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Pre-Boolean algebra, ordered DSmT and DSm continuous models

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Abstract: *When implementing the DSmT, a difficulty may arise from the possible huge dimension of hyper-power sets, which are indeed free structures. However, it is possible to reduce the dimension of these structures by involving logical constraints. In this chapter, the logical constraints will be related to a predefined order over the logical propositions. The use of such orders and their resulting logical constraints will ensure a great reduction of the model complexity. Such results will be applied to the definition of continuous DSm models. In particular, a simplified description of the continuous impreciseness is considered, based on impreciseness intervals of the sensors. From this viewpoint, it is possible to manage the contradictions between continuous sensors in a DSmT manner, while the complexity of the model stays handleable.*

5.1 Introduction

Recent advances [6] in the Dezert Smarandache Theory have shown that this theory was able to handle the contradiction between propositions in a quite flexible way. This new theory has been already applied in different domains; *e.g.*:

- Data association in target tracking [9] ,
- Environmental prediction [2] .

Although free DSm models are defined over hyper-power sets, which sizes evolve exponentially with the number of *atomic* propositions, it appears that the manipulation of the fusion rule is still manageable for practical problems reasonably well shaped. Moreover, the hybrid DSm models are of lesser complexity.

If DSmT works well for discrete spaces, the manipulation of continuous DSm models is still an unknown. Nevertheless, the management of continuous data is an issue of main interest. It is necessary for implementing a true fusion engine for localization informations; and associated with a principle of conditioning, it will be a main ingredient for implementing filters for the localization. But a question first arises: *what could be an hyper-power set for a continuous DSm model?* Such first issue does not arises so dramatically in Dempster Shafer Theory or for Transfer Belief Models [7]. In DST, a continuous proposition could just be a measurable subset. On the other hand, a free DSm model, defined over an hyper-power set, will imply that any pair of propositions will have a non empty intersection. This is disappointing, since the notion of *point* (a minimal non empty proposition) does not exist anymore in an hyper-power set.

But even if it is possible to define a continuous propositional model in DST/TBM, the manipulation of continuous basic belief assignment is still an issue [4, 8]. In [4] , Ristic and Smets proposed a restriction of the bba to intervals of $\mathbb{R}$. It was then possible to derive a mathematical relation between a continuous bba density and its Bel function.

In this chapter, the construction of continuous DSm models is proposed. This construction is based on a constrained model, where the logical constraints are implied by the definition of an order relation over the propositions.

A one-dimension DSm model will be implemented, where the definition of the basic belief assignment relies on a *generalized notion of intervals*. Although this construction has been fulfilled on a different ground, it shares some surprising similarities with Ristic and Smets viewpoint. As in [4], the bba will be seen as density defined over a 2-dimension measurable space. We will be able to derive the Belief function from the basic belief assignment, by applying an integral computation. At last, the conjunctive fusion operator, $\oplus$, is derived by a rather simple integral computation.

Section 5.2 makes a quick introduction of the Dezert Smarandache Theory. Section 5.3 is about ordered DSm models. In section 5.4, a continuous DSm model is defined. This method is restricted to only one dimension. The related computation methods are detailed. In section 5.5, our algorithmic implementation is described and an example of computation is given. The paper is then concluded.

5.2 A short introduction to the DSmT

The theory and its meaning are widely explained in [6]. However, we will particularly focus on the notion of hyper-power sets, since this notion is fundamental subsequently.

The *Dezert Smarandache Theory* belongs to the family of *Evidence Theories*. As the *Dempster Shafer Theory* [3] [5] or the *Transferable Belief Models* [7], the DSmT is a framework for fusing belief informations, originating from independent sensors. However, free DSm models are defined over Hyper-power sets, which are *fully open-world extensions* of sets. It is possible to restrict this full open-world hypothesis by adding propositional constraints, resulting in the definition of an *hybrid Dezert Smarandache model*.

The notion of hyper-power set is thus a fundamental ingredient of the DSmT. Hyper-power sets could be considered as a free pre-Boolean algebra. As these structures will be of main importance subsequently, the next sections are devoted to introduce them in details. As a prerequisite, the notion of Boolean algebra is quickly introduced now.

5.2.1 Boolean algebra

Definition. A Boolean algebra is a sextuple $(\Phi, \wedge, \vee, \neg, \bot, \top)$ such that:

- Φ is a set, called set of propositions,
- $\bot, \top$ are specific propositions of Φ, respectively called *false* and *true*,
- $\neg : \Phi \to \Phi$ is a unary operator,
- $\wedge : \Phi \times \Phi \to \Phi$ and $\vee : \Phi \times \Phi \to \Phi$ are binary operators,

and verifying the following properties:

A1. $\wedge$ and $\vee$ are commutative:
$$\forall \phi, \psi \in \Phi,\ \phi \wedge \psi = \psi \wedge \phi \text{ and } \phi \vee \psi = \psi \vee \phi\ ,$$

A2. $\wedge$ and $\vee$ are associative:
$$\forall \phi, \psi, \eta \in \Phi,\ (\phi \wedge \psi) \wedge \eta = \phi \wedge (\psi \wedge \eta) \text{ and } (\phi \vee \psi) \vee \eta = \phi \vee (\psi \vee \eta)\ ,$$

A3. $\top$ is neutral for $\wedge$ and $\bot$ is neutral for $\vee$:
$$\forall \phi \in \Phi,\ \phi \wedge \top = \phi \text{ and } \phi \vee \bot = \phi\ ,$$

A4. $\wedge$ and $\vee$ are distributive for each other:
$$\forall \phi, \psi, \eta \in \Phi,\ \phi \wedge (\psi \vee \eta) = (\phi \wedge \psi) \vee (\phi \wedge \eta) \text{ and } \phi \vee (\psi \wedge \eta) = (\phi \vee \psi) \wedge (\phi \vee \eta)\ ,$$

A5. $\neg$ defines the complement of any proposition:
$$\forall \phi \in \Phi,\ \phi \wedge \neg\phi = \bot \text{ and } \phi \vee \neg\phi = \top\ .$$

The Boolean algebra $(\Phi, \wedge, \vee, \neg, \bot, \top)$ will be also referred to as the *Boolean algebra* Φ, the structure being thus implied. An order relation $\subset$ is defined over Φ by:
$$\forall \phi, \psi \in \Phi\,, \quad \phi \subset \psi \overset{\Delta}{\iff} \phi \wedge \psi = \phi\ .$$

Fundamental examples. The following examples are two main conceptions of Boolean algebra.

***Example* 1.** *Let Ω be a set and $\mathcal{P}(\Omega)$ be the set of its subsets. For any $A \subset \Omega$, denote $\sim A = \Omega \setminus A$ its complement. Then $\big(\mathcal{P}(\Omega), \cap, \cup, \sim, \emptyset, \Omega\big)$ is a Boolean algebra.*

The proof is immediate by verifying the properties A1 to A5.

***Example* 2.** *For any $i \in \{1, \ldots, n\}$, let $\theta_i = \{0,1\}^{i-1} \times \{0\} \times \{0,1\}^{n-i}$. Let $\Theta = \{\theta_1, \ldots, \theta_n\}$ and denote $\bot = \emptyset$, $\top = \{0,1\}^n$ and $\mathcal{B}(\Theta) = \mathcal{P}\big(\{0,1\}^n\big)$. Define the operators $\wedge$, $\vee$ and $\neg$ by $\phi \wedge \psi = \phi \cap \psi$, $\phi \vee \psi = \phi \cup \psi$ and $\neg\phi = \top \setminus \phi$ for any $\phi, \psi \in \mathcal{B}(\Theta)$. Then $\big(\mathcal{B}(\Theta), \wedge, \vee, \neg, \bot, \top\big)$ is a Boolean algebra.*

The second example seems just like a rewriting of the first one, but it is of the most importance. It is called the *free Boolean algebra generated by the set of atomic propositions* Θ. Figure 5.1 shows the structure of such algebra, when $n = 2$. The free Boolean algebra $\mathcal{B}(\Theta)$ is deeply related to the classical propositional logic: it gives the (logical) equivalence classes of the propositions generated from the atomic propositions of Θ. Although we give here an explicit definition of $\mathcal{B}(\Theta)$ by means of its binary coding $\mathcal{P}\big(\{0,1\}^n\big)$, the truly rigorous definition of $\mathcal{B}(\Theta)$ is made by means of the logical equivalence (which is out of the scope of this presentation). Thus, the binary coding of the atomic propositions $\theta_i \in \Theta$ is only implied.

