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Contents

A. A. K. Majumdar: A note on the Smarandache cyclic geometric determinant sequences 1

A. A. K. Majumdar: On the Smarandache LCM ratio 4

Prajisha Adunkudi, J. López Bonilla and Arjun K. Rathie: On an extension of Kummer's second theorem 12

S. Balasubramanian and P. Arun Swathi Vyjayanthi: On $\nu-$Lindeloff space 19

S. Balasubramanian: Somewhat $\nu g-$closed mappings 25

Jianwei Xu and Yu Huang: The mean value of $\widetilde{P}(n)$ over cube-full numbers 33

V. Ramachandran and C. Sekar: Gracefulness and one modulo N gracefulness of $L_n \bigotimes S_m$ 38

Niraj Kumar and Garima Manocha: A study on prime one-sided ideals, socles of entire Dirichlet series as a gamma ring 49

Scientia Magna
Vol. 11 (2016), No. 1, 1-3

A note on the Smarandache cyclic geometric determinant sequences

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Abstract This paper gives an alternative approach to find the determinant of the right circular matrix with geometric sequence, using the known results of the circulant matrix.

Keywords circulant matrix, right circulant matrix with geometric sequence.

§1. Introduction

In a recent paper, Bueno[1] has introduced the concept of the right circulant geometric matrix with geometric sequence, defined as follows:

Definition 1.1. *A right circulant matrix (of order n) with geometric sequence, denoted by RCIRC(n), is a matrix of the form*

$$RCIRC(n) = \begin{pmatrix} 1 & r & r^2 & \cdots & r^{n-2} & r^{n-1} \\ r^{n-1} & 1 & r & \cdots & r^{n-3} & r^{n-2} \\ r^{n-2} & r^{n-1} & 1 & \cdots & r^{n-4} & r^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r^2 & r^3 & r^4 & \vdots & 1 & r \\ r & r^2 & r^3 & \vdots & r^{n-1} & 1 \end{pmatrix}.$$

Using the elementary properties of matrices and determinants, Bueno[1] has found an explicit form of the associated determinant.

In this paper, we follow an alternative approach to derive the determinant of the matrix RCIRC(n). This is given in Section 3. Some preliminary results are given in Section 2.

§2. Some preliminary results

In this section, we give some well-known results that would be needed later in proving the main results of this paper in Section 3. We start with the following definition.

Definition 2.1. *The circulant matrix with the vector* $\boldsymbol{C} = (c_0,\ c_1,\ \ldots,\ c_{n-1})$*, denoted* *by C_{n}, is the matrix of the form*

$$C(n) = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \vdots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \vdots & c_{n-1} & c_0 \end{pmatrix}.$$

Lemma 2.1. *For any n (≥ 2),*

$$\begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \vdots & c_{n-1} & c_0 \end{vmatrix} = \prod_{j=0}^{n-1} (c_0 + c_1\omega_j + c_2\omega_j^2 + \ldots + c_{n-1}\omega_j^{n-1}),$$

where $\omega_0 \equiv 0$, $\omega_j = e^{\frac{2\pi i}{n}j}$ ($1 \leq j \leq n - 1$) are the n^{th} roots of unity.

§3. Main result

We now give the main result of this paper in the following theorem.

Lemma 3.1. *For $n \geq 1$, $\det(RCIRC(n)) = (1 - r^{n})^{n-1}$.*

Proof. From Lemma 2.1 with $c_j = r^j$ $(0 \leq j \leq n - 1)$, we see that

$$\det(RCIRC(n)) = \prod_{j=0}^{n-1} (1 + r\omega_j + r^2\omega_j^2 + \ldots + r^{n-1}\omega_j^{n-1}).$$

But, for any j with $0 \leq j \leq n - 1$,

$$1 + r\omega_j + r^2\omega_j^2 + \ldots + r^{n-1}\omega_j^{n-1} = \frac{1 - (r\omega_j)^n}{1 - r\omega_j} = \frac{1 - r^n}{1 - r\omega_j}. \tag{1}$$

Again, since

$x^n - 1 = (x - \omega_0)(x - \omega_1)(x - \omega_2) \ldots (x - \omega_{n-1})$,

for $x = \frac{1}{r}$, we get

$$\frac{1 - r^n}{r^n} = \frac{(1 - r\omega_0)(1 - r\omega_1)(1 - r\omega_2)\ldots(1 - r\omega_{n-1})}{r^n},$$

so that

$$(1-\omega_0)(1-\omega_1)(1-\omega_2)\ldots(1-\omega_{n-1}) = 1-r^n. \tag{2}$$

The lemma now follows by virtue of (1) and (2).

References

[1] A. C. F. Bueno, Smarandache Cyclic Geometric Determinant Sequences, *Sci. Magna,* 8 (2012), pp. 88–91.

[2] D. Geller, I. Kra, S. Popescu and S. Simanca, On Circulant Matrices, *Preprint,* pp. 1–13. Retrieved on June 9, 2014.

[3] Circulant Matrix, *http://en.wikipedia.org/wiki/Circulant_matrix*, pp. 1–3. Retrieved on June 9, 2014.

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On the Smarandache LCM ratio

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Abstract Two types of the Smarandache LCM ratio functions have been introduced by Murthy [1]. Recently, the second type of the Smarandache LCM ratio function has been considered by Khainar, Vyawahare and Salunke [2]. This paper establishes the relationships between these two forms of the Smarandache LCM ratio functions, and derives some reduction formulas and interesting properties in connection with these functions.

Keywords Smarandache LCM ratio functions (of two kinds), reduction formulas.

§1. Introduction

The Smarandache LCM ratio function, proposed by Murthy [1], is as follows :

Definition 1.1. *The Smarandache LCM ratio function of degree r, denoted by $T(n, r)$, is*

$$T(n,r) = \frac{[n, n+1, n+2, \cdots, n+r-1]}{[1, 2, 3, \cdots, r]}, \qquad n, r \in \mathbb{N},$$

where $[n_1, n_2, \cdots, n_k]$ denotes the least common multiple (LCM) of the integers $n_1, n_2, \cdots, n_k$.

The explicit expressions for $T(n, 1)$ and $T(n, 2)$ are already mentioned in Murthy [1], and are reproduced in the following two lemmas.

Lemma 1.1. *$T(n, 1) = n$ for all $n \geq 1$.*

Lemma 1.2. *For $n \geq 1$, $T(n, 2) = \frac{n(n+1)}{2}$.*

The following two lemmas, due to Maohua [3], give explicit expressions for $T(n, 3)$ and $T(n, 4)$ respectively.

