Poly-complex Clifford Algebra and grand unification

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Abstract. An algebra for unit multivector components for a manifold of five poly-complex dimensions is presented. The algebra has many properties that suggest it may provide a basis for a grand unification theory.

1. Introduction

Different types of imaginary numbers in physics can be used in physics. The Pauli wave equation uses two complex functions, as does the Higgs mechanism. The standard unit imaginary is used in the Dirac equation describing the electron using spinors with four complex valued components, but is not used in the geometric algebra approach proposed by David Hestenes[1]. Quaternions can be used in a form of Maxwell's equations. Octonionic unit imaginary numbers have been used in several proposed models of reality.

The range of types of unit imaginary that can be used suggests that the dimensionality of the universe is subject to complexification using more than one unit imaginary, as generated by the higher dimensional Cayley-Dickson algebras, such as the quaternions, octonions, sedenions or trigintaduonions. In this paper these types of complexification are referred to as poly-complex, and a poly-complex algebra including all of these elements as subalgebras is used to represent unit elements for a multivector for a manifold with five poly-complex unit vectors.

2. Construction of an algebra of unit elements for a poly-complex multivector

Matrices are useful for representing elements of associative algebras such as Clifford algebras and quaternions. For octonionic and higher Cayley-Dickson algebras which are non-associative, their use is problematic. A form of a construction used to construct Moufang loops[2] can be used to construct the octonion algebra which is non-associative but alternative. This form of construction can be extended to allow construction of higher Cayley-Dickson algebras which are not only non-associative, but also non-alternative. This construction can then be merged with algebras of matrices representing unit multivector elements of a Clifford algebra to assemble algebras, referred to as Kaotic algebras[3], that can be used represent to unit multivector elements of poly-complex Clifford algebras.

2.1. Notation

The multivector for a manifold of 5+5 poly-complex dimensions has $2^{10} = 1024$ unit elements. The multiplication table for these unit elements generates more than a million products, making investigation of their properties using a table with elements labelled e_0 to e_{1023} impractical.

A more useful approach is to assign labels to unit multivector components which identify components as a combination of sub-components having smaller product tables.

2.2. Real, imaginary, poly-real and poly-imaginary components

For the poly-complex Clifford algebra, poly-real components are multivector components of a Clifford algebra. These include the pseudoscalar which, for many purposes, is equivalent to the standard unit imaginary. However, it commutes with quaternions, so it differs from the unit imaginaries in the octonions, sedenions and trigintaduonions. In this paper it is represented by ι and may be present in poly-real components.

2.3. Labels for unit components

This paper uses a scheme whereby unit kaotic algebra components are designated using combinations of letters such as $\nu j \iota X$ which denote a dis-association operator, a unit quaternion and a 4×4 unit matrix which can have real or imaginary entries.

A greek first letter (ν) designates a dis-association operator which dictates multiplication procedures applicable for the component. The next letter (j) identifies a unit quaternion, to be in the form of a real 4×4 matrix with entries that are 0, or ± 1. Together with the disassociation operator it represents the poly-imaginary status of the component.

The next letters (ιX) denote a poly-real matrix which is a real or imaginary matrix from table 1.

It and the unit quaternion are expanded into a 16×16 matrix by inserting 0, or ± 1 times the 4×4 unit matrix specified by the next letters (ιX) into each entry of the quaternion matrix as appropriate.

2.4. Poly-real components

Poly-real components are represented by matrices which may be real or standard imaginary. The real matrices are shown in table 1.

Components whose labels include ι are assigned the corresponding imaginary matrix. Note that the positive and negative forms of matrices R, Q, L, X, Y, Z are opposite to those used in a previous paper, The Pattern of Reality[4], so that matrices with a positive entry in the first row are the positive form.

2.5. Poly-imaginary components

For each poly-real component there are 31 poly-imaginary components obtained by combining them with the 31 unit trigintaduonions that square to -1. These components are obtained using three applications of the Cayley-Dickson construction to the unit quaternions. As documented in a previous paper by this author[3], three applications of the Cayley-Dickson construction can be condensed into a modified Moufang loop construction.

Table 1. Notation used to label real 4×4 unit matrices

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad M = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad X = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad V = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

2.6. Moufang Loop construction for octonions

For the Moufang loop construction of octonions, based on quaternion pairs, a dis-association operator, μ, is assigned to the second pair, and the product rule is, for $p = (p, \mu p)$ and $q = (q, \mu q)$:
$p.q = (pq)$
$p.\mu q = \mu(p^{-1}q)$
$\mu p.q = \mu(qp)$
$\mu p.\mu q = -(qp^{-1})$

2.7. Modified Moufang Loop construction for higher Cayley-Dickson algebras

The Moufang loop construction for octonions uses one dis-association operator. Further dis-association operators can be used to define sedenions and trigintaduonions, where products are constructed in accordance with table 1.