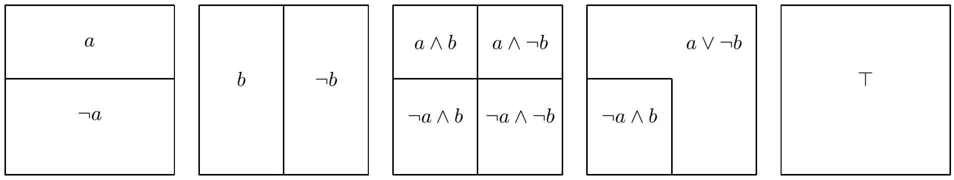

Figure 5.1: Boolean algebra $\mathcal{B}(\{a, b\})$; (partial)

Fundamental proposition.

Proposition 3. *Any Boolean algebra is isomorph to a Boolean algebra derived from a set,* i.e. $(\mathcal{P}(\Omega), \cap, \cup, \sim, \emptyset, \Omega)$.

Proofs should be found in any good reference; see also [1].

5.2.2 Hyper-power sets

Definition of hyper-power set. Let's consider a finite set Θ of atomic propositions, and denote $(\mathcal{B}(\Theta), \wedge, \vee, \neg, \bot, \top)$ the free Boolean algebra generated by Θ. For any $\Sigma \subset \mathcal{P}(\Theta)$, define $\varphi(\Sigma)$, element of $\mathcal{B}(\Theta)$, by[1] $\varphi(\Sigma) = \bigvee_{\sigma\in\Sigma} \bigwedge_{\theta\in\sigma} \theta$. The set $< \Theta >= \{\varphi(\Sigma) \;/\; \Sigma \subset \mathcal{P}(\Theta)\}$ is called hyper-power set generated by Θ.

It is noticed that both $\bot = \varphi(\emptyset)$ and $\top = \varphi(\mathcal{P}(\Theta))$ are elements of $< \Theta >$. Figure 5.2 shows the structure of the hyper-power set, when $n = 2$. Typically, it appears that the elements of the hyper-power set are built only from $\neg$-free components.

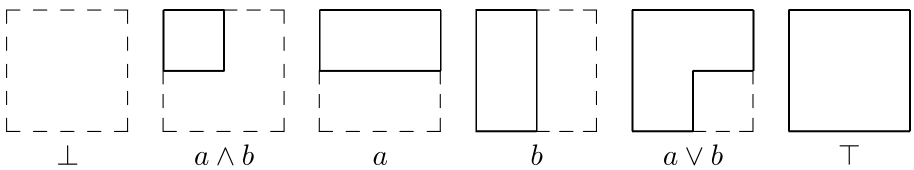

Figure 5.2: Hyper-power set $< a, b >= \{\bot, a \wedge b, a, b, a \vee b, \top\}$

***Example* 3.** *Hyper-power set generated by* $\Theta = \{a, b, c\}$.

$$
\begin{aligned}
< a, b, c >= \{&\bot, a, b, c, a \wedge b \wedge c, a \wedge b, b \wedge c, c \wedge a, a \vee b \vee c, \top \\
&a \vee b, b \vee c, c \vee a, (a \wedge b) \vee c, (b \wedge c) \vee a, (c \wedge a) \vee b, \\
&(a \vee b) \wedge c, (b \vee c) \wedge a, (c \vee a) \wedge b, (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)\}
\end{aligned}
$$

The following table associates some $\Sigma \subset \mathcal{P}(\Theta)$ *to their related hyper-power element* $\varphi(\Sigma)$.

[1] It is assumed $\bigvee_{\phi\in\emptyset} = \bot$ and $\bigwedge_{\phi\in\emptyset} = \top$.

This table is partial; there is indeed 256 *possible choices for* Σ. *It appears that* φ *is not one-to-one:*

Σ	$\varphi(\Sigma)$	*reduced form in* $<\Theta>$
$\emptyset$	$\bot$	$\bot$
$\{\emptyset\}$	$\top$	$\top$
$\big\{\{a\};\{b\};\{c\}\big\}$	$a\vee b\vee c$	$a\vee b\vee c$
$\big\{\{a,b\};\{b,c\};\{c,a\}\big\}$	$(a\wedge b)\vee(b\wedge c)\vee(c\wedge a)$	$(a\wedge b)\vee(b\wedge c)\vee(c\wedge a)$
$\big\{\{a,c\};\{b,c\};\{a,b,c\}\big\}$	$(a\wedge c)\vee(b\wedge c)\vee(a\wedge b\wedge c)$	$(a\vee b)\wedge c$
$\big\{\{a,c\};\{b,c\}\big\}$	$(a\wedge c)\vee(b\wedge c)$	$(a\vee b)\wedge c$

Remark. In the DSmT book 1 [6], the hyper-power sets have been defined by means of the Smarandache encoding. Our definition is quite related to this encoding. In fact this encoding is just implied in the definition of φ.

Hyper-power set as a free pre-Boolean algebra. It is easy to verify on example 3 that $<\Theta>$ is left unchanged by any application of the operators $\wedge$ and $\vee$. For example:

$$(a\wedge b)\wedge\big((b\wedge c)\vee a\big)=(a\wedge b\wedge b\wedge c)\vee(a\wedge b\wedge a)=a\wedge b\,.$$

This result is formalized by the following proposition.

Proposition 4. *Let* $\phi,\psi\in<\Theta>$. *Then* $\phi\wedge\psi\in<\Theta>$ *and* $\phi\vee\psi\in<\Theta>$.

Proof. Let $\phi,\psi\in<\Theta>$.

There are $\Sigma\subset\mathcal{P}(\Theta)$ and $\Gamma\subset\mathcal{P}(\Theta)$ such that $\phi=\varphi(\Sigma)$ and $\psi=\varphi(\Gamma)$.

By applying the definition of φ, it comes immediately:

$$\varphi(\Sigma)\vee\varphi(\Gamma)=\bigvee_{\sigma\in\Sigma\cup\Gamma}\bigwedge_{\theta\in\sigma}\theta\,.$$

It is also deduced:

$$\varphi(\Sigma)\wedge\varphi(\Gamma)=\left(\bigvee_{\sigma\in\Sigma}\bigwedge_{\theta\in\sigma}\theta\right)\wedge\left(\bigvee_{\gamma\in\Gamma}\bigwedge_{\theta\in\gamma}\theta\right).$$

By applying the distributivity, it comes:

$$\varphi(\Sigma)\wedge\varphi(\Gamma)=\bigvee_{\sigma\in\Sigma}\bigvee_{\gamma\in\Gamma}\left(\left(\bigwedge_{\theta\in\sigma}\theta\right)\wedge\left(\bigwedge_{\theta\in\gamma}\theta\right)\right)=\bigvee_{(\sigma,\gamma)\in\Sigma\times\Gamma}\bigwedge_{\theta\in\sigma\cup\gamma}\theta\,.$$

Then $\varphi(\Sigma)\wedge\varphi(\Gamma)=\varphi(\Lambda)$, with $\Lambda=\big\{\sigma\cup\gamma\;/\;(\sigma,\gamma)\in\Sigma\times\Gamma\big\}$.

□□□

Corollary and definition. Proposition 4 implies that $\wedge$ and $\vee$ infer inner operations within $<\Theta>$. As a consequence, $\big(<\Theta>,\wedge,\vee,\bot,\top\big)$ is an algebraic structure by itself. Since it does not contains the negation $\neg$, this structure is called the *free pre-Boolean algebra generated by* Θ.

5.2.3 Pre-Boolean algebra

Generality. Typically, a free algebra is an algebra where the only constraints are the intrinsic constraints which characterize its fundamental structures. For example in a free Boolean algebra, the only constraints are A1 to A5, and there are no other constraints put on the propositions. But conversely, it is indeed possible to derive any algebra by constraining its free counterpart. This will be our approach for defining pre-Boolean algebra in general: a pre-Boolean algebra will be a *constrained* free pre-Boolean algebra. Constraining a structure is a quite intuitive notion. However, a precise mathematical definition needs the abstract notion of equivalence relations and classes. Let us start with the intuition by introducing an example.

