Some questions related with elementary analysis

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abstract

In this note we give some formulas related with: pi constant pi=3.1415..., Catalan’s constant G=0.9159... , Euler´s constant γ=0.5772... .

Keywords:pi constant , Catalan’s constant ,Euler’s constant,formulas.

1. Introducción

En esta nota mostramos una colección de fórmulas relacionadas con algunas constantes matemáticas, como son :

La constante pi : $\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.1415\ldots$, la constante de Catalan : $G = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = 0.9159\ldots$, la

constante gamma de Euler : $\gamma = \lim_{k\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}-\ln k\right) = 0.5772\ldots$, la constante e de Euler :

$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.7182\ldots$.

2. Fórmulas

- **Question 1.**

$$G = \sum_{n=0}^{\infty}(-1)^n (n+1)\int_0^1\int_0^1\left(\sqrt{1+x^2y^2}-1\right)^n dx\, dy \tag{1}$$

$$G = 1+\sum_{n=1}^{\infty}(n+1)\left\{1+\sum_{k=1}^{n}(-1)^k\binom{n}{k}F\left(\left\{\frac{1}{2},\frac{1}{2},-\frac{k}{2}\right\},\left\{\frac{3}{2},\frac{3}{2}\right\},-1\right)\right\} \tag{2}$$

$$G = 1+\sum_{n=1}^{\infty}(n+1)f(n) \tag{3}$$

donde

$$f(n) = \sum_{k=0}^{[n/2]}\binom{n}{2k}\sum_{m=0}^{k}\binom{k}{m}(2m+1)^{-2} - \sum_{k=0}^{[(n-1)/2]}\binom{n}{2k+1}F\left(\left\{\frac{1}{2},\frac{1}{2},-\frac{2k+1}{2}\right\},\left\{\frac{3}{2},\frac{3}{2}\right\},-1\right) \tag{4}$$

$$f(n)=\sum_{k=0}^{[n/2]}\binom{n}{2k}\sum_{m=0}^{k}\binom{k}{m}(2m+1)^{-2}-\sum_{k=0}^{[(n-1)/2]}\binom{n}{2k+1}\sum_{m=0}^{k}\binom{k}{m}(2m+1)^{-2}F\left(\left\{-\frac{1}{2},\frac{1}{2}+m,\frac{1}{2}+m\right\},\left\{\frac{3}{2}+m,\frac{3}{2}+m\right\},-1\right) \tag{5}$$

$$G=\sum_{n=0}^{\infty}(-1)^n\binom{n+k-1}{n}\int_0^1\int_0^1\left(\sqrt[k]{1+x^2y^2}-1\right)^n dx\,dy\ ,\ k\in\mathbb{N} \tag{6}$$

En (2), (4), (5), F representa una función hipergeométrica.

■ Question 2.

$$G=\frac{2}{3}\sum_{n=0}^{\infty}3^{-n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k 2^k(2k+1)^{-2} \tag{7}$$

$$G=\sqrt{\frac{2}{5}}\sum_{n=0}^{\infty}\binom{2n}{n}2^{-2n}\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{2}{5}\right)^k\sum_{m=0}^{2k}\binom{2k}{m}(2m+1)^{-2} \tag{8}$$

$$G=\sqrt[m]{\frac{2}{2^m+1}}\sum_{n=0}^{\infty}\frac{(1/m)_n}{n!}\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{2}{2^m+1}\right)^k\sum_{s=0}^{mk}\binom{mk}{s}(2s+1)^{-2}\ ,\ m\in\mathbb{N} \tag{9}$$

■ Question 3.

Sean c_n , $n\in\mathbb{N}\cup\{0\}$, los números definidos por :

$$c_0=1\ ,\ c_n=-\sum_{k=1}^{n}\frac{(-1)^k}{k+1}c_{n-k}\ ,\ n\in\mathbb{N} \tag{10}$$

$$c_n=\frac{1}{n!}\sum_{k=0}^{n}\frac{S(n,k)}{k+1}\ ,\ S(n,k)\ :\text{números de Stirling de primer orden.} \tag{11}$$

se tiene :

$$\frac{2\sqrt{2}}{\pi}=\sum_{n=0}^{\infty}(-1)^n c_n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\cos\left(\frac{k\pi}{4}\right) \tag{12}$$

$$\frac{4-2\sqrt{2}}{\pi}=\sum_{n=0}^{\infty}(-1)^n c_n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\operatorname{sen}\left(\frac{k\pi}{4}\right) \tag{13}$$

$$\frac{4}{\pi}=\sum_{n=0}^{\infty}(-1)^n c_n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\operatorname{sen}\left(\frac{k\pi}{4}\right)+\cos\left(\frac{k\pi}{4}\right)\right) \tag{14}$$

$$\frac{4\left(\sqrt{2}-1\right)}{\pi}=\sum_{n=0}^{\infty}(-1)^n c_n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\cos\left(\frac{k\pi}{4}\right)-\operatorname{sen}\left(\frac{k\pi}{4}\right)\right) \tag{15}$$

$$\frac{3}{\pi}=\sum_{n=0}^{\infty}(-1)^n c_n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\cos\left(\frac{k\pi}{6}\right) \tag{16}$$

$$\frac{6-3\sqrt{3}}{\pi}=\sum_{n=0}^{\infty}(-1)^n c_n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\operatorname{sen}\left(\frac{k\pi}{6}\right) \tag{17}$$

$$\frac{9-3\sqrt{3}}{\pi}=\sum_{n=0}^{\infty}(-1)^n c_n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\operatorname{sen}\left(\frac{k\pi}{6}\right)+\cos\left(\frac{k\pi}{6}\right)\right) \tag{18}$$