This table allows codification of three application of the Cayley Dickson construction to the quaternions, which already comprise two applications of the Cayley Dickson construction to the reals, so can be used to generate the result of five applications of the Cayley Dickson construction, the trigintaduonions.

TABLE 2. Multiplication procedures for non-associative components

	q	μq	νq	λq	αq	βq	γq	δq
p	$+pq$	$+\mu qp$	$+\nu qp$	$+\lambda qp^{-1}$	$+\alpha qp$	$+\beta qp^{-1}$	$+\gamma qp^{-1}$	$+\delta qp$
μp	$+\mu pq^{-1}$	$-q^{-1}p$	$+\lambda pq$	$-\nu p^{-1}q$	$+\beta pq$	$-\alpha p^{-1}q$	$-\delta pq$	$+\gamma p^{-1}q$
νp	$+\nu pq^{-1}$	$-\lambda qp$	$-q^{-1}p$	$+\mu qp^{-1}$	$+\gamma pq$	$+\delta qp$	$-\alpha p^{-1}q$	$-\beta qp^{-1}$
λp	$+\lambda pq$	$+\nu q^{-1}p$	$-\mu pq^{-1}$	$-p^{-1}q$	$+\delta pq^{-1}$	$-\gamma q^{-1}p$	$+\beta pq^{-1}$	$-\alpha q^{-1}p$
αp	$+\alpha pq^{-1}$	$-\beta qp$	$-\gamma qp$	$-\delta qp^{-1}$	$-q^{-1}p$	$+\mu qp^{-1}$	$+\nu qp^{-1}$	$+\lambda qp$
βp	$+\beta pq$	$+\alpha q^{-1}p$	$-\delta pq$	$+\gamma p^{-1}q$	$-\mu pq^{-1}$	$-p^{-1}q$	$-\lambda pq$	$+\nu p^{-1}q$
γp	$+\gamma pq$	$+\delta qp$	$+\alpha q^{-1}p$	$-\beta qp^{-1}$	$-\nu pq^{-1}$	$+\lambda qp$	$-p^{-1}q$	$-\mu qp^{-1}$
δp	$+\delta pq^{-1}$	$-\gamma q^{-1}p$	$+\beta pq^{-1}$	$+\alpha p^{-1}q$	$-\lambda pq$	$-\nu q^{-1}p$	$+\mu pq^{-1}$	$-q^{-1}p$

3. Assembling a poly-complex Clifford algebra multivector

There is a wide range of possible ways to combine poly-real and poly-imaginary components of the Kaotic algebra[3] $Ka^5Cl(0,5)$. To assemble a poly-complex Clifford algebra multivector, the latin square for the poly-real multivector elements has to to correspond to the latin square for the poly-imaginary elements, so that the product of two complex elements generates a third complex element, and elements of the same grade should have consistent properties.

Consider allocation of poly-real and poly-imaginary unit components of $Ka^5Cl(0,5)$ to poly-complex vector components as follows:
$V+\mu$
$T+\nu$
$\iota X+\beta i$
$\iota Y+\gamma j$
$\iota Z+\delta k$
The poly-real unit vector elements generate the Clifford algebra $Cl(0,5)$
Scalar: S
Vector: $V, T, \iota X, \iota Y, \iota Z$
Bi-vector: $U, \iota P, \iota Q, \iota R, \iota D, \iota E, \iota F, N, M, L$
Tri-vector:$\iota U, P, Q, R, D, E, F, \iota N, \iota M, \iota L$
Quadri-vector: $\iota V, \iota T, X, Y, Z$
Pseudoscalar: ι

The poly-imaginary vector elements generate the trigintaduonion algebra. If organised by association with the corresponding poly-real component, components of this algebra can be assigned to multivector components as follows:
Vector: $\mu, \nu, \beta i, \gamma j, \delta k$
Bi-vector: $\lambda, \alpha i, \delta j, \gamma k, \delta i, \alpha j, \beta k, \lambda k, \nu j, \mu i$
Tri-vector: $\alpha, \lambda i, j, \mu k, i, \lambda j, \nu k, \alpha k, \beta j, \gamma i$
Quadri-vector: $\gamma, \beta, \nu i, \mu j, k$
Pseudoscalar: δ

All of these poly-imaginary components anti-commute with each other.
The poly-imaginary and poly-real scalar components are the same and commute.
The poly-imaginary and poly-real vector components all anti-commute.
The poly-imaginary and poly-real bi-vector components all anti-commute.
The poly-imaginary and poly-real tri-vector components all commute.
The poly-imaginary and poly-real quadri-vector components all commute.
The poly-imaginary and poly-real pseudoscalar components anti-commute.

3.1. Signature

The poly-real vector elements for the poly-complex components chosen in the preceding section have (- - - - -) signature. The quadrivector components could have been chosen to be the vector components which would then have (+++++) signature.