Lemma 1.3. *For $n \geq 1$,*

$$T(n,3) = \begin{cases} \frac{n(n+1)(n+2)}{6}, & \text{if } n \text{ is odd} \\ \frac{n(n+1)(n+2)}{12}, & \text{if } n \text{ is even} \end{cases}$$

Lemma 1.4. *For $n \geq 1$,*

$$T(n,\,4) = \begin{cases} \frac{n(n+1)(n+2)(n+3)}{72}, & \text{if } 3 \text{ divides } n \\ \frac{n(n+1)(n+2)(n+3)}{24}, & \text{if } 3 \text{ does not divide } n \end{cases}$$

Finally, the expression for $T(n, 5)$ is given by Wang Ting [4].

[1] On ADL from: Ritsumeikan Asia-Pacific University, Japan.

Lemma 1.5. *For $n \geq 1$,*

$$T(n,\,5)=\begin{cases}\frac{n(n+1)(n+2)(n+3)(n+4)}{1440}, & if\ \ n=12m,\ 12m+8\\ \frac{n(n+1)(n+2)(n+3)(n+4)}{120}, & if\ \ n=12m+1,\ 12m+7\\ \frac{n(n+1)(n+2)(n+3)(n+4)}{720}, & if\ \ n=12m+2,\ 12m+6\\ \frac{n(n+1)(n+2)(n+3)(n+4)}{360}, & if\ \ n=12m+3,\ 12m+5,\ 12m+9,\ 12m+11\\ \frac{n(n+1)(n+2)(n+3)(n+4)}{480}, & if\ \ n=12m+4\\ \frac{n(n+1)(n+2)(n+3)(n+4)}{240}, & if\ \ n=12m+10\end{cases}$$

Recently, Khairnar, Vyawahare and Salunke [2] treated the Smarandache LCM ratio function, defined as follows :

Definition 1.2. *The Smarandache LCM ratio function, denoted by $SL(n,r)$, is*

$$SL(n,r)=\frac{[n,\,n\text{-}1,\,n\text{-}2,\,\ldots,\,n\text{-}r\text{+}1]}{[1,\,2,\,3,\,\ldots,\,r]},\,r\leq n;\ n,\ r\in N.$$

The function $SL(n,r)$, given in Definition 1.2 above, may be called the Smarandache LCM ratio function of the second type.

In Section 2, we derive the relationships between the two functions $T(n,r)$ and $SL(n,r)$, and hence, derive the reduction formulas for $SL(n,3)$, $SL(n,4)$ and $SL(n,5)$, using the known expressions for $T(n,3)$, $T(n,4)$ and $T(n,5)$. Some more properties, together with some open problems involving these functions, are given in Section 3.

§2. Reduction formulas

The following lemma gives the relationship between $T(n,r)$ and $SL(n,r)$.

Lemma 2.1. *$SL(n,r)=T(n-r+1,r)$.*

Proof. This is evident from Definition 1.1 and Definition 1.2.

Note that, in Lemma 2.1 above, the condition $n-r+1\geq 1$ requires that $SL(n,r)$ is defined only for $r\leq n$.

The explicit expressions for the functions $SL(n,1)$, $SL(n,2)$, $SL(n,3)$ and $SL(n,4)$ are given in Theorems 2.1 - 2.4 below.

Theorem 2.1. *For any $n\geq 1$, $SL(n,\,1)=n$.*

Theorem 2.2. *For any $n\geq 2$, $SL(n,\,2)=\frac{n(n\text{-}1)}{2}$.*

Proof. By Lemma 1.2 and Lemma 2.1,

$$\mathrm{SL(n,\,2)}=\mathrm{T(n\,\text{-}\,1,\,2)}=\frac{\mathrm{(n\text{-}1)n}}{2}.$$

Theorem 2.3. *For any $n\geq 3$,*

$$SL(n,3)=\begin{cases}\frac{n(n\text{-}1)(n\text{-}2)}{6}, & if\,n\,is\,odd\\ \frac{n(n\text{-}1)(n\text{-}2)}{12}, & if\,n\,is\,even\end{cases}$$

Proof. Using Lemma 1.3, together with Lemma 2.1,

$$SL(n,\, 3) = T(n-2,\, 3) = \begin{cases} \frac{\text{(n-2)(n-1)n}}{6}, \text{ if } \text{n}-2 \text{ is odd} \\ \frac{\text{(n-2)(n-1)n}}{12}, \text{ if } \text{n}-2 \text{ is even} \end{cases}$$

Now, since n is odd or even according as $n-2$ is odd or even respectively, the result follows.

Theorem 2.4. *For any* $n \geq 4$,

$$SL(n, 4) = \begin{cases} \frac{n(n\text{-}1)(n\text{-}2)(n\text{-}3)}{72}, \text{ if } 3 \text{ divides } n \\ \frac{n(n\text{-}1)(n\text{-}2)(n\text{-}3)}{24}, \text{ if } 3 \text{ does not divide } n \end{cases}$$

Proof. By Lemma 1.4 and Lemma 2.1,

$$SL(n,\, 4) = T(n-3,\, 4) = \begin{cases} \frac{\text{(n-3)(n-2)(n-1)n}}{72}, \text{ if } 3 \text{ divides } \text{n}-3 \\ \frac{\text{(n-3)(n-2)(n-1)n}}{24}, \text{ if } 3 \text{ does not divide } \text{n}-3 \end{cases}$$

But, 3 divides n − 3 if and only if 3 divides n. This, in turn, establishes the theorem.

Theorem 2.5. *For any* $n \geq 5$,

$$SL(n,\, 5) = \begin{cases} \frac{n(n\text{-}1)(n\text{-}2)(n\text{-}3)(n\text{-}4)}{1440}, \text{ if } n = 12m,\ 12m+4 \\ \frac{n(n\text{-}1)(n\text{-}2)(n\text{-}3)(n\text{-}4)}{120}, \text{ if } n = 12m+1,\ 12m+3,\ 12m+7,\ 12m+9 \\ \frac{n(n\text{-}1)(n\text{-}2)(n\text{-}3)(n\text{-}4)}{720}, \text{ if } n = 12m+2 \\ \frac{n(n\text{-}1)(n\text{-}2)(n\text{-}3)(n\text{-}4)}{360}, \text{ if } n = 12m+5,\ 12m+11 \\ \frac{n(n\text{-}1)(n\text{-}2)(n\text{-}3)(n\text{-}4)}{480}, \text{ if } n = 12m+6,\ 12m+10 \\ \frac{n(n\text{-}1)(n\text{-}2)(n\text{-}3)(n\text{-}4)}{240}, \text{ if } n = 12m+8 \end{cases}$$

Proof. By virtue of Lemma 1.5 and Lemma 2.1,

$$SL(n,\, 5) = T(n-4,\, 5)$$

$$= \begin{cases} \frac{\text{(n-4)(n-3)(n-2)(n-1)n}}{1440}, \text{ if } \text{n}-4 = 12m,\ 12\text{m}+8 \\ \frac{\text{(n-4)(n-3)(n-2)(n-1)n}}{120}, \text{ if } \text{n}-4 = 12m+1,\ 12m+7 \\ \frac{\text{(n-4)(n-3)(n-2)(n-1)n}}{720}, \text{ if } \text{n}-4 = 12m+2,\ 12\text{m}+6 \\ \frac{\text{(n-4)(n-3)(n-2)(n-1)n}}{360}, \text{ if } \text{n}-4 = 12\text{m}+3,\ 12m+5,\ 12\text{m}+9,\ 12m+11 \\ \frac{\text{(n-4)(n-3)(n-2)(n-1)n}}{480}, \text{ if } \text{n}-4 = 12m+4 \\ \frac{\text{(n-4)(n-3)(n-2)(n-1)n}}{240}, \text{ if } \text{n}-4 = 12m+10 \end{cases}$$

Now, since n − 4 is of the form 12m if and only if n is of the form 12m + 4, n − 4 is of the form 12m + 8 if and only if n is of the form 12m, n − 4 is of the form 12m + 9 if and only if n is of the form 12m + 1, n − 4 is of the form 12m + 11 if and only if n is of the form 12m + 3, n − 4 is of the form 12m + 10 if and only if n is of the form 12m + 2, etc., the result follows.