***Example* 4.** *Pre-Boolean algebra generated by* $\Theta = \{a, b, c\}$ *and constrained by* $a \wedge b = a \wedge c$ *and* $a \wedge c = b \wedge c$*.*
For coherence with forthcoming notations, these constraints will be designated by using the set of propositional pairs $\Gamma = \big\{(a \wedge b, a \wedge c), (a \wedge c, b \wedge c)\big\}$.
The idea is to start from the free pre-Boolean algebra $< a, b, c >$*, propagate the constraints, and then reduce the propositions identified by the constraints.*

It is first deduced $a \wedge b = a \wedge c = b \wedge c = a \wedge b \wedge c$*.*
It follows $(a \wedge b) \vee c = c$*,* $(b \wedge c) \vee a = a$ *and* $(c \wedge a) \vee b = b$*.*
Also holds $(a \vee b) \wedge c = (b \vee c) \wedge a = (c \vee a) \wedge b = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = a \wedge b \wedge c$ *.*
By discarding these cases from the free structure $< a, b, c >$*, it comes the following constrained pre-Boolean algebra:*

$$< a, b, c >_\Gamma = \big\{\bot, a \wedge b \wedge c, a, b, c, a \vee b, b \vee c, c \vee a, a \vee b \vee c, \top\big\}$$

Of course, it is necessary to show that there is actually no further reduction in $< a, b, c >_\Gamma$*. This is done by explicating a model; for example the structure of figure 5.3.*

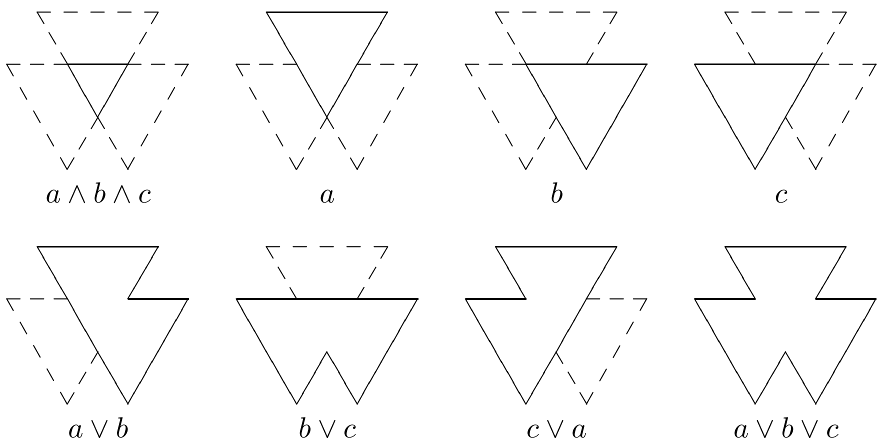

Figure 5.3: Pre-Boolean algebra $< a, b, c >_\Gamma$; ($\bot$ and $\top$ are omitted)

For the reader not familiar with the notion of equivalence classes, the following construction is just a mathematical formalization of the constraint propagation which has been described in example 4. Now, it is first introduced the notion of morphism between structures.

Magma. A $(\wedge,\vee,\bot,\top)$-*magma*, also called *magma* for short, is a quintuple $(\Phi,\wedge,\vee,\bot,\top)$ where Φ is a set of propositions, $\wedge$ and $\vee$ are binary operators on Φ, and $\bot$ and $\top$ are two elements of Φ.

The magma $(\Phi,\wedge,\vee,\bot,\top)$ may also be referred to as the magma Φ, the structure being thus implied. Notice that an hyper-power set is a magma.

Morphism. Let $(\Phi,\wedge,\vee,\bot,\top)$ and $(\Psi,\wedge,\vee,\bot,\top)$ be two magma. A morphism μ from the magma Φ to the magma Ψ is a mapping from Φ to Ψ such that:

- $\mu(\phi\wedge\psi)=\mu(\phi)\wedge\mu(\psi)$ and $\mu(\phi\vee\psi)=\mu(\phi)\vee\mu(\psi)$,
- $\mu(\bot)=\bot$ and $\mu(\top)=\top$.

A morphism is an isomorphism if it is a bijective mapping. In such case, the magma Φ and the magma Ψ are said to be isomorph, which means that they share the same structure.

The notions of $(\wedge,\vee)$*-magma and of* $(\wedge,\vee)$*-morphism are defined similarly by discarding* $\bot$ *and* $\top$.

Propagation relation. Let $<\Theta>$ be a free pre-Boolean algebra. Let $\Gamma\subset<\Theta>\times<\Theta>$ be a set of propositional pairs; for any pair $(\phi,\psi)\in\Gamma$ is defined the constraint $\phi=\psi$. The propagation relation associated to the constraints, and also denoted Γ, is defined recursively by:

- $\phi\Gamma\phi$, for any $\phi\in<\Theta>$,
- If $(\phi,\psi)\in\Gamma$, then $\phi\Gamma\psi$ and $\psi\Gamma\phi$,
- If $\phi\Gamma\psi$ and $\psi\Gamma\eta$, then $\phi\Gamma\eta$,
- If $\phi\Gamma\eta$ and $\psi\Gamma\zeta$, then $(\phi\wedge\psi)\Gamma(\eta\wedge\zeta)$ and $(\phi\vee\psi)\Gamma(\eta\vee\zeta)$

The relation Γ is thus obtained by propagating the constraint over $<\Theta>$. It is obviously reflexive, symmetric and transitive; it is an equivalence relation. An equivalence class for Γ contains propositions which are identical in regards to the constraints.

It is now time to define the pre-Boolean algebra.

Pre-Boolean algebra.

Proposition 5. *Let be given a free pre-Boolean algebra* $<\Theta>$ *and a set of propositional pairs* $\Gamma\subset<\Theta>\times<\Theta>$ *. Then, there is a magma* $<\Theta>_\Gamma$ *and a morphism* $\mu:<\Theta>\rightarrow<\Theta>_\Gamma$ *such that:*

$$\begin{cases}\mu\big(<\Theta>\big)=<\Theta>_\Gamma\ ,\\ \forall\phi,\psi\in<\Theta>,\ \mu(\phi)=\mu(\psi)\iff\phi\Gamma\psi\ .\end{cases}$$

The magma $<\Theta>_\Gamma$ *is called the pre-Boolean algebra generated by* Θ *and constrained by the constraints* $\phi=\psi$ *where* $(\phi,\psi)\in\Gamma$ *.*

Proof. For any $\phi\in<\Theta>$, define $\phi_\Gamma=\big\{\psi\in<\Theta>\ /\ \psi\Gamma\phi\big\}$; this set is called the class of ϕ for Γ .

It is a well known fact, and the proof is immediate, that $\phi_\Gamma=\psi_\Gamma$ or $\phi_\Gamma\cap\psi_\Gamma=\emptyset$ for any

$\phi, \psi \in <\Theta>$; in particular, $\phi_\Gamma = \psi_\Gamma \iff \phi\Gamma\psi$.
Now, assume $\eta_\Gamma = \phi_\Gamma$ and $\zeta_\Gamma = \psi_\Gamma$, that is $\eta\Gamma\phi$ and $\zeta\Gamma\psi$.
It comes $(\eta \wedge \zeta)\Gamma(\phi \wedge \psi)$ and $(\eta \vee \zeta)\Gamma(\phi \vee \psi)$.
As a consequence, $(\eta \wedge \zeta)_\Gamma = (\phi \wedge \psi)_\Gamma$ and $(\eta \vee \zeta)_\Gamma = (\phi \vee \psi)_\Gamma$.
At last:

$$\Big(\eta_\Gamma = \phi_\Gamma \text{ and } \zeta_\Gamma = \psi_\Gamma\Big) \Rightarrow \Big((\eta \wedge \zeta)_\Gamma = (\phi \wedge \psi)_\Gamma \text{ and } (\eta \vee \zeta)_\Gamma = (\phi \vee \psi)_\Gamma\Big)$$

The proof is then concluded easily, by setting:

$$\begin{cases} <\Theta>_\Gamma = \big\{\phi_\Gamma \;/\; \phi \in <\Theta>\big\}\ , \\ \forall \phi, \psi \in <\Theta>,\ \phi_\Gamma \wedge \psi_\Gamma = (\phi \wedge \psi)_\Gamma \text{ and } \phi_\Gamma \vee \psi_\Gamma = (\phi \vee \psi)_\Gamma\ , \\ \forall \phi \in <\Theta>,\ \mu(\phi) = \phi_\Gamma\ . \end{cases}$$

□□□

From now on, the element $\mu(\phi)$, where $\phi \in <\Theta>$, will be denoted ϕ as if ϕ were an element of $<\Theta>_\Gamma$. In particular, $\mu(\phi) = \mu(\psi)$ will imply $\phi = \psi$ in $<\Theta>_\Gamma$ (but not in $<\Theta>$).

Proposition 6. *Let be given a free pre-Boolean algebra $<\Theta>$ and a set of propositional pairs $\Gamma \subset <\Theta> \times <\Theta>$. Let $<\Theta>_\Gamma$ and $<\Theta>'_\Gamma$ be pre-Boolean algebras generated by Θ and constrained by the family Γ . Then $<\Theta>_\Gamma$ and $<\Theta>'_\Gamma$ are isomorph.*

Proof. Let $\mu :<\Theta>\to<\Theta>_\Gamma$ and $\mu' :<\Theta>\to<\Theta>'_\Gamma$ be as defined in proposition 5.
For any $\phi \in <\Theta>$, define $\nu\big(\mu(\phi)\big) = \mu'(\phi)$.
Then, $\nu\big(\mu(\phi)\big) = \nu\big(\mu(\psi)\big)$ implies $\mu'(\phi) = \mu'(\psi)$.
By definition of μ', it is derived $\phi\Gamma\psi$ and then $\mu(\phi) = \mu(\psi)$.
Thus, ν is one-to-one.
By definition, it is also implied that ν is onto.