$$\frac{3\left(\sqrt{3}-1\right)}{\pi}=\sum_{n=0}^{\infty}(-1)^n c_n \sum_{k=0}^{n}(-1)^k \binom{n}{k}\left(\cos\left(\frac{k\pi}{6}\right)-\operatorname{sen}\left(\frac{k\pi}{6}\right)\right) \tag{19}$$

$$\frac{4\sqrt{2-\sqrt{2}}}{\pi}=\sum_{n=0}^{\infty}(-1)^n c_n \sum_{k=0}^{n}(-1)^k \binom{n}{k}\cos\left(\frac{k\pi}{8}\right) \tag{20}$$

$$\frac{8-4\sqrt{2+\sqrt{2}}}{\pi}=\sum_{n=0}^{\infty}(-1)^n c_n \sum_{k=0}^{n}(-1)^k \binom{n}{k}\operatorname{sen}\left(\frac{k\pi}{8}\right) \tag{21}$$

$$\frac{3\left(\sqrt{6}-\sqrt{2}\right)}{\pi}=\sum_{n=0}^{\infty}(-1)^n c_n \sum_{k=0}^{n}(-1)^k \binom{n}{k}\cos\left(\frac{k\pi}{12}\right) \tag{22}$$

$$\frac{12-3\left(\sqrt{6}+\sqrt{2}\right)}{\pi}=\sum_{n=0}^{\infty}(-1)^n c_n \sum_{k=0}^{n}(-1)^k \binom{n}{k}\operatorname{sen}\left(\frac{k\pi}{12}\right) \tag{23}$$

$$\frac{12-6\sqrt{2}}{\pi}=\sum_{n=0}^{\infty}(-1)^n c_n \sum_{k=0}^{n}(-1)^k \binom{n}{k}\left(\operatorname{sen}\left(\frac{k\pi}{12}\right)+\cos\left(\frac{k\pi}{12}\right)\right) \tag{24}$$

$$\frac{6\sqrt{6}-12}{\pi}=\sum_{n=0}^{\infty}(-1)^n c_n \sum_{k=0}^{n}(-1)^k \binom{n}{k}\left(\cos\left(\frac{k\pi}{12}\right)-\operatorname{sen}\left(\frac{k\pi}{12}\right)\right) \tag{25}$$

■ Question 4.

Para $z>0$, $y>x\geq 0$, $z(y-x)-z^2-x\,y=0$, se tiene :

$$\pi=\int_x^y \frac{4z}{z^2+t^2}\,dt \tag{26}$$

■ Question 5.

Para $0\leq z\leq 1$, $n\in\mathbb{N}$, se tiene :

$$\tan^{-1} z=\sum_{k=0}^{n-1}\frac{(-1)^k z^{2k+1}}{2k+1}+2(-1)^n z^{2n+3}\int_0^1\int_0^1 \frac{x^{2n+2}\,y}{(z^2x^2+y^2)^2}\,dy\,dx \tag{27}$$

$$\pi=4\sum_{k=0}^{n-1}\frac{(-1)^k}{2k+1}+8(-1)^n\int_0^1\int_0^1\frac{x^{2n+2}\,y}{(x^2+y^2)^2}\,dy\,dx\ ,\ n\in\mathbb{N} \tag{28}$$

$$\pi=4\sum_{k=0}^{n-1}\frac{(-1)^k\left(2^{-2k-1}+3^{-2k-1}\right)}{2k+1}+8(-1)^n\int_0^1\int_0^1 x^{2n+2}\,y\left(\frac{2^{-2n+1}}{(x^2+4y^2)^2}+\frac{3^{-2n+1}}{(x^2+9y^2)^2}\right)dy\,dx\ ,\ n\in\mathbb{N} \tag{29}$$

■ Question 6.

La constante gamma de Euler – Mascheroni se define por :

$$\gamma=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right)=0.577215\ldots \tag{30}$$

se tiene :

$$\gamma=\int_{-\infty}^{\infty}x\,e^{-x-e^{-x}}\,dx \tag{31}$$

$$\gamma=\int_0^{\infty}x\,e^{-x-e^{-x}}\,dx-\int_0^{\infty}x\,e^{x-e^{x}}\,dx \tag{32}$$

$$\gamma = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!\,n} - \int_1^{\infty} e^{-x} \ln x \, dx \tag{33}$$

$$\gamma = -\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!\,(2n)} - \int_1^{\infty} \left(\cos x - \frac{1}{1+x} \right) \frac{1}{x} \, dx \tag{34}$$

$$\gamma = 1 - \ln 2 - \sum_{n=1}^{\infty} \frac{(-1)^{n}}{(2n+1)!\,(2n)} - \int_1^{\infty} \left(\frac{\operatorname{sen} x}{x} - \frac{1}{1+x} \right) \frac{1}{x} \, dx \tag{35}$$

$$\gamma = -\ln 2 - \sum_{n=1}^{\infty} \frac{(-1)^{n}}{n!\,n} - \int_1^{\infty} \left(e^{-x} - \frac{1}{1+x} \right) \frac{1}{x} \, dx \tag{36}$$

$$\gamma = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!\,n} - 2^k \int_1^{\infty} \frac{e^{-x^{2^k}}}{x} \, dx \ , \ k \in \mathbb{Z} \tag{37}$$

$$\gamma - \sum_{k=1}^{n} \frac{1}{k} + \ln(n+1) = \int_0^1 \int_0^1 \frac{x^n (1 - x^y)}{1-x} \, dy \, dx \ , \ n \in \mathbb{N} \tag{38}$$

$$(p-q)\,\gamma = \sum_{n=1}^{\infty} \frac{(-1)^n \,(q\, a^{p\,n} - p\, a^{q\,n})}{n!\,n} + p\,q \int_a^{\infty} \frac{e^{-x^p} - e^{-x^q}}{x} \, dx \tag{39}$$

$$p > 0 \, , \ q > 0 \, , \ a > 0$$

■ Question 7.