§3. Some open problems

In this section, we give some open problems involving the functions $SL(n, r)$.

First, we state and prove the following two results.

Lemma 3.1. *For any integer $n \geq 1$, SL(n, n) = 1.*

Proof. This is evident from Definition 1.2.

Lemma 3.2. *If p is a prime, then p divides SL(p, r) for all r <p.*

Proof. By definition,

$$SL(p,\ r) = \frac{[\text{p, p-1, p-2, ..., p-r+1}]}{[1, 2, 3, ..., \text{r}]},\ r \leq p.$$

Now, p divides [p, p − 1, p − 2, . . . , p − r + 1] for all r <p, while p does not divide [1, 2, 3, . . . , r]. Thus, p divides SL(p, r).

Using the values of SL(n, r), the following table, called the Smarandache-Amar LCM triangle, is formed as follows :

The 1st column contains the elements of the sequence $\{SL(n,\ 1)\}_{n=1}^{\infty}$, the 2nd column is formed with the elements of the sequence $\{SL(n,\ 2)\}_{n=2}^{\infty}$, and so on, and in general, the k-th column contains the elements of the sequence $\{SL(n,\ \mathrm{k})\}_{n=k}^{\infty}$,

Note that, the 1st column contains the natural numbers, and the 2nd column contains the triangular numbers.

The Smarandache-Amar LCM triangle is

	1-st column SL(n, 1)	2-nd column SL(n, 2)	3-rd column SL(n, 3)	4-th column SL(n, 4)	5-th column SL(n, 5)	6-th column SL(n, 6)	7-th column SL(n, 7)	8-th column SL(n, 8)	9-th column SL(n, 9)
1-st row	1								
2-nd row	2	1							
3-rd row	3	3	1						
4-th row	4	6	2	1					
5-th row	5	10	10	5	1				
6-th row	6	15	10	5	1	1			
7-th row	7	21	35	35	7	7	1		
8-th row	8	28	28	70	14	14	2	1	
9-th row	9	36	84	42	42	42	6	3	1
10-th row	10	45	60	210	42	42	6	3	1
11-th row	11	55	165	330	462	462	66	33	55
12-th row	12	66	440	165	66	462	66	33	11
13-th row	13	78	286	715	429	858	858	429	143

Note that, by Lemma 3.1, the leading diagonal contains all unity.

Lemma 3.3. *If p is a prime, then sum of the elements of the p-th row ≡1 (mod p).*

Proof. The sum of the p-th row is

$$\begin{aligned}&\mathrm{SL(p, 1) + SL(p, 2) + \ldots + SL(p, p-1) + SL(p, p)}\\&= \mathrm{[SL(p, 1) + SL(p, 2) + \ldots + SL(p, p-1)] + 1}\\&\equiv \quad 1 \pmod{p},\end{aligned}$$

by virtue of Lemma 3.2.

Lemma 3.4. *If p is a prime, then p does not divide SL(2p, r) for any $p \leq r \leq 2p$. Moreover, if q is the prime next to p, then q divides SL(2p, r) for all $p \leq r \leq q - 1$.*

Proof. If p is a prime, then p divides [2p, 2p – 1, . . . , 2p – r + 1] for all r ≤ 2p, and p divides [1, 2, . . . , r] for all r ≥ p. Hence, p does not divide

$$SL(2p,\, r) = \frac{[2p,\, 2p-1,\, ...,\, 2p-r+1]}{[1,\, 2,\, ...,\, r]},\ \mathrm{p} \leq r \leq 2p. \tag{1}$$

To prove the remaining part of the lemma, first note that, by Bertrand's postulate (see, for Example, Hardy and Wright [5]), there is at least one prime, say, q, such that p <q <2p. Now, from (1), q divides the numerator if p ≤ r ≤ q – 1, but q does not divide the denominator. As such, q divides SL(2p, r) for all p ≤ r ≤ q – 1.

Open Problem # 1 : Is it possible to find a congruence property for the sum of the elements of the k-th row when k is a composite?

Open Problem # 2 : Is it possible to find the sum of the elements of the k-th row?

Note that, by Lemma 3.2 and Lemma 3.4, some of the elements of the (2p)-th row is divisible by p, and some elements are not divisible by p but are divisible by q, where q is the next larger prime to p.

Looking at the 9^{th} row of the triangle, we observe that the number 42 appears in three consecutive places. Note that, 42 is divisible by the prime next to 7 in the interval (p, 2p) with p = 5.

Open Problem # 3 : In the Smarandache-Amar triangle, is it possible to find (in some row) repeating values of arbitrary length?

Note that, the above problem is related to the problem of finding the solutions of the equation

$$SL(n, r) = SL(n, r + 1). \tag{2}$$

A necessary and sufficient condition that (2) holds is

$$([n,\, n-1,\, ...,\, n-r+1],\, n-r)(r+1) = ([1,\, 2,\, ...,\, r],\, r+1)(n-r). \tag{3}$$

The proof is as follows : The equation (2) holds for some n and r if and only if

$$\frac{[n,\, n-1,\, ...,\, n-r+1]}{[1,\, 2,\, ...,\, r]} = \frac{[n,\, n-1,\, ...,\, n-r]}{[1,\, 2,\, ...,\, r,\, r+1]}$$

that is, if and only if

$$[n,\, n-1,\, ...,\, n-r+1].[1,\, 2,\, ...,\, r,\, r+1] = [n,\, n-1,\, ...,\, n-r].[1,\, 2,\, ...,\, r]$$

that is, if and only if

$$[n,\, n-1,\, ...,\, n-r+1].\frac{[1,\, 2,\, ...,\, r](r+1)}{([1,\, 2,\, ...,\, r],\, r+1)} = \frac{[n,\, n-1,\, ...,\, n-r+1](n-r)}{([n,\, n-1,\, ...,\, n-r+1],\, n-r)}.[1,\, 2,\, ...,\, r],$$

which reduces to (3) after simplification.