□□□

This property thus says that there is a structural uniqueness of $<\Theta>_\Gamma$.

Example 5. *Let us consider again the pre-Boolean algebra generated by $\Theta = \{a, b, c\}$ and constrained by $a \wedge b = a \wedge c$ and $a \wedge c = b \wedge c$. In this case, the mapping $\mu :<\Theta>\to<\Theta>_\Gamma$ is defined by:*

- $\mu\big(\{\bot\}\big) = \{\bot\}$, $\mu\Big(\big\{a, (b \wedge c) \vee a\big\}\Big) = \{a\}$, $\mu\Big(\big\{b, (c \wedge a) \vee b\big\}\Big) = \{b\}$,
  $\mu\Big(\big\{c, (a \wedge b) \vee c\big\}\Big) = \{c\}$, $\mu\big(\{a \vee b \vee c\}\big) = \{a \vee b \vee c\}$, $\mu\big(\{\top\}\big) = \{\top\}$,
- $\mu\big(\{a \vee b\}\big) = \{a \vee b\}$, $\mu\big(\{b \vee c\}\big) = \{b \vee c\}$, $\mu\big(\{c \vee a\}\big) = \{c \vee a\}$,
- $\mu\Big(\big\{a \wedge b \wedge c, a \wedge b, b \wedge c, c \wedge a, (a \vee b) \wedge c, (b \vee c) \wedge a, (c \vee a) \wedge b,$
  $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a)\big\}\Big) = \{a \wedge b \wedge c\}$.

Between sets and hyper-power sets.

Proposition 7. *The Boolean algebra* $\big(\mathcal{P}(\Theta),\cap,\cup,\sim,\emptyset,\Theta\big)$*, considered as a* $(\wedge,\vee,\bot,\top)$*-magma, is isomorph to the pre-Boolean algebra* $<\Theta>_\Gamma$*, where* Γ *is defined by:*

$$\Gamma=\big\{(\theta\wedge\vartheta,\bot)\;/\;\theta,\vartheta\in\Theta\ \text{and}\ \theta\neq\vartheta\big\}\cup\left\{\left(\bigvee_{\theta\in\Theta}\theta,\top\right)\right\}.$$

Proof. Recall the notation $\varphi(\Sigma)=\bigvee_{\sigma\in\Sigma}\bigwedge_{\theta\in\sigma}\theta$ for any $\Sigma\subset\mathcal{P}(\Theta)$.

Define $\mu:<\Theta>\to\mathcal{P}(\Theta)$ by setting[2] $\mu\big(\varphi(\Sigma)\big)=\bigcup_{\sigma\in\Sigma}\bigcap_{\theta\in\sigma}\{\theta\}$ for any $\Sigma\subset\mathcal{P}(\Theta)$.

It is immediate that μ is a morphism.

Now, by definition of Γ, $\mu\big(\varphi(\Sigma)\big)=\mu\big(\varphi(\Lambda)\big)$ is equivalent to $\varphi(\Sigma)\Gamma\varphi(\Lambda)$.

The proof is then concluded by proposition 6.

□□□

Thus, sets, considered as Boolean algebra, and hyper-power sets are both extremal cases of the notion of pre-Boolean algebra. But while hyper-power sets extend the structure of sets, hyper-power sets are more complex in structure and size than sets. A practical use of hyper-power sets becomes quickly impossible. Pre-Boolean algebra however allows intermediate structures between sets and hyper-power sets.

A specific kind of pre-Boolean algebra will be particularly interesting when defining the DSmT. Such pre-Boolean algebra will forbid any interaction between the trivial propositions $\bot,\top$ and the other propositions. These algebra, called insulated pre-Boolean algebra, are characterized now.

Insulated pre-Boolean algebra. A pre-Boolean algebra $<\Theta>_\Gamma$ verifies the *insulation* property if $\Gamma\subset\big(<\Theta>\setminus\{\bot,\top\}\big)\times\big(<\Theta>\setminus\{\bot,\top\}\big)$.

Proposition 8. *Let* $<\Theta>_\Gamma$ *a pre-Boolean algebra verifying the insulation property. Then holds for any* $\phi,\psi\in<\Theta>_\Gamma$ *:*

$$\begin{cases}\phi\wedge\psi=\bot\Rightarrow(\phi=\bot\ \text{or}\ \psi=\bot)\,,\\ \phi\vee\psi=\top\Rightarrow(\phi=\top\ \text{or}\ \psi=\top)\,.\end{cases}$$

In other words, all propositions are independent *with each other in a pre-Boolean algebra with insulation property.*

The proof is immediate, since it is impossible to obtain $\phi\wedge\psi\Gamma\bot$ or $\phi\vee\psi\Gamma\top$ without involving $\bot$ or $\top$ in the constraints of Γ. Examples 3 and example 4 verify the insulation property. On the contrary, a non empty set does not.

Corollary and definition. Let $<\Theta>_\Gamma$ be a pre-Boolean algebra, verifying the insulation property. Define $\ll\Theta\gg_\Gamma=<\Theta>_\Gamma\setminus\{\bot,\top\}$. The operators $\wedge$ and $\vee$ restrict to $\ll\Theta\gg_\Gamma$, and $\big(\ll\Theta\gg_\Gamma,\wedge,\vee\big)$ is an algebraic structure by itself, called *insulated* pre-Boolean algebra. This structure is also referred to as the insulated pre-Boolean algebra $\ll\Theta\gg_\Gamma$.

[2]It is defined $\bigcap_{\theta\in\emptyset}\theta=\Theta$.

Proposition 9. *Let* $< \Theta >_\Gamma$ *and* $< \Theta >'_\Gamma$ *be pre-Boolean algebras with insulation properties. Assume that the insulated pre-Boolean algebra* $\ll \Theta \gg_\Gamma$ *and* $\ll \Theta \gg'_\Gamma$ *are* $(\wedge, \vee)$*-isomorph. Then* $< \Theta >_\Gamma$ *and* $< \Theta >'_\Gamma$ *are isomorph.*

Deduced from the insulation property.

All ingredients are now gathered for the definition of Dezert Smarandache models.

5.2.4 The *free* Dezert Smarandache Theory

Dezert Smarandache Model. Assume that Θ is a finite set. A Dezert Smarandache model (DSmm) is a pair (Θ, m), where Θ is a set of propositions and the *basic belief assignment* m is a non negatively valued function defined over $< \Theta >$ such that:

$$\sum_{\phi \in <\Theta>} m(\phi) = \sum_{\phi \in \ll\Theta\gg} m(\phi) = 1 \ .$$

The property $\sum_{\phi \in \ll\Theta\gg} m(\phi) = 1$ implies that the propositions of Θ are exhaustive.

Belief Function. Assume that Θ is a finite set. The belief function Bel related to a bba m is defined by:

$$\forall \phi \in < \Theta >, \mathrm{Bel}(\phi) = \sum_{\psi \in <\Theta> : \psi \subset \phi} m(\psi) \ . \tag{5.1}$$

The equation (6.1) is invertible:

$$\forall \phi \in < \Theta >, m(\phi) = \mathrm{Bel}(\phi) - \sum_{\psi \in <\Theta> : \psi \subsetneq \phi} m(\psi) \ .$$

Fusion rule. Assume that Θ is a finite set. For a given universe Θ, and two basic belief assignments m_1 and m_2, associated to independent sensors, the fused basic belief assignment is $m_1 \oplus m_2$, defined by:

$$m_1 \oplus m_2(\phi) = \sum_{\psi_1, \psi_2 \in <\Theta> : \psi_1 \wedge \psi_2 = \phi} m_1(\psi_1) m_2(\psi_2) \ . \tag{5.2}$$

Remarks. It appears obviously that the previous definitions could be equivalently restricted to $\ll \Theta \gg$, owing to the insulation properties.

From the insulation property $(\phi \neq \bot$ and $\psi \neq \bot) \Rightarrow (\phi \wedge \psi) \neq \bot$ and the definition of the fusion rule, it appears also that these definitions could be generalized to any algebra $< \Theta >_\Gamma$ with the insulation property.

5.2.5 Extensions to any insulated pre-Boolean algebra

Let $\ll \Theta \gg_\Gamma$ be an insulated pre-Boolean algebra. The definition of bba m, belief Bel and fusion $\oplus$ is thus kept unchanged.