Sean :

$$\phi(x) = \int_0^1 e^{-x} \ln(1 + x\,t) \, dt \ , \ 0 \le x \le 1 \tag{40}$$

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) = 0.577215 \ldots \tag{41}$$

se tiene :

$$\gamma = -\left(1 - e^{-k}\right) \ln k + \sum_{n=0}^{\infty} \frac{(-1)^n \, k^{n+1}}{n!\,(n+1)^2} - \left(1 - e^{-1}\right) \sum_{n=k}^{\infty} e^{-n} \ln n + \sum_{n=k}^{\infty} e^{-n} \phi\!\left(\frac{1}{n}\right) \ , \ k \in \mathbb{N} \tag{42}$$

$$\phi(x) = e^{1/x} \operatorname{Ei}\!\left(1, \frac{1}{x}\right) - e^{-1} \ln(1+x) - e^{1/x} \operatorname{Ei}\!\left(1, \frac{1+x}{x}\right) \ , \ 0 \le x \le 1 \tag{43}$$

$$\operatorname{Ei}(1, x) = \int_1^{\infty} \frac{e^{-x\,t}}{t} \, dt \ , \ x > 0 \tag{44}$$

$$\phi(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^n \, x^{n-k+1}}{k!\,(n-k+1)\,(n+2)} \ , \ 0 \le x \le 1 \tag{45}$$

$$\phi(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \, c_n}{n} \, x^n \ , \ 0 \le x \le 1 \tag{46}$$

$$c_n = \int_0^1 e^{-t} \, t^n \, dt = n! \left(1 - e^{-1} \sum_{k=0}^{n} \frac{1}{k!} \right) \ , \ n \in \mathbb{N} \tag{47}$$

$$c_{n+1} = (n+1)\, c_n - e^{-1} \ , \ n \in \mathbb{N} \tag{48}$$

$$\phi(x) = x \int_0^1 \int_0^1 \frac{u\, e^{-u}}{1 + x\,u\,v} \, dv \, du \ , \ 0 \le x \le 1 \tag{49}$$

$$\gamma = -(1 - e^{-a}) \ln a + \sum_{n=0}^{\infty} \frac{(-1)^n a^{n+1}}{n!\,(n+1)^2} - \int_a^{\infty} e^{-t} \ln t \, dt \ , \ a > 0 \tag{50}$$

$$\gamma = -\ln a + \sum_{n=0}^{\infty} \frac{(-1)^n a^{n+1}}{n!\,(n+1)^2} - (1 - e^{-a}) \sum_{n=1}^{\infty} e^{-a n} \ln n - a \sum_{n=1}^{\infty} e^{-a n} \phi\left(a, \frac{1}{n}\right) \ , \ a > 0 \tag{51}$$

$$\phi(a, x) = \int_0^1 e^{-a t} \ln(1 + x t) \, dt \ , \ a > 0 \, , \ 0 \le x \le 1 \tag{52}$$

$$\gamma = -\ln a + \sum_{n=0}^{\infty} \frac{(-1)^n a^{n+1}}{n!\,(n+1)^2} + e^{-a} \ln m - \left(1 - e^{-a/m}\right) \sum_{n=m}^{\infty} e^{-a n/m} \ln n - \frac{a}{m} \sum_{n=m}^{\infty} e^{-a n/m} \phi\left(\frac{a}{m}, \frac{1}{n}\right) \tag{53}$$

$$a > 0 \, , \ m \in \mathbb{N}$$

■ Question 8.

Para $m \in \mathbb{N}$, $\pi = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 3.1415 \ldots$, $\zeta(x) = \sum_{n=1}^{\infty} n^{-x}$, $x > 1$, se tiene :

$$I(m) = \left(\int_{-\infty}^{\infty} \frac{e^{(2m+1)x/2}}{1 + e^{e^x}} \, dx \right)^2 \tag{54}$$

$$I(m) = \int_1^{\infty} \int_1^{\infty} \frac{(\ln x \ln y)^{m-1/2}}{x\,y(1 + x)(1 + y)} \, dx \, dy \tag{55}$$

$$I(m) = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{(2m+1)x}}{\left(1 + e^{e^{x+y}}\right)\left(1 + e^{e^{x-y}}\right)} \, dx \, dy \tag{56}$$

$$I(m) = \int_0^1 \int_0^1 \frac{(\ln x \ln y)^{m-1/2}}{(1 + x)(1 + y)} \, dx \, dy \tag{57}$$

$$I(m) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{(m+1/2)(x+y)}}{\left(1 + e^{e^x}\right)\left(1 + e^{e^y}\right)} \, d\Box \, d\Box \tag{58}$$

$$I(m) = \pi \left(\frac{1}{2}\right)_m^2 \left(1 - 2^{-m+1/2}\right)^2 \left(\zeta(2m+1) + 2 \sum_{n=1}^{\infty} \sum_{k=1}^{n} (k(n+1))^{-m-1/2} \right) \tag{59}$$

■ Question 9.