Lemma 3.5. *If n is an odd (positive) integer, then the equation (2) has always a solution.*

Proof. We show that

$$\mathrm{SL}(2r + 1, r) = \mathrm{SL}(2r + 1, r + 1) \text{ for any integer } r \geq 1.$$

In this case, the necessary and sufficient condition (3) takes the form

$$([2r+1, 2r, ..., r+2], r+1)(r+1) = ([1, 2, ..., r], r+1)(r+1).$$

Now, since

$$([2r+1, 2r, ..., r+2], r+1) = ([1, 2, ..., r], r+1) \text{ for any integer r} \geq 1,$$

we see that (3) is satisfied, which, in turn, establishes the result.

If n is an even integer, the equation (2) may not have a solution. A counter-example is the case when n = 4. However, we have the following result.

Lemma 3.6. *If n is an integer of the form n = 2p + 1, where p is a prime, then*

$$SL(2p, p) = SL(2p, p + 1)$$

if and only if

$$([1, 2, ..., p], p+1) = p+1.$$

Proof. If n = 2p + 1, then the l.h.s. of the condition (3) is

$$([2p, 2p-1, ..., p+1], p)(p+1) = p(p+1),$$

which, together with the r.h.s. of (3), gives the desired condition.

Conjecture 3.1. *The equation SL(n, r) = SL(n, r + 1) has always a solution for any n ≥ 5.*

In the worst case, SL(n, n − 1) = SL(n, n) = 1, and the necessary and sufficient condition is that n divides [1, 2, . . . , n − 1].

Another interesting problem is to find the solution of the equation

$$SL(n+1, r) = SL(n, r). \tag{4}$$

The equation (4) holds for some n and r if and only if

$$\frac{[n, n-1, ..., n-r+1]}{[1, 2, ..., r]} = \frac{[n+1, n, ..., n-r+2]}{[1, 2, ..., r]}$$

that is, if and only if

$$\frac{[n, n-1, ..., n-r+2].(n-r+1)}{([n, n-1, ..., n-r+2], n-r+1)} = \frac{(n+1).[n, \mathrm{n}-1, ..., n-r+2]}{([n, \mathrm{n}-1, ..., n-r+2], n+1)},$$

which, after simplification, leads to

$$(n-r+1).([n, \mathrm{n}-1, ..., n-r+2], n+1) = (n+1).([n, n-1, ..., n-r+2], n-r+1), \tag{5}$$

which is the necessary and sufficient condition for (4).

From (5), we observe the following facts :

1. n + 1 cannot be prime, for otherwise,

$$([n, \text{n}-1, ..., n-r+2], n+1) = 1,$$

which leads to a contradiction.

2. In (5),

$$[n, \text{n}-1, ..., n-r+2], n+1) = n+1 \ \Leftrightarrow \ ([n, n-1, ..., n-r+2], n-r+1) = n-r+1.$$

3. In (5), if n − r + 1 = 2, then

$$([n, n-1, ..., n-r+2], n-r+1) = 2 \ \Rightarrow \ [n, \text{n}-1, ..., n-r+2], n+1) = n+1.$$

4. If n − r + 1 $\neq$ 2 is prime, then

$$\begin{aligned} &([n, n-1, ..., n-r+2], n-r+1) = 1 \\ \Rightarrow \ &\text{n}+1 = (n-r+1).([n, \text{n}-1, ..., n-r+2], n+1) \\ \Rightarrow \ &\text{n}+1 = \tfrac{([n, n-1, ..., n-r+2], n+1)}{([n, n-1, ..., n-r+2], n+1)-1} r, \end{aligned} \tag{6}$$

after simplification, showing that $([n, \text{n}-1, ..., n-r+2], n+1) - 1$ must divide r.

5. In (5), if $([n, \text{n}-1, ..., n-r+2], n+1) = n+1$, then $n-r+1$ cannot be an odd prime, for otherwise, by (6),

$$\text{n}+1 = \frac{n+1}{(n+1)-1} r \ \Rightarrow \ \text{n} = r,$$

which leads to a contradiction.

Conjecture 3.2. *The equation SL(n + 1, r) = SL(n, r) has always a solution for any r ≥ 3.*

In the worst case, SL(r + 1, r) = SL(r, r) = 1, and the necessary and sufficient condition is that $([1, 2, ..., r], r+1) = r+1$.

Remark 3.1. In [2], Khairnar, Vyawahare and Salunke mention some identities involving the ratio and sum of reciprocals of two consecutive LCM ratios. The validity of these results depends on the fact that SL(n, r) can be expressed as

$$SL(n, r) = \frac{n(n-1) .... (n-r+1)}{r!}. \tag{7}$$

If SL(n, r) can be represented as in (7), it can be deduced that

$$\frac{SL(n, r+1)}{SL(n, r)} = \frac{n-r}{r+1}, \ \frac{1}{SL(n, r)} + \frac{1}{SL(n, r+1)} = \frac{n+1}{(r+1).SL(n, r+1)}.$$

However, the above results are valid only under certain conditions on n and r. For example, for r = 2, the above two identities are valid only for odd (positive) integers n.

Thus, the next question is : What are the conditions on n and r for (7)?

If r = p, where p is a prime, then SL(p! − 1, p) can be expressed as in (7), because in such a case

$$SL(p! - 1, p) = \frac{[\text{p!-1, p!-2, ..., p!-p}]}{[1, 2, ..., \text{p}]} = \frac{\text{(p!-1)(p!-2) ... (p!-p)}}{\text{p!}}.$$

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References

[1] A. Murthy, Some notions on least common multiples, *Smarandache Notions J.,* 12 (2004), pp. 307–308.

[2] S. M. Khairnar, A. W. Vyawahare and J. N. Salunke, On Smarandache least common multiple ratio, *Sci. Magna,* 5 (2009), no.1, pp. 29–36.

[3] M. Le, Two formulas for Smarandache LCM ratio sequences, *Smarandache Notions J.,* 12 (2004), pp. 183–185.

[4] T. Wang, A formula for Smarandache LCM ratio sequence, *Research on Smarandache Problems in Number Theory (Vol. II),* Edited by Zang Wengpeng, Li Junzhuang and Liu Duansen, Hexis, Phoenix AZ, 2005, pp. 45–46.

[5] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford University Press, 2002.

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On an extension of Kummer's second theorem

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Abstract : In recent years, various extensions of the well known and useful Kummer's second theorem have been given. In this paper, we aim to give another extension of Kummer's second theorem.
Keywords : Generalized Hypergeometric Function, Kummer's 1st and 2nd theorems.