- A *basic belief assignment* m is a non negatively valued function defined over $\ll \Theta \gg_\Gamma$ such that:

$$\sum_{\phi \in \ll\Theta\gg_\Gamma} m(\phi) = 1 \ .$$

- The belief function Bel related to a bba m is defined by:

$$\forall \phi \in \ll \Theta \gg_\Gamma,\ \mathrm{Bel}(\phi) = \sum_{\psi \in \ll \Theta \gg_\Gamma : \psi \subset \phi} m(\psi)\ .$$

- Being given two basic belief assignments m_1 and m_2, the fused basic belief assignment $m_1 \oplus m_2$ is defined by:

$$m_1 \oplus m_2(\phi) = \sum_{\psi_1, \psi_2 \in \ll \Theta \gg_\Gamma : \psi_1 \wedge \psi_2 = \phi} m_1(\psi_1) m_2(\psi_2)\ .$$

These extended definitions will be applied subsequently.

5.3 Ordered DSm model

From now on, we are working only with insulated pre-Boolean structures.

In order to reduce the complexity of the free DSm model, it is necessary to introduce logical constraints which will lower the size of the pre-Boolean algebra. Such constraints may appear clearly in the hypotheses of the problem. In this case, constraints come naturally and approximations may not be required. However, when the model is too complex and there are no explicit constraints for reducing this complexity, it is necessary to approximate the model by introducing some new constraints. Two rules should be applied then:

- Only weaken informations[3]; do not produce information from nothing,
- minimize the information weakening.

First point guarantees that the approximation does not introduce false information. But some significant informations (*e.g.* contradictions) are possibly missed. This drawback should be avoided by second point.

In order to build a good approximation policy, some external knowledge, like distance or order relation among the propositions could be used. Behind these relations will be assumed some kind of distance between the informations: *more are the informations distant, more are their conjunctive combination valuable.*

5.3.1 Ordered atomic propositions

Let $(\Theta, \leq)$ be an ordered set of atomic propositions. This order relation is assumed to describe the relative distance between the information. For example, the relation $\phi \leq \psi \leq \eta$ implies that ϕ and ψ are closer informations than ϕ and η. Thus, the information contained in $\phi \wedge \eta$ is stronger than the information contained in $\phi \wedge \psi$. Of course, this comparison does not matter when all the information is kept, but when approximations are necessary, it will be useful to be able to choose the best information.

[3]Typically, a constraint like $\phi \wedge \psi \wedge \eta = \phi \wedge \psi$ will weaken the information, by erasing η from $\phi \wedge \psi \wedge \eta$.

Sketchy example. Assume that 3 independent sensors are giving 3 measures about a continuous parameter, that is x, y and z. The parameters x, y, z are assumed to be real values, not of the *set* $\mathbb{R}$ but of its pre-Boolean extension (theoretical issues will be clarified later[4]). The fused information could be formalized by the proposition $x \wedge y \wedge z$ (in a DSmT viewpoint). What happen if we want to reduce the information by removing a proposition. Do we keep $x \wedge y$, $y \wedge z$ or $x \wedge z$? This is of course an information weakening. But it is possible that one information is better than an other. At this stage, the order between the values x, y, z will be involved. Assume for example that $x \leq y < z$. It is clear that the proposition $x \wedge z$ indicates a greater contradiction than $x \wedge y$ or $y \wedge z$. Thus, the proposition $x \wedge z$ is the one which should be kept! The discarding constraint $x \leq y \leq z \Rightarrow x \wedge y \wedge z = x \wedge z$ is implied then.

5.3.2 Associated pre-Boolean algebra and complexity.

In regard to the previous example, the insulated pre-Boolean algebra associated to the ordered propositions $(\Theta, \leq)$ is $\ll \Theta \gg_\Gamma$, where Γ is defined by:

$$\Gamma = \big\{(\phi \wedge \psi \wedge \eta, \phi \wedge \eta) / \phi, \psi, \eta \in \Theta \text{ and } \phi \leq \psi \leq \eta\big\} \ .$$

The following property give an approximative bound of the size of $\ll \Theta \gg_\Gamma$ in the case of a total order.

Proposition 10. *Assume that* $(\Theta, \leq)$ *is totally ordered. Then,* $\ll \Theta \gg_\Gamma$ *is a substructure of the set* Θ^2 .

proof. Since the order is total, first notice that the added constraints are:

$$\forall \phi, \psi, \eta \in \Theta \, , \; \phi \wedge \psi \wedge \eta = \min\{\phi, \psi, \eta\} \wedge \max\{\phi, \psi, \eta\} \ .$$

Now, for any $\phi \in \Theta$, define $\breve{\phi}$ by[5]:

$$\breve{\phi} \triangleq \big\{(\varphi_1, \varphi_2) \in \Theta^2 \,/\, \varphi_1 \leq \phi \leq \varphi_2\big\}$$

It is noteworthy that:

$$\breve{\phi} \cap \breve{\psi} = \big\{(\varphi_1, \varphi_2) \in \Theta^2 \,/\, \varphi_1 \leq \min\{\phi, \psi\} \text{ and } \max\{\phi, \psi\} \leq \varphi_2\big\}$$

and

$$\breve{\phi} \cap \breve{\psi} \cap \breve{\eta} = \big\{(\varphi_1, \varphi_2) \in \Theta^2 \,/\, \varphi_1 \leq \min\{\phi, \psi, \eta\} \text{ and } \max\{\phi, \psi, \eta\} \leq \varphi_2\big\} \ .$$

By defining $m = \min\{\phi, \psi, \eta\}$ and $M = \max\{\phi, \psi, \eta\}$, it is deduced:

$$\breve{\phi} \cap \breve{\psi} \cap \breve{\eta} = \breve{m} \cap \breve{M} \ . \tag{5.3}$$

Figure 5.4 illustrates the construction of $\breve{\phi}$, $\breve{\phi} \cap \breve{\psi}$ and property (5.3).

[4]In particular, as we are working in a pre-Boolean algebra, $x \wedge y$ makes sense and it is possible that $x \wedge y \neq \bot$ even when $x \neq y$.

[5]*wherethesymbol*$\triangleq$ means *equals by definition.*

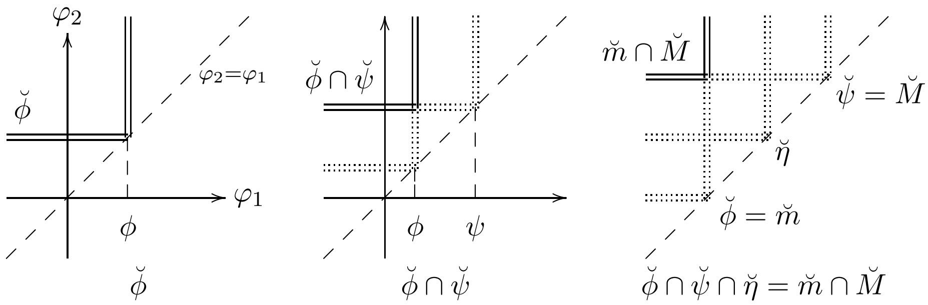

Figure 5.4: Construction of $\breve{\phi}$

Let $\mathcal{A} \subset \mathcal{P}(\Theta^2)$ be generated by $\breve{\phi}|_{\phi\in\Theta}$ with $\cap$ and $\cup$, *ie.*:

$$\mathcal{A} = \bigcup_{n\geq 0} \left\{ \bigcup_{k=1}^{n} (\breve{\phi}_k \cap \breve{\psi}_k) \Big/ \forall k\,,\ \breve{\phi}_k, \breve{\psi}_k \in \Theta \right\} .$$

A consequence of (5.3) is that $\mathcal{A}$ is an insulated pre-Boolean algebra which satisfies the constraints of Γ. Then, the mapping:

$$\smile : \begin{cases} \ll \Theta \gg_\Gamma \longrightarrow \mathcal{A} \\ \bigvee_{k=1}^{n} \bigwedge_{l=1}^{n_k} \phi_{k,l} \longmapsto \bigcup_{k=1}^{n} \bigcap_{l=1}^{n_k} \breve{\phi}_{k,l} \end{cases} , \quad \text{where } \phi_{k,l} \in \Theta$$

is an onto morphism of pre-Boolean algebra.
Now, let us prove that $\smile$ is a one-to-one morphism.

