Using binomial coefficients to find the sum of powers

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Introduction

This paper presents a method of calculating powers and sums of powers using binomial coefficients. The method involves finding analogues of Pascal's triangle for each power and then showing that powers and sums of powers are the sums of binomial coefficients multiplied by constants. The constants are unique for each power. This paper presents a general idea and not a formal proof.

The Tables

We can put Pascal's triangle into table form:

1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9
1	3	6	10	15	21	28	36	45
1	4	10	20	35	56	84	120	165
1	5	15	35	70	126	210	330	495
1	6	21	56	126	252	462	792	1287
1	7	28	84	210	462	924	1716	3003
1	8	36	120	330	792	1716	3432	6435
1	9	45	165	495	1287	3003	6435	12870

Where each element (x_{ij}) is the sum of the element to the left and the element above, so that:

$$x_{ij} = x_{(i-1)j} + x_{i(j-1)}$$

And $x_{ij} = 1$ if i = 1 or j = 1. We can create a similar table where we have $x_{ij} = i^n$ for some j. For the case when n = 2, the table looks like:

	1	1	1	1	1	1
2	3	4	5	6	7	8
2	5	9	14	20	27	35
2	7	16	30	50	77	112
2	9	25	55	105	182	294

2	11	36	91	196	378	672

In this table the elements of the third column are the squares of integers. The elements of the second column are the differences between consecutive squares and the elements of the first column are the differences between the differences.

The same rule applies for this table as Pascal's triangle:

$$x_{ij} = x_{(i-1)j} + x_{i(j-1)}$$

However, we now have $x_{1j} = 1$ and $x_{i1} = 2$.

If n = 3 then the table looks like:

		1	1	1	1	1
	6	7	8	9	10	11
6	12	19	27	36	46	57
6	18	37	64	100	146	203
6	24	61	125	225	371	574
6	30	91	216	441	812	1386

In this table the elements of the forth column are cubes of integers. And $x_{1j} = 1$, $x_{i1} = 6$ and $x_{22} = 6$.

For n = 4 the table is:

			1	1	1	1
		14	15	16	17	18
	36	50	65	81	98	116
24	60	110	175	256	354	470
24	84	194	369	625	979	1449
24	108	302	671	1296	2275	3724

Here, the fifth column are the powers of 4. $x_{1j} = 1$, $x_{i1} = 24$, $x_{23} = 14$, and $x_{32} = 36$.

The Formulas

The elements of Pascal's trianlge are the binomial coefficients, and we can use these to find the elements of the above tables. If we define some constants C_{nm} where m ranges from 1 to n and n is the power, then we can write an equation for an interger raised to that power:

$$y^n = \sum_{k=1}^{L(y,n)} C_{nk} \binom{y}{k}$$

Where L(y, n) = y if y < n, otherwise L(y, n) = n. Furthermore, we can use the properties of the binomial coefficients to write an equation for the sums of powers:

$$\sum_{a=1}^{y} a^n = \sum_{k=1}^{L(y,n)} C_{nk} \binom{y+1}{k+1}$$

Examples

For n = 2, the constants are: $C_{21} = 1$, and $C_{22} = 2$, so if we set y = 5 then:

$$5^2 = 1\binom{5}{1} + 2\binom{5}{2} = 1 \cdot 5 + 2 \cdot 10 = 25$$

And,

$$\sum_{a=1}^{5} a^2 = 1\binom{6}{2} + 2\binom{6}{3} = 1 \cdot 15 + 2 \cdot 20 = 55$$

For n = 3, the constants are: $C_{31} = 1$, $C_{32} = 6$, and $C_{33} = 6$, for y = 5:

$$5^3 = 1\binom{5}{1} + 6\binom{5}{2} + 6\binom{5}{3} = 1 \cdot 5 + 6 \cdot 10 + 6 \cdot 10 = 125$$

And,

$$\sum_{a=1}^{5} a^3 = 1\binom{6}{2} + 6\binom{6}{3} + 6\binom{6}{4} = 1 \cdot 15 + 6 \cdot 20 + 6 \cdot 15 = 225$$

For n = 4, the constants are: $C_{41} = 1$, $C_{42} = 14$, $C_{43} = 36$, and $C_{44} = 24$, for y = 5:

$$5^4 = 1\binom{5}{1} + 14\binom{5}{2} + 36\binom{5}{3} + 24\binom{5}{4} = 1 \cdot 5 + 14 \cdot 10 + 36 \cdot 10 + 24 \cdot 5 = 625$$

And,

$$\sum_{a=1}^{5} a^4 = 1\binom{6}{2} + 14\binom{6}{3} + 36\binom{6}{4} + 24\binom{6}{5}$$

$$= 1 \cdot 15 + 14 \cdot 20 + 36 \cdot 15 + 24 \cdot 6 = 979$$

A formula for the constants

In my notes I wrote a recursive formula for finding the constants:

$$C_{nm} = m(C_{(n-1)m} + C_{(n-1)(m-1)})$$

Where $C_{n1} = 1$ for all n, and $C_{0m} = 0$ for all m, and $C_{nm} = 0$ if m > n.

I didn't manage to prove this formula, but I'm sure the logic is sound.

Conclusion

Hopefully the logic in this paper is self evident as nothing has been proved. If everything works then it's possible to use these formulas to calculate the power of an interger as well as calculate the sum of powers for a power.