§1. Introduction

In the theory of hypergeometric and generalized hypergeometric series, summation and transformation formulas play an important role. For this, we start with the following Kummer's first theorem **[3,5]** for the series ${}_1F_1$,

$$e^{-x}\ {}_1F_1\left[\begin{matrix} a \\ b \end{matrix}\ ;\ x\right] = {}_1F_1\left[\begin{matrix} b-a \\ b \end{matrix}\ ;\ -x\right]. \tag{1.1}$$

Recently, Paris **[4]** generalized (1.1) in the form

$$e^{-x}\ {}_2F_2\left[\begin{matrix} a,\ 1+d \\ b,\ \ d \end{matrix}\ ;\ x\right] = {}_2F_2\left[\begin{matrix} b-a-1,\ f+1 \\ b,\ \ f \end{matrix}\ ;\ -x\right], \tag{1.2}$$

where

$$f = \frac{d(1+a-b)}{a-d}. \tag{1.3}$$

The well known Kummer's second theorem **[4]** is

$$e^{-x/2}\ {}_1F_1\left[\begin{matrix} a \\ 2a \end{matrix}\ ;\ x\right] = {}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix}\ ;\ \frac{x^2}{16}\right]. \tag{1.4}$$

Bailey[1] established (1.4) by using Gauss's second summation theorem. Rathie and Choi [8] derived (1.4) by employing Gauss's summation theorem[5].

Motivated by the extension of Kummer's first theorem (1.2) obtained by Paris [4], recently Rathie and Pogany [9] have given the following interesting extension of Kummer's second theorem in the form

$$e^{-x/2}\,{}_2F_2\left[\begin{matrix} a,\ 1+d \\ 2a+1,\ d \end{matrix} ;\, x\right] = {}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix} ;\, \frac{x^2}{16}\right] - \frac{x\,(1-\frac{2a}{d})}{2\,(2a+1)}\,{}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix} ;\, \frac{x^2}{16}\right]. \quad (1.5)$$

Recently, Kim et al.[2] have generalized the Kummer's second theorem and obtained explicit expression of

$$e^{-x/2}\,{}_1F_1\left[\begin{matrix} a \\ 2a+j \end{matrix} ;\, x\right] \qquad \text{for} \quad j = 0, \pm 1, \pm 2, ..., \pm 5. \quad (1.6)$$

Very recently, Rakha et al.[6] have given another extension of Kummer's second theorem (1.4) in the following form

$$\begin{aligned} e^{-x/2}\,{}_2F_2\left[\begin{matrix} a,\ 2+d \\ 2a+2,\ d \end{matrix} ;\, x\right] &= {}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix} ;\, \frac{x^2}{16}\right] + \frac{(\frac{2a}{d}-\frac{1}{2})x}{(a+1)}\,{}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix} ;\, \frac{x^2}{16}\right] \\ &\quad + \frac{c\,x^2}{2\,(2a+3)}\,{}_0F_1\left[\begin{matrix} - \\ a+\frac{5}{2} \end{matrix} ;\, \frac{x^2}{16}\right], \end{aligned} \quad (1.7)$$

where $d \neq 0, -1, -2...$ and c is given by

$$c = \left(\frac{1}{a+1}\right)\left(\frac{1}{2}-\frac{a}{d}\right) + \left(\frac{a}{d(a+1)}\right). \quad (1.8)$$

Also, very recently Rakha and Rathie [7] have given another extension of Kummer's second theorem in the following form

$$\begin{aligned} e^{-x/2}\,{}_2F_2\left[\begin{matrix} a,\ 3+d \\ 2a+3,\ d \end{matrix} ;\, x\right] &= {}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix} ;\, \frac{x^2}{16}\right] + c_1 x\,{}_0F_1\left[\begin{matrix} - \\ a+\frac{5}{2} \end{matrix} ;\, \frac{x^2}{16}\right] \\ &\quad + c_2 x^2\,{}_0F_1\left[\begin{matrix} - \\ a+\frac{5}{2} \end{matrix} ;\, \frac{x^2}{16}\right] + c_3\,x^3\,{}_0F_1\left[\begin{matrix} - \\ a+\frac{7}{2} \end{matrix} ;\, \frac{x^2}{16}\right] \end{aligned} \quad (1.9)$$

where c_1, c_2 and c_3 are given by

$$\left.\begin{aligned} c_1 &= \frac{3(\frac{a}{d} - \frac{1}{2})}{(2a+3)} \\ c_2 &= \frac{\left(1 - \frac{3a}{d} + \frac{3a\,(a+1)}{d\,(d+1)}\right)}{2(a\,+\,2)(2a\,+\,3)} \\ c_3 &= \frac{\left(\frac{3a}{2d} - \frac{1}{2} - \frac{3a\,(a+1)}{2d\,(d+1)} + \frac{a(a+1)\,(a+2)}{d\,(d+1)(d+2)}\right)}{2(a\,+\,2)\;(2a\,+\,3)\;(2a\,+\,5)} \end{aligned}\right\}. \tag{1.10}$$

In this paper, we aim to establish one more extension of Kummer's second theorem in the form

$$e^{-x/2}\;{}_2F_2\left[\begin{matrix} a,\ 4+d \\ 2a+4,\ d \end{matrix}\;;\;x\right] \tag{1.11}$$

The result is derived with the help of Kummer's second theorem (1.4) and its various contiguous results obtained by Kim et al **[2]**. For this, the following results given in [2] will be required in our present investigation.

$$e^{-x/2}\;{}_1F_1\left[\begin{matrix} a \\ 2a+1 \end{matrix}\;;\;x\right] = {}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix}\;;\;\frac{x^2}{16}\right] - \frac{x}{2\,(2a+1)}\,{}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix}\;;\;\frac{x^2}{16}\right], \tag{1.12}$$

$$\begin{aligned} e^{-x/2}\;{}_1F_1\left[\begin{matrix} a \\ 2a+2 \end{matrix}\;;\;x\right] &= {}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix}\;;\;\frac{x^2}{16}\right] - \frac{x}{2(a+1)}\;{}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix}\;;\;\frac{x^2}{16}\right] \\ &\quad + \frac{x^2}{4(a+1)\,(2a+3)}\;{}_0F_1\left[\begin{matrix} - \\ a+\frac{5}{2} \end{matrix}\;;\;\frac{x^2}{16}\right], \end{aligned} \tag{1.13}$$

$$\begin{aligned} e^{-x/2}\;{}_1F_1\left[\begin{matrix} a \\ 2a+3 \end{matrix}\;;\;x\right] &= {}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix}\;;\;\frac{x^2}{16}\right] - \frac{3x}{2(2a+3)}\;{}_0F_1\left[\begin{matrix} - \\ a+\frac{5}{2} \end{matrix}\;;\;\frac{x^2}{16}\right] \\ &\quad + \frac{x^2}{2(a+2)\;(2a+3)}\;{}_0F_1\left[\begin{matrix} - \\ a+\frac{5}{2} \end{matrix}\;;\;\frac{x^2}{16}\right] \\ &\quad - \frac{x^3}{4(a+2)\,(2a+3)\,(2a+5)}\;{}_0F_1\left[\begin{matrix} - \\ a+\frac{7}{2} \end{matrix}\;;\;\frac{x^2}{16}\right], \end{aligned} \tag{1.14}$$