Different signatures can be obtained by selecting a mixture of vector and quadrivector components as vector elements, but some signatures do not generate a multivector for a five dimensional poly-complex space, such as:
$V + \mu$
$\iota T + \beta$
$\iota X + \beta i$
$\iota Y + \gamma j$
$\iota Z + \delta k$
For these complex components one of the components can be obtained as the product of the other four, making it the pseudoscalar for a multivector for their 4+4 dimensional poly-complex manifold.

Consider the multivector for the 4+4 dimensional poly-complex manifold for which the unit vectors are:
$\iota T + \beta S$
$\iota X + \beta i$
$\iota Y + \gamma j$
$\iota Z + \delta k$

The poly-real unit vector elements generate the Clifford algebra $Cl(1, 3)$
Scalar: S
Vector: $\iota T, \iota X, \iota Y, \iota Z$
Bi-vector: D, E, F, N, M, L
Tri-vector:$\iota U, \iota P, \iota Q, \iota R$
Pseudoscalar: V

The poly-imaginary vector elements generate a sedenion algebra. If organised by association with the corresponding poly-real component, they can be assigned to multivector components as follows:

Scalar: S
Vector: $\beta, \beta i, \gamma j, \delta k$
Bi-vector: $i, \lambda j, \nu k, \lambda k, \nu j, \mu i$
Tri-vector: $\alpha, \alpha i, \delta j, \gamma k$
Pseudoscalar: μ
From these components the components of the 5+5 dimensional poly-complex multivector can be generated by multiplication by: $\iota + \delta$
$S + \iota + \delta + \delta\iota$ generate a group isomorphic to the quaternions, which can be used to represent a complex doublet.

4. Discussion

The 5+5 dimensional poly-complex Clifford algebra has many features suggesting that it could be useful in model building for grand unification theories.
Space-time can be associated with:
$\iota T + \beta$
$\iota X + \beta i$
$\iota Y + \gamma j$
$\iota Z + \delta k$

The Higgs mechanism[5] features a complex doublet, which can be associated with:
$S + \iota + \delta + \delta\iota$

The Georgi-Glashow model[6] is based on $SU(5)$, which is related to five standard complex dimensions. It may be able to be modified for five poly-complex dimensions in a way which changes its problematic prediction of proton decay.

Any two imaginary products of the trigintaduonion algebra can be used to generate an anti-commuting quaternionic algebra. The corresponding real components generate a commuting algebra isomorphic to that of 4×4 real diagonal matrices. This suggests a form of supersymmetry.

The five poly-real vector components and the five poly-imaginary vector components all anti-commute, so can be regarded as unit vectors for a ten dimensional space and used to assemble a ten-dimensional multivector. If sign is ignored, the latin square for the products of these multivector components is the same as for the multivector components of $Cl(0, 10)$, which relates to grand unification theories based on $SO(10)$, such as M-theory[7].

It incorporates many octonion sub-algebras, which relates to grand unification theories based on the octonions such as that proposed by G. Dixon[8].

The phenomenology of the fermions can be generated using dimensional permutations. David Hestenes' model of the electron[1] uses even components of the spacetime multivector, which has unit components:
Scalar: S
Bivector: L, M, N, D, E, F
Pseudoscalar: V
These components are products of:
$(S + L + M + N) \times (V)$
Or, in terms of vector components:
$(S + XY + XZ + YZ) \times (\iota X.\iota Y.\iota Z.\iota T)$

For a manifold with three poly-complex spatial dimensions:
$\iota X + \beta i$
$\iota Y + \gamma j$
$\iota Z + \delta k$
There are four ways which three dimensions can be chosen:
1. All poly-real - identify with the electron family
2. All poly-imaginary - identify with the neutrino family
3. One poly-real + Two poly imaginary - identify with one quark family
4. Two poly-real + One poly imaginary - identify with the other quark family
Then identify the weak force with an operation that takes all poly-reals into the associated poly-imaginaries and vice-versa, and identify the strong force with an operation: $(\iota X + \beta i) \to (\iota Y + \gamma j) \to (\iota Z + \delta k)$

Then alternative pseudoscalars can be found:
$(\iota X.\iota Y.\iota Z.\iota T) = V$
$(\iota X.\iota Y.\iota Z.\beta) = \beta\iota U$
$(\iota X.\iota Y.\iota Z.\beta i) = \beta i\iota U$
$(\iota X.\iota Y.\iota Z.\gamma j) = \gamma j\iota U$
$(\iota X.\iota Y.\iota Z.\delta i) = \delta k\iota U$
If the symmetry between spatial dimensions results in there being no distinction between $\beta i\iota U$, $\gamma j\iota U$ and $\delta k\iota U$, then the number of different types of pseudoscalar reduces to three, accounting for three generations for each family of fermion.

References

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