Para $n \in \mathbb{N}$, $s \in \mathbb{N} - \{1\}$, $\zeta(x) = \sum_{n=1}^{\infty} n^{-x}$, $x > 1$, se tiene :

$$\zeta\left(n + \frac{1}{s}\right) = \sum_{k=1}^{\infty} k^{-n-1/s} = \zeta(n s + 1) + \sum_{k \neq m^s,\, k, m \in \mathbb{N}} k^{-n-1/s} \tag{60}$$

$$\zeta\left(n + \frac{1}{2}\right) = \zeta(2n + 1) + \sum_{k \neq m^2, k, m \in \mathbb{N}} \frac{\sqrt{k}}{k^{n+1}} \tag{61}$$

$$\zeta\left(n + \frac{1}{3}\right) = \zeta(3n + 1) + \sum_{k \neq m^3, k, m \in \mathbb{N}} k^{-n-1/3} \tag{62}$$

$$\zeta\left(2n - \frac{2}{3}\right) = \frac{2^{6n-3} B_{3n-1}}{(6n-2)!} \pi^{6n-2} + \sum_{k \neq m^3,\, k, m \in \mathbb{N}} k^{-2n-4/3} \tag{63}$$

B_{3n-1} son los números de Bernoulli : $B_2 = 1/30$, $B_5 = 5/66$, $B_8 = 3617/510$, ...

■ Question 10.

Sea $n \in \mathbb{N}$, $y\, F(a,\, b\,;\, c\,;\, x)$ la función hipergeometrica :

$$F(a,\, b\,;\, c\,;\, x) = \sum_{k=0}^{\infty} \frac{(a)_k\, (b)_k}{(c)_k} \frac{x^k}{k!}\ ,\ \ -1 < x < 1 \tag{64}$$

se tiene :

$$I(n) = \frac{1}{2\,n-1} F\left(n - \frac{1}{2},\, n\,;\, n + \frac{1}{2}\,;\, -1\right) \tag{65}$$

$$I(n) = \sqrt{2}\ \frac{2^{-n}}{2\,n-1} F\left(n - \frac{1}{2},\, \frac{1}{2}\,;\ n + \frac{1}{2}\,;\, \frac{1}{2}\right) \tag{66}$$

$$I(n) = \frac{2^{-n}}{2\,n-1} F\left(n,\, 1\,;\ n + \frac{1}{2}\,;\, \frac{1}{2}\right) \tag{67}$$

$$I(n) = \frac{2^{-n+1}}{2\,n-1} F\left(1,\, \frac{1}{2}\,;\ n + \frac{1}{2}\,;\, -1\right) \tag{68}$$

$$I(1) = F\left(\frac{1}{2},\, 1;\, \frac{3}{2};\, -1\right) = \sum_{k=0}^{\infty} \frac{(-1)^k\, (1/2)_k}{(3/2)_k} = \frac{\pi}{4} \tag{69}$$

$$I(n+1) = \frac{2\,n-1}{2\,n} I(n) - \frac{2^{-n}}{2\,n} \tag{70}$$

$$I(n) = \frac{2^{-n}}{2\,n-1} + \frac{2\,n}{2\,n-1} I(n+1) \tag{71}$$

$$\lim_{n\to\infty} I(n) = 0 \tag{72}$$

$$I(n) = \int_0^{\infty} \frac{e^{-(2\,n-1)\,x}}{(1+e^{-2\,x})^n}\, dx \tag{73}$$

$$I(n) = \int_0^{1} \frac{x^{2\,n-2}}{(1+x^2)^n}\, dx \tag{74}$$

$$I(n) = \frac{1}{2}\int_0^{1} \frac{x^{n-3/2}}{(1+x)^n}\, dx \tag{75}$$

$$I(2) = \frac{\pi}{8} - \frac{1}{4} = \frac{1}{3} \sum_{k=0}^{\infty} \frac{(-1)^k\, (3/2)_k\, (2)_k}{k!\, (5/2)_k} \tag{76}$$

$$I(3) = \frac{3\,\pi}{32} - \frac{1}{4} = \frac{1}{5} \sum_{k=0}^{\infty} \frac{(-1)^k\, (5/2)_k\, (3)_k}{k!\, (7/2)_k} \tag{77}$$

$$I(4) = \frac{5\,\pi}{64} - \frac{11}{48} = \frac{1}{7} \sum_{k=0}^{\infty} \frac{(-1)^k\, (7/2)_k\, (4)_k}{k!\, (9/2)_k} \tag{78}$$

$$I(5) = \frac{35\,\pi}{512} - \frac{5}{24} = \frac{1}{9} \sum_{k=0}^{\infty} \frac{(-1)^k\, (9/2)_k\, (5)_k}{k!\, (11/2)_k} \tag{79}$$

$$\frac{\pi}{4} = \frac{1}{2} + 2\, I(2) \tag{80}$$

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{2^3}{3} I(3) \tag{81}$$

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 5} + \frac{2^4}{5} I(4) \tag{82}$$

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \frac{2^7}{7} I(5) \tag{83}$$

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \frac{2^2}{5\cdot 7\cdot 9} + \frac{2^2}{7\cdot 9\cdot 11} + \frac{2^3}{3\cdot 7\cdot 11\cdot 13} + \ldots \tag{84}$$

■ Question 11.