and

$$\begin{aligned} e^{-x/2}\;{}_1F_1\left[\begin{matrix} a \\ 2a+4 \end{matrix}\;;\;x\right] &= {}_0F_1\left[\begin{matrix} - \\ a+\frac{5}{2} \end{matrix}\;;\;\frac{x^2}{16}\right] - \frac{x}{(a+2)}\;{}_0F_1\left[\begin{matrix} - \\ a+\frac{5}{2} \end{matrix}\;;\;\frac{x^2}{16}\right] \\ &\quad + \frac{x^2}{(a+2)\;(2a+5)}\;{}_0F_1\left[\begin{matrix} - \\ a+\frac{7}{2} \end{matrix}\;;\;\frac{x^2}{16}\right] \end{aligned}$$

$$- \frac{x^3}{4(a+2)\,(a+3)(2a+5)}\ {}_0F_1\left[\begin{matrix} - \\ a+\frac{7}{2}\end{matrix}\ ;\ \frac{x^2}{16}\right]$$
$$+ \frac{x^4}{8(a+2)\,(a+3)\,(2a+5)\,(2a+7)}\ {}_0F_1\left[\begin{matrix} - \\ a+\frac{9}{2}\end{matrix}\ ;\ \frac{x^2}{16}\right]. \tag{1.15}$$

§2. Main Result

The following extension of Kummer's second transformation will be established in this paper.

$$\begin{aligned} e^{-x/2}\ {}_2F_1\left[\begin{matrix} a,\ 4+d \\ 2a+4,\ d\end{matrix}\ ;\ x\right] =\ & {}_0F_1\left[\begin{matrix} - \\ a+\frac{5}{2}\end{matrix}\ ;\ \frac{x^2}{16}\right] + c_1\,x\ {}_0F_1\left[\begin{matrix} - \\ a+\frac{5}{2}\end{matrix}\ ;\ \frac{x^2}{16}\right] \\ & + c_2\,x^2\ {}_0F_1\left[\begin{matrix} - \\ a+\frac{7}{2}\end{matrix}\ ;\ \frac{x^2}{16}\right] + c_3\,x^3\ {}_0F_1\left[\begin{matrix} - \\ a+\frac{9}{2}\end{matrix}\ ;\ \frac{x^2}{16}\right] \\ & + c_4\,x^4\ {}_0F_1\left[\begin{matrix} - \\ a+\frac{9}{2}\end{matrix}\ ;\ \frac{x^2}{16}\right] \end{aligned} \tag{2.1}$$

where

$$\left.\begin{aligned} c_1 &= \frac{\left(\frac{2a}{d}-1\right)}{(a+2)}, \\ c_2 &= \frac{\left(1-\frac{3a}{d}+\frac{3a\,(a+1)}{d\,(d+1)}\right)}{(a+2)\,(2a+5)}, \\ c_3 &= \frac{\left(\frac{a}{d}-\frac{1}{4}-\frac{3a\,(a+1)}{2d\,(d+1)}+\frac{a\,(a+1)\,(a+2)}{d\,(d+1)\,(d+2)}\right)}{(a+2)\,(a+3)\,(2a+5)}, \\ \text{and}\qquad c_4 &= \frac{\left(\frac{1}{4}-\frac{a}{d}+\frac{3a\,(a+1)}{2d\,(d+1)}-\frac{a\,(a+1)\,(a+2)}{d\,(d+1)\,(d+2)}+\frac{a\,(a+1)\,(a+2)\,(a+3)}{2d\,(d+1)\,(d+2)\,(d+3)}\right)}{2\,(a+2)\,(a+3)\,(2a+5)\,(2a+7)}. \end{aligned}\right\} \tag{2.2}$$

Proof. Using the definition of the Pochhammer's symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)},$$

we can see that

$$\frac{(d+4)_n}{(d)_n} = 1+\frac{4n}{d}+\frac{6n(n-1)}{d(d+1)}+\frac{4n(n-1)(n-2)}{d(d+1)(d+2)}+\frac{n(n-1)(n-2)(n-3)}{d(d+1)(d+1)(d+3)}. \tag{2.3}$$

Now, in order to establish to our main result (2.1), we proceed as follows. Denoting left hand side of (2.1) by S and expressing ${}_2F_2$ as series, we have

$$\begin{aligned} S &= e^{-x/2} \sum_{n=0}^{\infty} \frac{(a)_n \ (d+4)_n}{(2a+4)_n \ (d)_n} \frac{x^n}{n!} \\ &= e^{-x/2} \sum_{n=0}^{\infty} \frac{(a)_n}{(2a+4)_n} \frac{x^n}{n!} \left\{ \frac{(d+4)_n}{(d)_n} \right\}. \end{aligned}$$

Using (2.3), we have

$$\begin{aligned} S &= e^{-x/2} \sum_{n=0}^{\infty} \frac{(a)_n}{(2a+4)_n} \frac{x^n}{n!} \left\{ 1 + \frac{4n}{d} + \frac{6n\,(n-1)}{d(d+1)} + \frac{4n\,(n-1)\,(n-2)}{d(d+1)(d+2)} + \frac{n(n-1)\,(n-2)\,(n-3)}{d\,(d+1)\,(d+1)\,(d+3)} \right\} \\ &= e^{-x/2} \sum_{n=0}^{\infty} \frac{(a)_n}{(2a+4)_n} \frac{x^n}{n!} + \frac{4}{d} e^{-x/2} \sum_{n=1}^{\infty} \frac{(a)_n}{(2a+4)_n} \frac{x^n}{(n-1)!} \\ &\quad + \frac{6}{d(d+1)} \ e^{-x/2} \sum_{n=2}^{\infty} \frac{(a)_n}{(2a+4)_n} \frac{x^n}{(n-2)!} + \frac{4}{d(d+1)(d+2)} e^{-x/2} \sum_{n=3}^{\infty} \frac{(a)_n}{(2a+4)_n} \frac{x^n}{(n-3)!} \\ &\quad + \frac{1}{d(d+1)(d+2)(d+3)} \ e^{-x/2} \sum_{n=4}^{\infty} \frac{(a)_n}{(2a+4)_n} \frac{x^n}{(n-4)!}. \end{aligned}$$

Now replacing $n-1$ by N, $n-2$ by N, $n-3$ by N and $n-4$ by N in second, third, fourth and fifth series respectively, we have

$$\begin{aligned} S = e^{-x/2} & \sum_{n=0}^{\infty} \frac{(a)_n}{(2a+4)_n} \frac{x^n}{n!} \\ & + \frac{4}{d} \ e^{-x/2} \sum_{N=0}^{\infty} \frac{(a)_{N+1}}{(2a+4)_{N+1}} \ \frac{x^{N+1}}{N!} \\ & + \frac{6}{d\,(d+1)} \ e^{-x/2} \sum_{N=0}^{\infty} \frac{(a)_{N+2}}{(2a+4)_{N+2}} \frac{x^{N+2}}{N!} \\ & + \frac{4}{d(d+1)(d+2)} \ e^{-x/2} \sum_{N=0}^{\infty} \frac{(a)_{N+3}}{(2a+4)_{N+3}} \frac{x^{N+3}}{N!} \\ & + \frac{1}{d(d+1)(d+2)(d+3)} e^{-x/2} \sum_{N=0}^{\infty} \frac{(a)_{N+4}}{(2a+4)_{N+4}} \frac{x^{N+4}}{N!}. \end{aligned}$$