Lemma 11. *Assume:*

$$\bigcup_{k=1}^{n} (\breve{\phi}_k^1 \cap \breve{\phi}_k^2) \subset \bigcup_{l=1}^{m} (\breve{\psi}_l^1 \cap \breve{\psi}_l^2) \quad , \quad \textit{where } \phi_k^j\ ,\ \psi_l^j \in \Theta\ .$$

Then:

$$\forall k\,,\ \exists l\,,\ \min\{\phi_k^1, \phi_k^2\} \leq \min\{\psi_l^1, \psi_l^2\} \textit{ and } \max\{\phi_k^1, \phi_k^2\} \geq \max\{\psi_l^1, \psi_l^2\}$$

and

$$\forall k\,,\ \exists l\,,\ \breve{\phi}_k^1 \cap \breve{\phi}_k^2 \subset \breve{\psi}_l^1 \cap \breve{\psi}_l^2\ .$$

Proof of lemma. Let $k \in [\![1, n]\!]$.
Define $m = \min\{\phi_k^1, \phi_k^2\}$ and $M = \max\{\phi_k^1, \phi_k^2\}$.
Then holds $(m, M) \in \breve{\phi}_k^1 \cap \breve{\phi}_k^2$, implying $(m, M) \in \bigcup_{l=1}^{m} (\breve{\psi}_l^1 \cap \breve{\psi}_l^2)$.
Let l be such that $(m, M) \in \breve{\psi}_l^1 \cap \breve{\psi}_l^2$.
Then $m \leq \min\{\psi_l^1, \psi_l^2\}$ and $M \geq \max\{\psi_l^1, \psi_l^2\}$.
At last, $\breve{\phi}_k^1 \cap \breve{\phi}_k^2 \subset \breve{\psi}_l^1 \cap \breve{\psi}_l^2$.

□□

From inequalities $\min\{\phi_k^1,\phi_k^2\}\le\min\{\psi_l^1,\psi_l^2\}$ and $\max\{\phi_k^1,\phi_k^2\}\ge\max\{\psi_l^1,\psi_l^2\}$ is also deduced $(\phi_k^1\wedge\phi_k^2)\wedge(\psi_l^1\wedge\psi_l^2)=\phi_k^1\wedge\phi_k^2$ (definition of Γ).
This property just means $\phi_k^1\wedge\phi_k^2\subset\psi_l^1\wedge\psi_l^2$. It is lastly deduced:

Lemma 12. *Assume:*

$$\bigcup_{k=1}^{n}(\breve\phi_k^1\cap\breve\phi_k^2)\subset\bigcup_{l=1}^{m}(\breve\psi_l^1\cap\breve\psi_l^2)\quad,\quad\textit{where } \phi_k^j\,,\ \psi_l^j\in\Theta\,.$$

Then:

$$\bigvee_{k=1}^{n}(\phi_k^1\wedge\phi_k^2)\subset\bigvee_{l=1}^{m}(\psi_l^1\wedge\psi_l^2)\,.$$

From this lemma, it is deduced that $\smile$ is one to one.
At last $\smile$ is an isomorphism of pre-Boolean algebra, and $\ll\Theta\gg_\Gamma$ is a substructure of Θ^2.

□□□

5.3.3 General properties of the model

In the next section, the previous construction will be extended to the continuous case, *ie.* $(\mathbb{R},\le)$. However, a strict logical manipulation of the propositions is not sufficient and instead a measurable generalization of the model will be used. It has been seen that a proposition of $\ll\Theta\gg_\Gamma$ could be described as a subset of Θ^2. In this subsection, the proposition model will be characterized precisely. This characterization will be used and extended in the next section to the continuous case.

Proposition 13. *Let* $\phi\in\ll\Theta\gg_\Gamma$.
Then $\smile(\phi)\subset\mathcal{T}$, *where* $\mathcal{T}=\left\{(\phi,\psi)\in\Theta^2/\phi\le\psi\right\}$.

Proof. Obvious, since $\forall\phi\in\Theta\,,\ \breve\phi\subset\mathcal{T}$.

□□□

Definition 14. *A subset* $\theta\subset\Theta^2$ *is* increasing *if and only if:*

$$\forall\,(\phi,\psi)\in\theta\,,\ \forall\eta\le\phi\,,\ \forall\zeta\ge\psi\,,\ (\eta,\zeta)\in\theta\,.$$

Let $\mathcal{U}=\left\{\theta\subset\mathcal{T}/\theta\text{ is increasing and }\theta\ne\emptyset\right\}$ be the set of increasing non-empty subsets of $\mathcal{T}$. Notice that the intersection or the union of increasing non-empty subsets are increasing non-empty subsets, so that $(\mathcal{U},\cap,\cup)$ is an insulated pre-Boolean algebra.

Proposition 15. *For any choice of* Θ, $\left\{\smile(\phi)/\phi\in\ll\Theta\gg_\Gamma\right\}\subset\mathcal{U}$.
When Θ *is finite,* $\mathcal{U}=\left\{\smile(\phi)/\phi\in\ll\Theta\gg_\Gamma\right\}$.

Proof of $\supset$. Obvious, since $\breve\phi$ is increasing for any $\phi\in\Theta$.

Proof of $\subset$. Let $\theta\in\mathcal{U}$ and let $(a,b)\in\theta$.
Since $\breve a\cap\breve b=\left\{(\alpha,\beta)\in\Theta^2/\alpha\le a\text{ and }\beta\ge b\right\}$ and θ is increasing, it follows $\breve a\cap\breve b\subset\theta$.
At last, $\theta=\bigcup_{(a,b)\in\theta}\breve a\cap\breve b=\smile\left(\bigvee_{(a,b)\in\theta}a\wedge b\right)$.
Notice that $\bigvee_{(a,b)\in\theta}a\wedge b$ is actually defined, since θ is finite when Θ is finite.

□□□

Figure 5.5 gives an example of increasing subsets, element of $\mathcal{U}$.

When infinite $\vee$-ing are allowed, notice that $\mathcal{U}$ may be considered as a model for $\ll \Theta \gg_\Gamma$ even

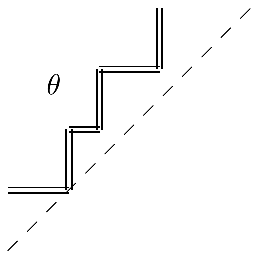

Figure 5.5: Example of increasing subset $\theta \in \mathcal{U}$

if Θ is infinite. In the next section, the *continuous* pre-Boolean algebra related to $(\mathbb{R}, \leq)$ will be modeled by the *measurable increasing subsets* of $\left\{(x,y) \in \mathbb{R}^2 / x \leq y\right\}$.

5.4 Continuous DSm model

In this section, the case $\Theta = \mathbb{R}$ is considered.

Typically, in a continuous model, it will be necessary to manipulate any measurable proposition, and for example intervals. It comes out that most intervals could not be obtained by a finite logical combination of the atomic propositions, but rather by infinite combinations. For example, considering the set formalism, it is obtained $[a,b] = \bigcup_{x\in[a,b]}\{x\}$, which suggests the definition of the infinite disjunction "$\bigvee_{x\in[a,b]} x$". It is known that infinite disjunctions are difficult to handle in a logic. It is better to manipulate the models directly. The pre-Boolean algebra to be constructed should verify the property $x \leq y \leq z \Rightarrow x \wedge y \wedge z = x \wedge z$. As discussed previously and since infinite disjunctions are allowed, a model for such algebra are the measurable increasing subsets.

5.4.1 Measurable increasing subsets

A measurable subset $A \subset \mathbb{R}^2$ is a measurable increasing subset if:

$$\begin{cases} \forall\,(x,y) \in A\,,\ x \leq y\,, \\ \forall\,(x,y) \in A\,,\ \forall a \leq x\,,\ \forall b \geq y\,,\ (a,b) \in A\,. \end{cases}$$

The set of measurable increasing subsets is denoted $\mathcal{U}$.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be a non decreasing measurable mapping such that $f(x) \geq x$ for any $x \in \mathbb{R}$. The set $\left\{(x,y) \in \mathbb{R}^2 / f(x) \leq y\right\}$ is a measurable increasing subset.

"Points". For any $x \in \mathbb{R}$, the measurable increasing subset $\breve{x}$ is defined by:

$$\breve{x} = \left\{(a,b) \in \mathbb{R}^2 \;/\; a \leq x \leq b\right\}.$$

The set $\breve{x}$ is of course a model for the point $x \in \mathbb{R}$ within the pre-Boolean algebra (refer to section 5.3).

Generalized intervals. A particular class of increasing subsets, the generalized intervals, will be useful in the sequel.

For any $x \in \mathbb{R}$, the measurable sets $\grave{x}$ and $\acute{x}$ are defined by:

$$\begin{cases} \grave{x} = \left\{(a,b) \in \mathbb{R}^2 \;/\; a \leq b \text{ and } x \leq b\right\} , \\ \acute{x} = \left\{(a,b) \in \mathbb{R}^2 \;/\; a \leq b \text{ and } a \leq x\right\} . \end{cases}$$

The following properties are derived:

$$\breve{x} = \grave{x} \cap \acute{x} \;,\;\; \grave{x} = \bigcup_{z \in [x,+\infty[} \breve{z} \quad \text{and} \quad \acute{x} = \bigcup_{z \in]-\infty,x]} \breve{z}$$

Moreover, for any x, y such that $x \leq y$, it comes:

$$\grave{x} \cap \acute{y} = \bigcup_{z \in [x,y]} \breve{z} \,.$$

As a conclusion, the set $\grave{x}$, $\acute{x}$ and $\grave{x} \cap \acute{y}$ (with $x \leq y$) are the respective models for the intervals $[x,+\infty[$, $]-\infty,x]$ and $[x,y]$ within the pre-Boolean algebra. Naturally, the quotation marks ` (opening) and ´ (closing) are used respectively for opening and closing the intervals. Figure 5.6 illustrates various cases of interval models.