Notación : Para $x,\ y \in \mathbb{R}$, $i = \sqrt{-1}$, $y = \mathrm{Im}(x + i\,y)$,

$$\pi = 4 \sum_{n=1}^{\infty} \frac{2^{-n}}{n} \mathrm{Im}((1+i)^n) \tag{85}$$

$$\pi = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \mathrm{Im}((a + i(1+a))^n) \ , \ -1 < a \le 0 \tag{86}$$

$$\pi = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left[\mathrm{Im}\left(\left(a + i\left(\frac{1+a}{2}\right)\right)^n\right) + \mathrm{Im}\left(\left(b + i\left(\frac{1+b}{3}\right)\right)^n\right)\right] \tag{87}$$

$$-1 < a < 3/5 \ , \ -1 < b < 4/5$$

$$\pi = 4 \sum_{n=0}^{\infty} \frac{\mathrm{Im}\left((1 - 2\,x - x^2 + i(2\,x(1-x)))^{2n+1}\right)}{(2\,n+1)\,(1+x^2)^{2n+1}} \ , \ \sqrt{2} - 1 < x < 1 \tag{88}$$

$$\pi = 4 \sum_{n=0}^{\infty} \frac{\mathrm{Im}\left((q^2 - 2\,p\,q - p^2 + i(2\,p\,(q-p)))^{2n+1}\right)}{(2\,n+1)\,(p^2+q^2)^{2n+1}} \tag{89}$$

$$p,\ q \in \mathbb{N} \ , \ \sqrt{2} - 1 < p/q < 1$$

$$\pi = 8 \sum_{n=0}^{\infty} \frac{1}{2\,n+1} \mathrm{Im}\left(\left(\frac{a - 1 + i\,a}{a + 1 + i\,a}\right)^{2n+1}\right) \ , \ a > 0 \tag{90}$$

$$\pi = 8 \sum_{n=0}^{\infty} \frac{\mathrm{Im}\left((2\,a^2 - 1 + 2\,a\,i)^{2n+1}\right)}{(2\,n+1)\,(2\,a^2 + 2\,a + 1)^{2n+1}} \ , \ a > 0 \tag{91}$$

$$\pi = 4 \sum_{n=1}^{\infty} \frac{(2\,a)^{-n}}{n} \mathrm{Im}((2\,a - 1 + i)^n) \ , \ a > 1/2 \tag{92}$$

$$\pi = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\,2^n} \mathrm{Im}((-1+i)^n) \tag{93}$$

$$\pi = 4 \sum_{n=1}^{\infty} \frac{1}{n\,2^{2n}} \mathrm{Im}((3+i)^n) \tag{94}$$

$$\pi = 8 \sum_{n=0}^{\infty} \frac{1}{(2\,n+1)\,5^{2n+1}} \mathrm{Im}\left((1+2\,i)^{2n+1}\right) \tag{95}$$

$$\pi = 4 \sum_{n=1}^{\infty} \frac{1}{n\,10^n} (\mathrm{Im}((6+2\,i)^n) + \mathrm{Im}((7+i)^n)) \tag{96}$$

$$\pi = 8 \sum_{n=0}^{\infty} \frac{1}{2\,n+1} \left[\mathrm{Im}\left(\left(\frac{1+i}{3+i}\right)^{2n+1}\right) + \mathrm{Im}\left(\left(\frac{2+i}{4+i}\right)^{2n+1}\right)\right] \tag{97}$$

■ Question 12.

Para $a > 0$, $0 < b < c$, se tiene :

$$\frac{\cosh\left(\frac{a\pi}{2c}\right)+\operatorname{sen}\left(\frac{b\pi}{2c}\right)}{\cosh\left(\frac{a\pi}{2c}\right)-\operatorname{sen}\left(\frac{b\pi}{2c}\right)}=\prod_{n=0}^{\infty}\left(\frac{a^2+(c+b+2cn)^2}{a^2+(c-b+2cn)^2}\right)^{(-1)^n} \tag{98}$$

■ Question 13.

Para $a>0,\ b>0$, se tiene :

$$\tan^{-1}\left(\frac{e^{a\pi}-e^{b\pi}}{1+e^{(a+b)\pi}}\right)=\sum_{n=0}^{\infty}(-1)^n\left(\tan^{-1}\left(\frac{2a}{2n+1}\right)-\tan^{-1}\left(\frac{2b}{2n+1}\right)\right) \tag{99}$$

■ Question 14.

Para $s>0$, $0\le x\le 1$, se tiene :

$$\tan^{-1}x+\sum_{n=2}^{\infty}\frac{1}{n^{s+1}}\tan^{-1}\left(\frac{x}{n^s}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}\zeta(2sn+2s+1)}{2n+1} \tag{100}$$

$$\frac{\pi}{4}+\sum_{n=2}^{\infty}\frac{1}{n^{s+1}}\tan^{-1}\left(\frac{1}{n^s}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n\zeta(2sn+2s+1)}{2n+1} \tag{101}$$

$$\frac{\pi}{6}+\sum_{n=2}^{\infty}\frac{1}{n^{s+1}}\tan^{-1}\left(\frac{1}{n^s\sqrt{3}}\right)=\frac{1}{\sqrt{3}}\sum_{n=0}^{\infty}\frac{(-1)^n 3^{-n}\zeta(2sn+2s+1)}{2n+1} \tag{102}$$

$$\frac{\pi}{4}+\sum_{n=2}^{\infty}\frac{1}{n^{s+1}}\left(\tan^{-1}\left(\frac{1}{2n^s}\right)+\tan^{-1}\left(\frac{1}{3n^s}\right)\right)=\sum_{n=0}^{\infty}\frac{(-1)^n(2^{-2n-1}+3^{-2n-1})\zeta(2sn+2s+1)}{2n+1} \tag{103}$$

■ Question 15.

Para $a,\ b,\ c\in(0,\ 1/2)$, $a+b+c=1$, se tiene :

$$\ln(\tan(a\pi)+\tan(b\pi)+\tan(c\pi))=3\ln\pi+\ln(abc)+2\sum_{n=1}^{\infty}\frac{(2^{2n-1}-1)(a^{2n}+b^{2n}+c^{2n})\zeta(2n)}{n} \tag{104}$$

■ Question 16.