Using the results

$$\begin{aligned} (a)_{N+1} &= a(a+1)_N \\ (a)_{N+2} &= a\,(a+1)\,(a+2)_N \\ (a)_{N+3} &= a\,(a+1)\,(a+2)\,(a+3)_N \\ (a)_{N+4} &= a\,(a+1)\,(a+2)\,(a+3)\,(a+4)_N \end{aligned}$$

and after some simplification, we have

$$S = e^{-x/2} \sum_{n=0}^{\infty} \frac{(a)_n}{(2a+4)_n} \frac{x^n}{n!} + \frac{4ax}{d\,(2a+4)}\; e^{-x/2} \sum_{N=0}^{\infty} \frac{(a+1)_N}{(2a+5)_N} \frac{x^N}{N!}$$
$$+ \frac{6a\,(a+1)\,x^2}{d\,(d+1)\,(2a+4)\,(2a+5)}\; e^{-x/2} \sum_{N=0}^{\infty} \frac{(a+2)_N}{(2a+6)_N} \frac{x^N}{N!}$$
$$+ \frac{4a\,(a+1)\,(a+2)\,x^3}{d\,(d+1)\,(d+2)\,(2a+4)\,(2a+5)\,(2a+6)}\; e^{-x/2} \sum_{N=0}^{\infty} \frac{(a+3)_N}{(2a+7)_N} \frac{x^N}{N!}$$
$$+ \frac{a\,(a+1)\,(a+2)\,(a+3)x^4}{d\,(d+1)\,(d+2)\,(d+3)\,(2a+4)\,(2a+5)\,(2a+6)\,(2a+7)}\; e^{-x/2} \sum_{N=0}^{\infty} \frac{(a+4)_N}{(2a+8)_N} \frac{x^N}{N!}.$$

Finally summing up the series, we have

$$S = e^{-x/2}\; {}_1F_1\left[\begin{matrix} a \\ 2a+4 \end{matrix};\ x\right] + \frac{4ax}{d(2a+4)}\; e^{-x/2}\; {}_1F_1\left[\begin{matrix} a+1 \\ 2a+5 \end{matrix};\ x\right]$$
$$+ \frac{6a(a+1)x^2}{d(d+1)(2a+4)(2a+5)}\; e^{-x/2}\; {}_1F_1\left[\begin{matrix} a+2 \\ 2a+6 \end{matrix};\ x\right]$$
$$+ \frac{4a(a+1)(a+2)x^3}{d(d+1)(d+2)(2a+4)(2a+5)(2a+6)} e^{-x/2}\; {}_1F_1\left[\begin{matrix} a+3 \\ 2a+7 \end{matrix};x\right]$$
$$+ \frac{a(a+1)(a+2)(a+3)x^4}{d(d+1)(d+2)(d+3)(2a+4)(2a+5)(2a+6)(2a+7)}\; e^{-x/2}\; {}_1F_1\left[\begin{matrix} a+4 \\ 2a+8 \end{matrix};\ x\right].$$

Now it is easy to see that the first, second, third, fourth and fifth expressions appearing on the right hand side can be evaluated with the help of the known results (1.15), (1.14), (1.13), (1.12) and (1.4) respectively, and after some simplification, we arrive at the desired result (2.1). This completes the proof of (2.1).

§3. Special Cases

Setting $d = 2a$ in (2.1), we see that

$$c_1 = c_3 = 0,$$
$$c_2 = \frac{1}{2(2a+1)(2a+5)}, \text{ and}$$
$$c_4 = \frac{1}{16(2a+1)(2a+3)(2a+5)(2a+7)}.$$

So we have

$$e^{-x/2}\ {}_1F_1\left[\begin{matrix} a \\ 2a \end{matrix}\ ;\ x\right] = {}_0F_1\left[\begin{matrix} - \\ a+\frac{5}{2} \end{matrix}\ ;\ \frac{x^2}{16}\right] + \frac{x^2}{2(2a+1)(2a+5)}\ {}_0F_1\left[\begin{matrix} - \\ a+\frac{7}{2} \end{matrix}\ ;\ \frac{x^2}{16}\right]$$
$$+\frac{x^4}{16(2a+1)(2a+3)(2a+5)(2a+7)}\ {}_0F_1\left[\begin{matrix} - \\ a+\frac{9}{2} \end{matrix}\ ;\ \frac{x^2}{16}\right]. \qquad (3.1)$$

Now, it is not difficult to see that the right hand side of (3.1) equals to ${}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix}\ ;\ \frac{x^2}{16}\right]$ and thus we arrive at the Kummer's second theorem (1.4). Thus our main result (2.1) may be regarded as an extension of (1.4).

References

[1] W. N. Bailey, Products of Generalized Hypergeomeric Series, *Proc. London Math. Soc.,* 28 (1928), pp. 242–254.

[2] Y. S. Kim, M. A. Rakha and A. K. Rathie, Generalization of Kummer's second theorem with applications, *Comput. Math. and Math. Phys.,* 50(3) (2010), pp. 387–402.

[3] E. E. Kummer, *Ü*ber die hypergeometrische Reihe $1+\frac{\alpha.\beta x}{1.\gamma}+\frac{\alpha(\alpha+1)\beta(\beta+1)x^2}{1.2.\gamma(\gamma+1)}+\cdots$, *J. Reine Angew Math.,* 15(1836), pp. 39–83, 127–172.

[4] R. B. Paris, A Kummer type transformation for a ${}_2F_2$ Hypergeometric Functions, *J. Comput. Appl. Math.,* 173(2005), no.2, pp. 379–382.

[5] E. D. Rainville, Special Functions, The Macmillan Company, NewYork, 1960.

[6] M. A. Rakha and A. K. Rathie, On an extension of Kummer type II transformation, *TWMS J. App. Eng. Math.,* 4(1) (2014), pp. 85–88.

[7] M. A. Rakha, M. M. Awad and A. K. Rathie, On an extension of Kummer's second theorem, Abstact and Applied Analysis. Volume 2013, Article ID 128458, 6 pages.

[8] A. K. Rathie and J. Choi, Another proof of Kummer's second theorem, *Commun. Korean Math. Soc.,* 13(1998), no.4, pp. 933-936.

[9] A. K. Rathie and T. K. Pogany, New summation formula for ${}_3F_2(\frac{1}{2})$ and a Kummer type II transformation of ${}_2F_2(x)$, *Math. Commun.,* (2008), no.1, pp. 63-66.