At last, the set $\grave{x} \cap \acute{y}$, where $x, y \in \mathbb{R}$ are not constrained, constitutes a generalized definition of

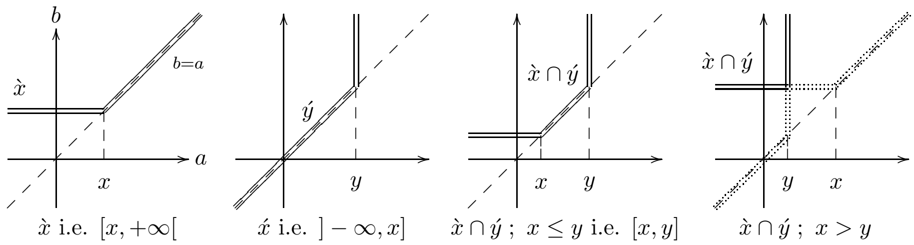

Figure 5.6: Interval models

the notion of interval. In the case $x \leq y$, it works like "classical" interval, but in the case $x > y$, it is obtained a new class of intervals with negative width (last case in figure 5.6). Whatever, $\grave{x} \cap \acute{y}$ comes with a non empty inner, and may have a non zero measure.

The width $\delta = \frac{y-x}{2}$ of the interval $\grave{x} \cap \acute{y}$ could be considered as a measure of contradiction associated with this proposition, while its center $\mu = \frac{x+y}{2}$ should be considered as its median value. The interpretation of the measure of contradiction is left to the human. Typically, a possible interpretation could be:

- $\delta < 0$ means contradictory informations,
- $\delta = 0$ means exact informations,
- $\delta > 0$ means imprecise informations.

It is also noteworthy that the set of generalized intervals

$$\mathcal{I} = \{\grave{x} \cap \acute{y} / x, y \in \mathbb{R}\}$$

is left unchanged by the operator $\cap$, as seen in the following proposition 16:

Proposition 16 (Stability). Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$.
Define $x = \max\{x_1, x_2\}$ and $y = \min\{y_1, y_2\}$.
Then $(\grave{x}_1 \cap \acute{y}_1) \cap (\grave{x}_2 \cap \acute{y}_2) = \grave{x} \cap \acute{y}$.

Proof is obvious.

This last property make possible the definition of basic belief assignment over generalized intervals only. This assumption is clearly necessary in order to reduce the complexity of the evidence modeling. Behind this assumption is the idea that a continuous measure is described by an imprecision/contradiction around the measured value. Such hypothesis has been made by Smets and Ristic [4]. From now on, all the defined bba will be zeroed outside $\mathcal{I}$. Now, since $\mathcal{I}$ is invariant by $\cap$, it is implied that all the bba which will be manipulated, from sensors or after fusion, will be zeroed outside $\mathcal{I}$. This makes the basic belief assignments equivalent to a density over the 2-dimension space $\mathbb{R}^2$.

5.4.2 Definition and manipulation of the belief

The definitions of bba, belief and fusion result directly from section 5.2, but of course the bba becomes density and the summations are replaced by integrations.

Basic Belief Assignment. As discussed previously, it is hypothesized that the measures are characterized by a precision interval around the measured values. In addition, there is an uncertainty about the measure which is translated into a basic belief assignment over the precision intervals.

According to these hypotheses, a bba will be a non negatively valued function m defined over $\mathcal{U}$, zeroed outside $\mathcal{I}$ (set of generalized intervals), and such that:

$$\int_{x,y \in \mathbb{R}} m(\grave{x} \cap \acute{y})\,dxdy = 1 .$$

Belief function. The function of belief, Bel, is defined for any measurable proposition $\phi \in \mathcal{U}$ by:

$$\text{Bel}(\phi) = \int_{\grave{x} \cap \acute{y} \subset \phi} m(\grave{x} \cap \acute{y})\,dxdy .$$

In particular, for a generalized interval $\grave{x} \cap \acute{y}$:

$$\text{Bel}(\grave{x} \cap \acute{y}) = \int_{u=x}^{+\infty} \int_{v=-\infty}^{y} m(\grave{u} \cap \acute{v})\,dudv .$$

Fusion rule. Being given two basic belief assignments m_1 and m_2, the fused basic belief assignment $m_1 \oplus m_2$ is defined by the curvilinear integral:

$$m_1 \oplus m_2\big(\grave{x} \cap \acute{y}\big) = \int_{\mathcal{C}=\{(\phi,\psi)/\phi\cap\psi=\grave{x}\cap\acute{y}\}} m_1(\phi)m_2(\psi)\, d\mathcal{C}\ .$$

Now, from hypothesis it is assumed that m_i is positive only for intervals of the form $\grave{x}_i \cap \acute{y}_i$. Proposition 16 implies:

$$\grave{x}_1 \cap \acute{y}_1 \cap \grave{x}_2 \cap \acute{y}_2 = \grave{x} \cap \acute{y} \text{ where } \begin{cases} x = \max\{x_1, x_2\}\ , \\ y = \min\{y_1, y_2\}\ . \end{cases}$$

It is then deduced:

$$\begin{aligned} m_1 \oplus m_2\big(\grave{x} \cap \acute{y}\big) &= \int_{x_2=-\infty}^{x}\int_{y_2=y}^{+\infty} m_1\big(\grave{x} \cap \acute{y}\big)m_2\big(\grave{x}_2 \cap \acute{y}_2\big)dx_2dy_2 \\ &+ \int_{x_1=-\infty}^{x}\int_{y_1=y}^{+\infty} m_1\big(\grave{x}_1 \cap \acute{y}_1\big)m_2\big(\grave{x} \cap \acute{y}\big)dx_1dy_1 \\ &+ \int_{x_1=-\infty}^{x}\int_{y_2=y}^{+\infty} m_1\big(\grave{x}_1 \cap \acute{y}\big)m_2\big(\grave{x} \cap \acute{y}_2\big)dx_1dy_2 \\ &+ \int_{x_2=-\infty}^{x}\int_{y_1=y}^{+\infty} m_1\big(\grave{x} \cap \acute{y}_1\big)m_2\big(\grave{x}_2 \cap \acute{y}\big)dx_2dy_1\ . \end{aligned}$$

In particular, it is now justified that a bba, from sensors or fused, will always be zeroed outside $\mathcal{I}$.

5.5 Implementation of the continuous model

Setting. In this implementation, the study has been restricted to the interval $[-1,1]$ instead of $\mathbb{R}$. The previous results still hold by truncating over $[-1,1]$. In particular, any bba m is zeroed outside $\mathcal{I}_{-1}^{1} = \{\grave{x} \cap \acute{y}/x, y \in [-1,1]\}$ and its related belief function is defined by:

$$\text{Bel}\,\big(\grave{x} \cap \acute{y}\big) = \int_{u=x}^{1}\int_{v=-1}^{y} m\big(\grave{u} \cap \acute{v}\big)dudv\ ,$$

for any generalized interval of $\mathcal{I}_{-1}^{1}$. The bba resulting of the fusion of two bba's m_1 and m_2 is defined by:

$$\begin{aligned} m_1 \oplus m_2\big(\grave{x} \cap \acute{y}\big) &= \int_{x_2=-1}^{x}\int_{y_2=y}^{1} m_1\big(\grave{x} \cap \acute{y}\big)m_2\big(\grave{x}_2 \cap \acute{y}_2\big)dx_2dy_2 \\ &+ \int_{x_1=-1}^{x}\int_{y_1=y}^{1} m_1\big(\grave{x}_1 \cap \acute{y}_1\big)m_2\big(\grave{x} \cap \acute{y}\big)dx_1dy_1 \\ &+ \int_{x_1=-1}^{x}\int_{y_2=y}^{1} m_1\big(\grave{x}_1 \cap \acute{y}\big)m_2\big(\grave{x} \cap \acute{y}_2\big)dx_1dy_2 \\ &+ \int_{x_2=-1}^{x}\int_{y_1=y}^{1} m_1\big(\grave{x} \cap \acute{y}_1\big)m_2\big(\grave{x}_2 \cap \acute{y}\big)dx_2dy_1\ . \end{aligned}$$

Method. A theoretical computation of these integrals seems uneasy. An approximation of the densities and of the integrals has been considered. More precisely, the densities have been approximated by means of 2-dimension *Chebyshev polynomials* , which have several good properties:

- The approximation grows quickly with the degree of the polynomial, without oscillation phenomena,
- The Chebyshev transform is quite related to the Fourier transform, which makes the parameters of the polynomials really quickly computable by means of a Fast Fourier Transform,
- Integration is easy to compute.