Para $a,\ b,\ c\in(0,\ 1)$, $a+b+c=1$, se tiene :

$$2\ln 2+3\ln\pi+\ln(abc)=\ln|\operatorname{sen}(2a\pi)+\operatorname{sen}(2b\pi)+\operatorname{sen}(2c\pi)|+\sum_{n=1}^{\infty}\frac{(a^{2n}+b^{2n}+c^{2n})\zeta(2n)}{n} \tag{105}$$

■ Question 17.

Para $-1<a<1$, se tiene :

$$a\pi\left(\frac{a\pi+\operatorname{sen}(a\pi)}{1+\cos(a\pi)}\right)=8\sum_{n=1}^{\infty}\left(1-2^{-2n}\right)n\,a^{2n}\zeta(2n) \tag{106}$$

$$\frac{\pi}{2}\left(\frac{\pi}{2}+1\right)=8\sum_{n=1}^{\infty}\left(1-2^{-2n}\right)n\,2^{-2n}\zeta(2n) \tag{107}$$

$$\pi\left(\pi\sqrt{2}+4\right)\left(\sqrt{2}-1\right)=32\sum_{n=1}^{\infty}\left(1-2^{-2n}\right)n\,2^{-4n}\zeta(2n) \tag{108}$$

$$\pi\left(2\pi+3\sqrt{3}\right)=216\sum_{n=1}^{\infty}\left(1-2^{-2n}\right)n\,3^{-2n}\zeta(2n) \tag{109}$$

$$\pi(\pi+3)\left(2-\sqrt{3}\right)=144\sum_{n=1}^{\infty}\left(1-2^{-2n}\right)n\,6^{-2n}\zeta(2n) \tag{110}$$

■ Question 18.

Sea $k \in \mathbb{N}$, se tiene :

$$\pi\left(\frac{\pi + 2^k a_k}{2 + b_k}\right) = 2^{2k+4} \sum_{n=1}^{\infty} \left(1 - 2^{-2n}\right) n \, 2^{-2(k+1)n} \zeta(2n) \qquad (111)$$

donde

$$a_k = \underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{\longleftarrow \longleftarrow \longleftarrow \; k\text{-radicales} \; \longrightarrow \longrightarrow \longrightarrow} \qquad (112)$$

$$b_k = \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{\longleftarrow \longleftarrow \longleftarrow \; k\text{-radicales} \; \longrightarrow \longrightarrow \longrightarrow} \qquad (113)$$

■ Question 19.

Para $-1 < a < 1$, se tiene :

$$\frac{\pi^2 a^2}{1 + \cos(a\pi)} = 4 \sum_{n=1}^{\infty} \left(1 - 2^{-2n}\right) (2n - 1) \, a^{2n} \zeta(2n) \qquad (114)$$

$$\frac{\pi^2}{1 + \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{\longleftarrow \longleftarrow \longleftarrow \; k\text{-radicales} \; \longrightarrow \longrightarrow \longrightarrow}} = 2^{2k+3} \sum_{n=1}^{\infty} \frac{(1 - 2^{-2n})(2n - 1)\zeta(2n)}{2^{(2k+2)n}}, \quad k \in \mathbb{N} \qquad (115)$$

■ Question 20.

Para $\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n\right) = 0.577215 \ldots$, se tiene :

$$\gamma = 1 - \ln 2 + \sum_{n=2}^{\infty} (\zeta(n) - \zeta(n+1)) \sum_{k=2}^{n} \frac{(-1)^k}{k} \qquad (116)$$

$$\gamma = \ln 2 - \frac{3}{4} \sum_{n=1}^{\infty} 2^{-2n} \sum_{k=1}^{n} \frac{\zeta(2k+1)}{2k+1} \qquad (117)$$

$$\gamma = \ln\left(\frac{4}{\pi}\right) + \frac{1}{3} \sum_{n=2}^{\infty} \left(1 + (-1)^n 2^{-n+1}\right) \left(\frac{\zeta(n)}{n} - \frac{\zeta(n+1)}{n+1}\right) \qquad (118)$$

$$\gamma = \ln\left(\frac{4}{\pi}\right) + 2 \sum_{n=2}^{\infty} 2^{-n} \left(\zeta(n) - \frac{\zeta(n+1)}{2}\right) \sum_{k=2}^{n} \frac{(-1)^k}{k} \qquad (119)$$

$$\gamma = \ln\left(\frac{4}{\pi}\right) + 2 \sum_{n=2}^{\infty} (-2)^{-n} \left(\zeta(n) + \frac{\zeta(n+1)}{2}\right) \sum_{k=2}^{n} \frac{1}{k} \qquad (120)$$

$$\gamma = \ln\left(\frac{4}{\pi}\right) + \sum_{n=2}^{\infty} 2^{-n} \sum_{k=2}^{n} \frac{(-1)^k \zeta(k)}{k} \qquad (121)$$

$$\gamma = \prod_{n=1}^{\infty} \left(1 - \frac{n! \left(\ln\left(\frac{n+1}{n}\right) - \frac{1}{n+1}\right)}{a_n - n! \ln n}\right) \qquad (122)$$

donde

$$a_n = n!\, H_n \ , \ \ H_n = \sum_{k=1}^{n} \frac{1}{k} \tag{123}$$

$$a_{n+1} = (n+1)\, a_n + n! \ , \ \ a_1 = 1 \tag{124}$$

$$\gamma = 3 - \Gamma\left(\frac{1}{3}\right) + \sum_{n=3}^{\infty}\left(1 - \Gamma\left(1 + \frac{1}{n+1}\right) + n\left(\Gamma\left(1 + \frac{1}{n}\right) - \Gamma\left(1 + \frac{1}{n+1}\right)\right)\right) \tag{125}$$

donde $\Gamma(x)$ es la función gamma usual.

