1) = \frac{(-1)^n}{2(2n)!} \int_0^{\frac{\pi}{2}} \frac{x^{2n}}{\sin x} dx + \sum_{l=0}^{n-1} \frac{(-1)^l \beta(2n-2l)}{(2l+1)!} \left(\frac{\pi}{2}\right)^{2l+1} \tag{21}$$

and

$$\beta(2n+2) = \frac{(-1)^n}{2(2n+1)!} \int_0^{\frac{\pi}{2}} \frac{x^{2n+1}}{\sin x} dx + \sum_{l=0}^{n-1} \frac{(-1)^l \beta(2n-2l)}{(2l+2)!} \left(\frac{\pi}{2}\right)^{2l+2}. \tag{22}$$

Applying of Lemma 3 for each integral term in (20), (21) and (22) yields

$$\lambda(2n+1) = \frac{\beta(2n+1)}{A'_{2n}} \sum_{m=0}^{\infty} \sum_{l=0}^{n} \binom{2n}{2l} \frac{(-1)^l A'_{2n-2l} A'_{2m}}{(2l+2m+1)(2m)!} \left(\frac{\pi}{2}\right)^{2m}, \tag{23}$$

$$\beta(2n) = \frac{\lambda(2n)}{A_{2n-1}} \sum_{m=0}^{\infty} \left(\left(\sum_{l=0}^{n-1} \binom{2n-1}{2l} \frac{(-1)^l A_{2n-2l-1}}{2l+2m+1} \right) - \frac{(-1)^{n-1}}{2m+2n} \right) \frac{A'_{2m}}{(2m)!} \left(\frac{\pi}{2}\right)^{2m}. \tag{24}$$

Applying of Lemma 1 for $\beta(2n+1)$ in (23) and for $\lambda(2n)$ in (24) provides

$$\lambda(2n+1) = \beta(1) \frac{A_{2n}}{(2n)!} \left(\frac{\pi}{2}\right)^{2n}, \tag{25}$$

$$\beta(2n) = \beta(1) \frac{A'_{2n-1}}{(2n-1)!} \left(\frac{\pi}{2}\right)^{2n-1} \tag{26}$$

which are corresponded with Lemma 1. In other words, A_{2n} in (25) and A'_{2n-1} in (26) follow that

$$A_{2n} = \sum_{m=0}^{\infty} \sum_{l=0}^{n} \binom{2n}{2l} \frac{(-1)^l A'_{2n-2l} A'_{2m}}{(2l+2m+1)(2m)!} \left(\frac{\pi}{2}\right)^{2m}, \tag{27}$$

$$A'_{2n-1} = \sum_{m=0}^{\infty} \left(\left(\sum_{l=0}^{n-1} \binom{2n-1}{2l} \frac{(-1)^l A_{2n-2l-1}}{2l+2m+1} \right) - \frac{(-1)^{n-1}}{2m+2n} \right) \frac{A'_{2m}}{(2m)!} \left(\frac{\pi}{2}\right)^{2m}. \tag{28}$$

Applying of Lemma 4 for each integral term in (20), (21) and (22) becomes

$$\beta(2) = \sum_{m=0}^{\infty} \frac{A'_{2m}}{2(2m+2)!} \left(\frac{\pi}{2}\right)^{2m+2}, \tag{29}$$

$$\lambda(2n+1)=\sum_{l=0}^{n-1}\frac{(-1)^l\beta(2n-2l)}{(2l+1)!}\left(\frac{\pi}{2}\right)^{2l+1}+\sum_{m=0}^{\infty}\frac{(-1)^nA'_{2m}}{2(2m+2n+1)!}\left(\frac{\pi}{2}\right)^{2m+2n+1} \tag{30}$$

and

$$\beta(2n+2)=\sum_{l=0}^{n-1}\frac{(-1)^l\beta(2n-2l)}{(2l+2)!}\left(\frac{\pi}{2}\right)^{2l+2}+\sum_{m=0}^{\infty}\frac{(-1)^nA'_{2m}}{2(2m+2n+2)!}\left(\frac{\pi}{2}\right)^{2m+2n+2}. \tag{31}$$