In our tests, we have chosen a Chebyshev approximation of degree 128×128 , which is more than sufficient for an almost exact computation.

Example. Two bba m_1 and m_2 have been constructed by normalizing the following functions mm_1 and mm_2 defined over $[-1, 1]^2$:

$$mm_1\big(\grave{x} \cap \acute{y}\big) = \exp\big(-(x+1)^2 - y^2\big)$$

and

$$mm_2\big(\grave{x} \cap \acute{y}\big) = \exp\big(-x^2 - (y-1)^2\big) \ .$$

The fused bba $m_1 \oplus m_2$ and the respective belief function $b_1, b_2, b_1 \oplus b_2$ have been computed. This computation has been instantaneous. All functions have been represented in the figures 5.7 to 5.14.

Interpretation. The bba m_1 is a density centered around the interval $[-1, 0]$, while m_2 is a density centered around $[0, 1]$. This explains why the belief b_1 increases faster from the interval $[-1, -1]$ to $[-1, 1]$ than from the interval $[1, 1]$ to $[-1, 1]$. And this property is of course inverted for b_2 .

A comparison of the fused bba $m_1 \oplus m_2$ with the initial bba's m_1 and m_2 makes apparent a global forward move of the density. This just means that the fused bba is put on intervals with less imprecision, and possibly on some intervals with negative width (*ie.* associated with a degree of contradiction). Of course there is nothing surprising here, since information fusion will reduce imprecision and produce some contradiction! It is also noticed that the fused bba is centered around the interval $[0, 0]$. This result matches perfectly the fact that m_1 and m_2 , and their related sensors, put more belief respectively over the interval $[-1, 0]$ and the interval $[0, 1]$; and of course $[-1, 0] \cap [0, 1] = [0, 0]$.

5.6 Conclusion

A problem of continuous information fusion has been investigated and solved in the DSmT paradigm. The conceived method is based on the generalization of the notion of hyper-power set. It is versatile and is able to specify the typical various degrees of contradiction of a DSm model. It has been implemented efficiently for a bounded continuous information. The work

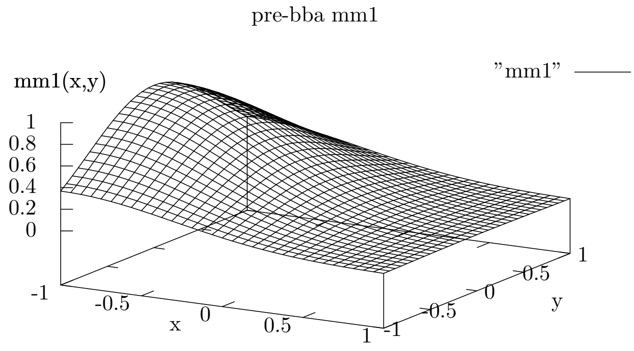

Figure 5.7: Non normalized bba mm_1

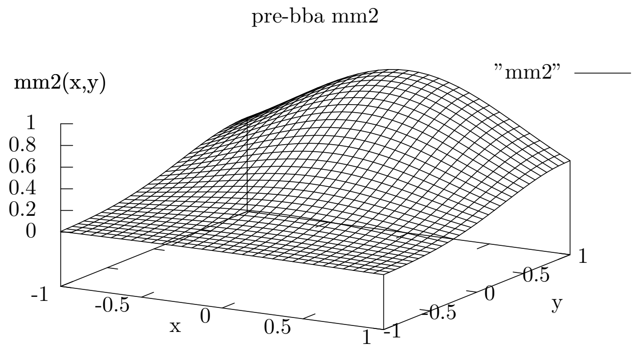

Figure 5.8: Non normalized bba mm_2

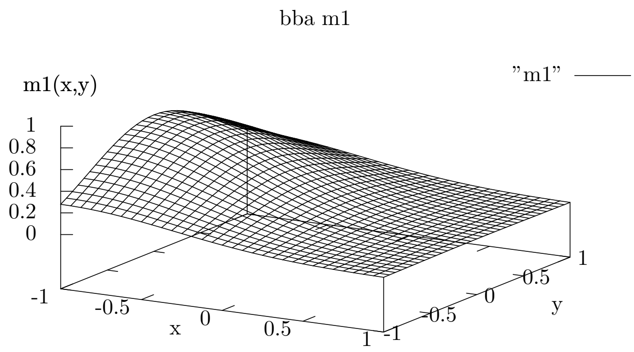

Figure 5.9: Basic belief assignment m_1

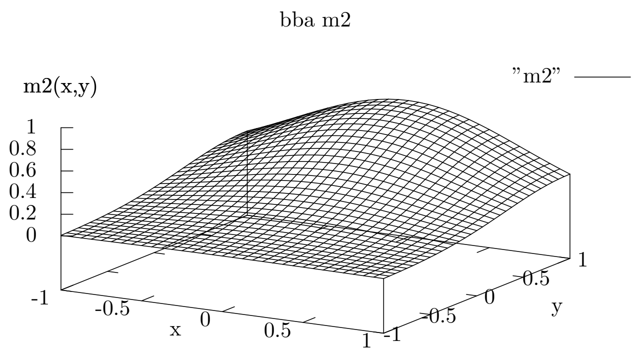

Figure 5.10: Basic belief assignment m_2

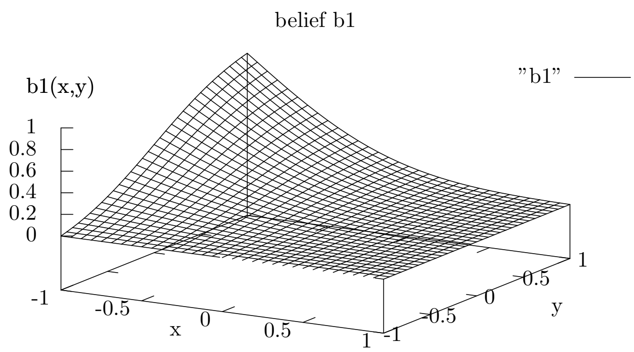

Figure 5.11: Belief function b_1

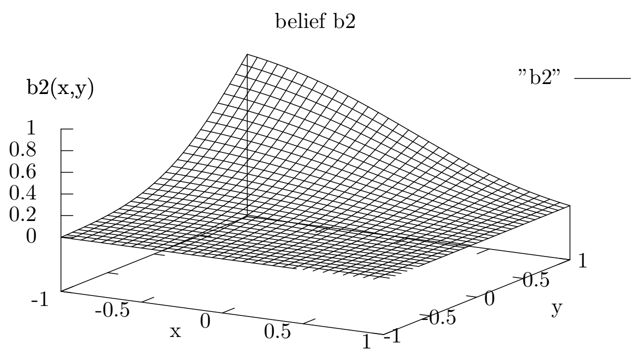

Figure 5.12: Belief function b_2

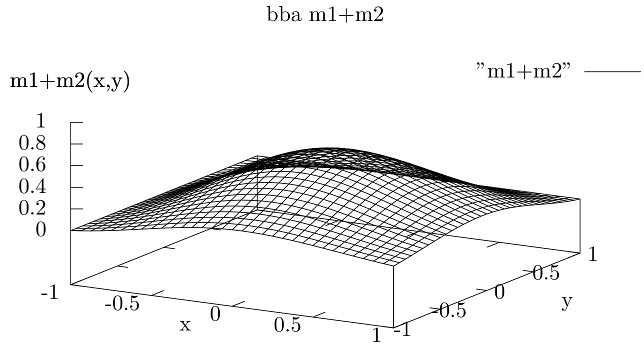

Figure 5.13: Fused bba $m_1 \oplus m_2$

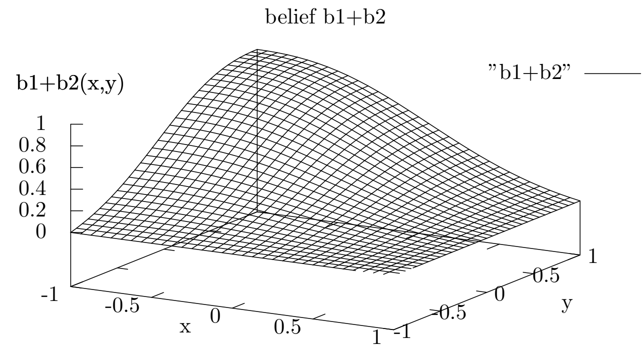

Figure 5.14: Fused bba $b_1 \oplus b_2$

is still prospective, but applications should be done in the future on localization problems. At this time, the concept is restricted to one-dimension informations. However, works are now accomplished in order to extend the method to multiple-dimensions domains.

5.7 References

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