$$\gamma = \frac{3}{2} - \ln 2 - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, n\, (\zeta(n+1) - 1)}{n+1} \tag{126}$$

$$\gamma = \frac{3}{2} - \ln 2 - \sum_{n=2}^{\infty}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\, k}{(k+1)\, n^{k+1}} \tag{127}$$

$$\gamma = \frac{3}{2} - \ln 2 - \sum_{k=1}^{\infty}\left(\ln 2 - \sum_{n=1}^{2^k} \frac{1}{2^k + n}\right) \tag{128}$$

$$e^{\gamma} = \prod_{n=1}^{\infty} \frac{n\, e^{1/n}}{n+1} \tag{129}$$

$$e^{\gamma} = e^2 \left(\prod_{n=0}^{\infty} \frac{2^n}{2^n + 1}\right)\left(\prod_{n=1}^{\infty}\prod_{k=2^n+1}^{2^{n+1}-1} \frac{k\, e^{1/k}}{k+1}\right) \tag{130}$$

$$e^{\gamma} = e^{1/(m-1)} \left(\prod_{n=1}^{m-1} \frac{n\, e^{1/n}}{n+1}\right)\left(\prod_{n=1}^{\infty} \frac{m^n}{m^n + 1}\right)\left(\prod_{n=1}^{\infty}\prod_{k=m^n+1}^{m^{n+1}-1} \frac{k\, e^{1/k}}{k+1}\right), \ \ m \in \mathbb{N} - \{1\} \tag{131}$$

■ Question 21.

Para $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = 2.71828\ldots$, se tiene :

$$e^e = e^2 \sqrt{e \sqrt[3]{e \sqrt[4]{e \sqrt[5]{e \ldots}}}} \tag{132}$$

$$e^e = 1 + \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{m^{n-m}}{m!\,(n-m)!} = 1 + \sum_{n=1}^{\infty}\sum_{m=1}^{n} \binom{n}{m} \frac{m^{n-m}}{n!} \tag{133}$$

$$e^e = e\left(\prod_{n=1}^{\infty} \cosh\left(\frac{1}{n!}\right)\right)\left(\prod_{n=1}^{\infty}\left(1 + \tanh\left(\frac{1}{n!}\right)\right)\right) \tag{134}$$

$$e^e = \frac{1 + \tanh 1}{1 - \tanh 1} \prod_{n=2}^{\infty} \frac{1 + \tanh(1/(2\,n!))}{1 - \tanh(1/(2\,n!))} \tag{135}$$

$$e^e = \frac{2\,\mathrm{senh}(e\,\pi)}{\pi}\left(\frac{1}{2\,e} + \sum_{n=1}^{\infty} \frac{(-1)^n\,(e \cos n - n \,\mathrm{sen}\, n)}{e^2 + n^2}\right) \tag{136}$$

$$e^e = \frac{2\,\mathrm{senh}(\pi)}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n\,(\cos(e\,n) - n \,\mathrm{sen}\,(e\,n))}{1 + n^2}\right) \tag{137}$$

$$e^e = e \sum_{n=0}^{\infty} \frac{a_n}{n!} \tag{138}$$

donde

$$a_0 = 1\ ,\ \ a_{n+1} = \sum_{k=0}^{n} \binom{n}{k} a_k\ ,\ \ n \in \mathbb{N} \cup \{0\} \tag{139}$$

$$e^e = \left(1 + \frac{1}{a}\left(1 + \frac{1}{a}(1 + \ldots)^{3/4}\right)^{3/4}\right)^{1/4} \tag{140}$$

donde

$$a = \frac{e^e}{e^{2e} - e^{-2e}} = \frac{1}{4e} + \frac{\pi}{\pi^2 + 4e^2} - \frac{2e}{(2\pi)^2 + 4e^2} - \frac{3\pi}{(3\pi)^2 + 4e^2} + \ldots \tag{141}$$

■ Question 22.

Para $a = \sqrt{\sqrt{5} - 1}$, $b = \sqrt{\sqrt{5} + 1}$, se tiene :

$$\frac{\pi}{8} a + \frac{a}{2}\left(\tan^{-1}\left(\frac{b}{5+a}\right) - \tan^{-1}\left(\frac{5-(b+a)}{5+(b-a)}\right)\right) + \frac{b}{4}\ln\left(\frac{2 + 2a\sqrt{5} + 5\sqrt{5}}{2 - 2a\sqrt{5} + 5\sqrt{5}}\right) = \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^{2n} \binom{2n}{k} \frac{2^{-k}\, 5^{-2n+k}}{4n - 2k + 1} \tag{142}$$

■ Question 23.

$$\pi = \int_0^{\pi} \ln\left(3 + 2\sqrt{3e - e^2}\,\cos x\right) dx \tag{143}$$

$$\pi = \int_0^{\pi} \ln\left(2\sqrt{2} + 2\sqrt{2\sqrt{2}\,e - e^2}\,\cos x\right) dx \tag{144}$$

$$\pi = \int_0^{\pi} \ln\left(\sqrt{14} - 1 + 2\sqrt{\left(\sqrt{14} - 1\right)e - e^2}\,\cos x\right) dx \tag{145}$$

■ Question 24.

$$\pi = \int_0^{\pi/2} \ln\left(e^2 \tan x\right) dx \tag{146}$$

$$\pi = \int_0^{\infty} \left(\frac{\pi}{2} - \tan^{-1}\left(e^{x-2}\right)\right) dx - \int_0^{\infty} \tan^{-1}\left(e^{-2-x}\right) dx \tag{147}$$

$$\pi = \int_0^{\infty} \tan^{-1}\left(e^{2-x}\right) dx - \int_0^{\infty} \tan^{-1}\left(e^{-2-x}\right) dx \tag{148}$$

$$\pi = \int_0^{\infty} \tan^{-1}\left(\frac{\operatorname{senh} 2}{\cosh x}\right) dx \tag{149}$$

■ Question 25.