Left side of (30) is replaced with right side of (25). Also, using right side of (26) in left side of (29), (31) and $\beta(2n-2l)$ each of (30), (31) obtains

$$\sum_{m=0}^{\infty}\frac{A'_{2m}}{(2m+2n)!}\left(\frac{\pi}{2}\right)^{2m}=\sum_{l=1}^{n}\frac{(-1)^{l+n}A'_{2n-2l+1}}{(2l-2)!(2n-2l+1)!}, \tag{32}$$

$$\sum_{m=0}^{\infty}\frac{A'_{2m}}{(2m+2n+1)!}\left(\frac{\pi}{2}\right)^{2m}=\frac{(-1)^nA_{2n}}{(2n)!}+\sum_{l=1}^{n}\frac{(-1)^{l+n}A'_{2n-2l+1}}{(2l-1)!(2n-2l+1)!} \tag{33}$$

respectively. Therefore, (32) and (33) can be combined to

$$\sum_{m=0}^{\infty}\frac{A'_{2m}}{(2m+n+1)!}\left(\frac{\pi}{2}\right)^{2m}=\frac{A_n}{n!}\cos\left(\frac{n}{2}\pi\right)+\sum_{l=1}^{n}\frac{A'_l}{(l)!(n-l)!}\sin\left(\frac{l}{2}\pi\right).$$

□

4. Proof of Theorem 2

Although $\zeta(1)$ diverges, we assume that A_0 has a constant.

Theorem 2. *For every $n\in\mathbb{N}$,*

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{A_{2m+1}}{(2m+n+2)!}\left(\frac{\pi}{2}\right)^{2m+1}&=\frac{1}{n!}\left(\frac{2}{\pi}\right)\ln 2+\frac{A_n}{2^nn!}\cos\left(\frac{n}{2}\pi\right)\\&+\sum_{l=0}^{n-1}\frac{(2^l-1)A_l}{2^l(2^{l+1}-1)(l)!(n-l)!}\cos\left(\frac{l}{2}\pi\right).\end{aligned} \tag{34}$$

Proof. Multiplying x^{n-1} and integrating 0 to $\pi/2$ for the both terms of (12) is

$$\int_0^{\frac{\pi}{2}}x^{n-1}\ln(\sin x)dx=-\int_0^{\frac{\pi}{2}}x^{n-1}(\ln 2)dx-\int_0^{\frac{\pi}{2}}\sum_{m=1}^{\infty}\frac{x^{n-1}}{m}\cos(2mx)dx. \tag{35}$$

Carrying out partial integral the left side of (35), equation (35) will be

$$\int_0^{\frac{\pi}{2}}x^n\cot x dx=\left(\frac{\pi}{2}\right)^n\ln 2+\int_0^{\frac{\pi}{2}}\sum_{m=1}^{\infty}\frac{nx^{n-1}}{m}\cos(2mx)dx. \tag{36}$$

Calculating the integral terms in (36), we find the following expressions:

$$\frac{\pi}{2}\eta(1)=\int_0^{\frac{\pi}{2}}x\cot x dx, \tag{37}$$

$$\frac{\pi}{2}\eta(2n+1)=\frac{(-1)^n2^{2n}}{(2n+1)!}\int_0^{\frac{\pi}{2}}x^{2n+1}\cot x dx+\sum_{l=1}^{n}\frac{(-1)^{l-1}2^{2l}\eta(2n-2l+1)}{(2l+1)!}\left(\frac{\pi}{2}\right)^{2l+1} \tag{38}$$

and

$$\lambda(2n+1)=\frac{(-1)^n 2^{2n-1}}{(2n)!}\int_0^{\frac{\pi}{2}} x^{2n}\cot x dx+\sum_{l=1}^{n}\frac{(-1)^{l-1}2^{2l-1}\eta(2n-2l+1)}{(2l)!}\left(\frac{\pi}{2}\right)^{2l}. \tag{39}$$

Applying of Lemma 5 for each integral term in (37), (38) and (39) becomes

$$\frac{\pi}{2}\eta(1)=\sum_{m=0}^{\infty}\frac{A_{2m+1}}{(2m+3)!}\left(\frac{\pi}{2}\right)^{2m+3}, \tag{40}$$

$$\frac{\pi}{2}\eta(2n+1)=\sum_{m=0}^{\infty}\frac{(-1)^n 2^{2n}A_{2m+1}}{(2m+2n+3)!}\left(\frac{\pi}{2}\right)^{2m+2n+3}+\sum_{l=1}^{n}\frac{(-1)^{l-1}2^{2l}\eta(2n-2l+1)}{(2l+1)!}\left(\frac{\pi}{2}\right)^{2l+1} \tag{41}$$

and

$$\lambda(2n+1)=\sum_{m=0}^{\infty}\frac{(-1)^n 2^{2n-1}A_{2m+1}}{(2m+2n+2)!}\left(\frac{\pi}{2}\right)^{2m+2n+2}+\sum_{l=1}^{n}\frac{(-1)^{l-1}2^{2l-1}\eta(2n-2l+1)}{(2l)!}\left(\frac{\pi}{2}\right)^{2l}. \tag{42}$$

From the relation with the (2), (3) and (25), $\eta(2n+1)$ becomes

$$\eta(2n+1)=\frac{(2^{2n+1}-2)A_{2n}}{(2^{2n+2}-2)(2n)!}\left(\frac{\pi}{2}\right)^{2n+1}. \tag{43}$$

Left side of (42) is replaced with right side of (25). Also, using right side of (43) in left side of (41) and $\eta(2n-2l+1)$ each of (41), (42) obtains

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{A_{2m+1}}{(2m+2n+3)!}\left(\frac{\pi}{2}\right)^{2m+1} &= \frac{1}{(2n+1)!}\left(\frac{2}{\pi}\right)\ln 2\\ &+\sum_{l=0}^{n-1}\frac{(-1)^{l+n}2^{2l}(2^{2n-2l+1}-2)A_{2n-2l}}{2^{2n+1}(2^{2n-2l+1}-1)(2l+1)!(2n-2l)!},\end{aligned} \tag{44}$$

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{A_{2m+1}}{(2m+2n+2)!}\left(\frac{\pi}{2}\right)^{2m+1} &= \frac{1}{(2n)!}\left(\frac{2}{\pi}\right)\ln 2+\frac{(-1)^n A_{2n}}{2^{2n}(2n)!}\\ &+\sum_{l=1}^{n}\frac{(-1)^{l+n}2^{2l-1}(2^{2n-2l+1}-2)A_{2n-2l}}{2^{2n}(2^{2n-2l+1}-1)(2l)!(2n-2l)!}\end{aligned} \tag{45}$$

respectively. Therefore, (40), (44) and (45) can be combined to

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{A_{2m+1}}{(2m+n+2)!}\left(\frac{\pi}{2}\right)^{2m+1} &= \frac{1}{n!}\left(\frac{2}{\pi}\right)\ln 2+\frac{A_n}{2^n n!}\cos\left(\frac{n}{2}\pi\right)\\ &+\sum_{l=0}^{n-1}\frac{(2^l-1)A_l}{2^l(2^{l+1}-1)(l)!(n-l)!}\cos\left(\frac{l}{2}\pi\right).\end{aligned}$$

□

References

[1] JeonWon Kim. Functional equations related to the dirichlet lambda and beta functions. *arXiv preprint arXiv:1404.5467*, 2014.

[2] Shin-ya Koyama and Nobushige Kurokawa. Euler's integrals and multiple sine functions. *Proceedings of the American Mathematical Society*, pages 1257–1265, 2005.