Para $\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n\right) = 0.577215\ \ldots$, se tiene :

$$\int_0^{\infty} e^{-x\, e^{-\gamma}} \ln x\, dx = 0 \tag{150}$$

$$\alpha = \int_1^{\infty} e^{-x\, e^{-\gamma}} \ln x\, dx \tag{151}$$

$$\alpha = -\int_0^{1} e^{-x\, e^{-\gamma}} \ln x\, dx \tag{152}$$

$$\alpha = \int_1^{\infty} \frac{\ln x}{x^2}\, e^{-e^{-\gamma}/x}\, dx \tag{153}$$

$$\alpha = \sum_{n=0}^{\infty} \frac{(-1)^n e^{-n\gamma}}{n!\,(n+1)^2} \tag{154}$$

■ Question 26.

$$\pi = 1 + 8\sqrt{2}\int_u^1 \sqrt{\sqrt{192x^4 - 24x^2 + 1} - 14x^2 + 1}\, dx\ ,\ u = \frac{1}{2\sqrt{2}} \tag{155}$$

$$\pi = 6\sqrt{6}\int_0^1 \sqrt{\sqrt{192x^4 + 32x^2 + 1} - 14x^2 - 1}\, dx \tag{156}$$

$$\pi = \frac{3\left(9 + 4\sqrt{3}\right)}{11} - \frac{48\sqrt{2}\left(9 + 4\sqrt{3}\right)}{11}\int_0^{1/4} \sqrt{1 - 14x^2 - \sqrt{192x^4 - 28x^2 + 1}}\, dx \tag{157}$$

$$\pi = \frac{3\left(9 - 4\sqrt{3}\right)}{11} + \frac{96\left(9 - 4\sqrt{3}\right)}{11}\int_{1/4}^1 \sqrt{x\sqrt{48x^2 + 1} - 7x^2}\, dx \tag{158}$$

■ Question 27.

$$\pi G = \sum_{n=0}^{\infty} 2^{-n} \sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{s=0}^{n-k} (-1)^{m+s}\binom{k}{m}\binom{n-k}{s}(2m+1)^{-2}(2s+1)^{-1} \tag{159}$$

■ Question 28.

$$G = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{(2k-1)^2} + (-1)^n \int_0^1\int_0^1 \frac{(xy)^{2n}}{1 + x^2y^2}\, dx\, dy\ ,\ n \in \mathbb{N} \cup \{0\} \tag{160}$$

■ Question 29.

$$\pi = -\frac{2(1 + \ln 6)}{3} + \frac{2}{3}\int_{1/2}^3 \left(\sqrt[3]{f(x)} + \sqrt[3]{g(x)}\right)\frac{1}{x}\, dx \tag{161}$$

donde

$$f(x) = -1 - 3x + 42x^2 - 10x^3 + 3x\sqrt{3\left(4x^4 - 32x^3 + 68x^2 - 8x - 3\right)} \tag{162}$$

$$g(x) = -1 - 3x + 42x^2 - 10x^3 - 3x\sqrt{3\left(4x^4 - 32x^3 + 68x^2 - 8x - 3\right)} \tag{163}$$

■ Question 30.

Sean $c(n)$, $n \in \mathbb{N} \cup \{0\}$, $a \in \mathbb{R}$ definidos como sigue :

$$c(n) = \frac{(-1)^{[n/2]}}{2[n/2] + 1} - \frac{(-1)^n}{2n + 1} - \frac{(-1)^n}{2n + 3} \tag{164}$$

$$a = -\frac{1}{3} + \frac{1}{3}\sqrt[3]{26 + 6\sqrt{33}} + \frac{1}{3}\sqrt[3]{26 - 6\sqrt{33}} = 0.2955\ ... \tag{165}$$

se tiene :

$$\pi = \frac{4}{1 + a}\left(1 + \sum_{n=0}^{\infty} c(n)\, a^{n+1}\right) \tag{166}$$

$$a = -\frac{1}{3} + \frac{1}{3}\left\{\frac{13}{6} - \frac{1}{24}\left(\frac{13}{6} - \frac{1}{24}\left(\frac{13}{6} - ...\right)^3\right)^3\right\} \tag{167}$$

$$\frac{4}{1 + a} = \frac{2}{3}\left(2 - \sqrt[3]{3\sqrt{33} - 17} + \sqrt[3]{3\sqrt{33} + 17}\right) \tag{168}$$

$$\left(\frac{4}{1 + a}\right)^{-1} = \frac{1}{6} + \left(\frac{13}{72} - 6\left(\frac{13}{72} - 6\left(\frac{13}{72} - ...\right)^3\right)^3\right) \tag{169}$$

$$\sum_{n=0}^{\infty} c(n) = \frac{\pi - 2}{2} \tag{170}$$

$$c(n) = \left\{-\frac{1}{3},\ \frac{23}{15},\ -\frac{71}{105},\ -\frac{5}{63},\ -\frac{1}{495},\ \frac{263}{715},\ -\frac{391}{1365},\ \ldots\right\} \tag{171}$$

$$\pi = 4 \sum_{n=0}^{\infty} (-a)^n\, b(n) \tag{172}$$

donde

$$b(n) = \sum_{k=0}^{n} (-1)^k\, c(k-1)\ ,\ \ n \in \mathbb{N} \bigcup \{0\}\,,\ \ c(-1) = 1 \tag{173}$$

Observación. $[x]$ es la función parte entera de x.

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