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On $\nu-$Lindeloff space

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Abstract In this paper we discuss a few properties of $\nu-$lindeloff space and its relation with $\nu-$compact space and other such covering axioms.

Keywords Compact, Lindeloff, $\nu-$compact and $\nu-$Lindeloff spaces.
2000 Mathematics Subject Classification: 54C10; 54C08; 54C05.

§1. Introduction

Compactness and Lindeloffness are important tools in the theory of analysis especially in Topology and Functional Analysis. With this interesting idea of Compactness and Lindeloffness, many mathematicians defined and studied these concepts for every type of open sets defined in General Point set Topology. During the years 2009 and 2010 the authors of this present paper together with Smt. C. Sandhya studied about $\nu-$compact spaces and $\nu-$Lindeloff spaces.

Inspired with these developments, the authors of the present paper further studied few interreltion between $\nu-$lindeloff space and $\nu-$compact spaces. Verified few basic properties of these two spaces. Throughout the paper a space X means a topological space (X,τ).

§2. Preliminaries

Definition 2.1. *A subset $A \subset X$ is said to be*
(i) regular open if $A = (\overline{A})^o$
(ii) semi-open [ν[4]-open] if there exists an open [r-open] set U such that $U \subset A \subset \overline{U}$.
(iii) regular closed [semi-closed; $\nu-$closed[4]] of its complement is regular open [semi-open; $\nu-$open].

Definition 2.2. *Let $A \subset X$. A point $x \in X$ is said to be ω-accumulation [$\nu-$accumulation[4]] point of A if every regular-open [$\nu-$open] neighborhood of x intersects A and the union of A and the set of all ω-accumulation [$\nu-$accumulation[4]] points of A is called ω-closed [$\nu-$closed] set.*

Definition 2.3. *$A \subset X$ is said to be*

(i) Compact[nearly-compact; $\nu-$compact[4]] if every open[regular-open; $\nu-$open] cover of A has a finite subcover.

(ii) Countably compact[countably nearly-compact; countably $\nu-$compact[4]*] if every countable open [countable regular-open; countable $\nu-$open] in A has a finite sub cover.*

(iii) σ-compact[σ-nearly-compact; σ-$\nu-$compact[4]*] if A is the countable union of compact [nearly-compact; $\nu-$compact] spaces.*

(iv) Weak almost regular [Almost regular] iff for any point $a \in A$ and any regular-open set U containing a, there exist a regular-open [an open] set V such that $a \in V \subset \overline{V} \subset U$.

(v) Lindeloff [nearly-lindeloff; $\nu-$lindeloff[4]*] if every open [regular-open; $\nu-$open] cover of A has a countable subcover.*

(vi) σ-lindeloff [σ-nearly-lindeloff; σ-$\nu-$lindeloff[4]*] if A is the countable union of lindeloff [nearly-lindeloff; $\nu-$lindeloff] spaces.*

Lemma 2.1[4]**.** *If f is almost continuous then for each $A \subset Y$, $\overline{f^{-1}(A))} \subset f^{-1}(\overline{(A)})$.*

Note 1. *(i) Every compact space is locally lindeloff.*

(ii) Every lindeloff space is locally lindeloff.

§3. Relation between ν-compact spaces and ν-Lindeloff spaces

Remark 1. *We have the following*

(i) Every ν-compact space is a $\nu-$Lindeloff space

(ii) σ-nearly-compact space is σ-$\nu-$compact

(iii) σ-nearly-compact space is σ-$\nu-$lindeloff

(iv) σ-$\nu-$compact space is σ-$\nu-$lindeloff

Theorem 3.1. *Let A be r-open and if $A \subseteq X$ is a $\nu-$compact subset of X then the subspace $(A, \tau_{/A})$ is $\nu-$lindeloff.*

Proof. Assume A is $\nu-$compact. Let $\{G_i : i \in I\}$ be any $\nu-$open cover in $(A, \tau_{/A})$. Each G_i is $\nu-$open in A and $G_i \subseteq A \subseteq X$, each G_i is $\nu-$open in X. By $\nu-$compactness of X, this cover has a finite subcover. Hence $(A, \tau_{/A})$ is $\nu-$compact. Therefore $(A, \tau_{/A})$ is $\nu-$lindeloff.

Theorem 3.2.

(i) $\nu-$closed subset of a (countably) $\nu-$compact space is $\nu-$lindeloff

(ii) $\nu-$irresolute image of a (countably) $\nu-$compact space is $\nu-$lindeloff

(iii) countable product of (countably) $\nu-$compact spaces is $\nu-$lindeloff

(iv) countable union of (countably) $\nu-$compact spaces is $\nu-$lindeloff

Proof. (i) Let X be $\nu-$compact and $A \subset X$ be $\nu-$closed. Let $\{U_\alpha : \alpha \in \Delta\}$ be any $\nu-$open cover for A. Let $U_0 = X - A$ and $\Delta' = \Delta \cup \{0\}$. Then $\{U_\alpha : \alpha \in \Delta'\}$ is a $\nu-$open cover for X. Since X is $\nu-$compact, there exists a finite subset $\Delta'' \subset \Delta' \ni X \subset \bigcup_{\alpha \in \Delta''} U_\alpha$. Since $A \subset X$ and $A \cap U_0 = \phi$, also $\Delta'' - \{0\} \subset \Delta$. $A \subset \bigcup_{\alpha \in \Delta'' - \{0\}} U_\alpha$. Hence $\{U_\alpha : \alpha \in \Delta'' - \{0\}\}$ is a finite subcover of A from $\{U_\alpha : \alpha \in \Delta\}$. Hence A is $\nu-$compact and so ν-lindeloff.

If X is countably $\nu-$compact, Δ will be countable set.

(ii) Let $f : (X, \tau) \to (Y, \sigma)$ be $\nu-$irresolute and let X be $\nu-$compact. Let $\{V_i : i \in I\}$ be any $\nu-$open cover for f(X), then each V_i is $\nu-$open in f(X). Since f is $\nu-$ irresolute, each $f^{-1}(V_i)$ is $\nu-$open in X. By $\nu-$compactness of X, we have $X \subseteq \cup_{n=1}^{\infty} f^{-1}(V_n)$ implies $X \subseteq \cup_{i=1}^{n} f^{-1}(V_i)$. Thus $f(X) \subseteq \cup_{i=1}^{n} V_i$.

(iii) Let $\{X_n\}$ be a countable family of $\nu-$ compact spaces and let $X = \Pi_n X_n$. Let $\{U_j = \Pi_{\alpha\neq\alpha_{ij}} X_\alpha \times U_{\alpha 1j} \times ... \times U_{\alpha nj} : U_{\alpha ij}$ is $\nu-$open in $X_{\alpha ij}$ for each i = 1 to n, $j \in I\}$ be a $\nu-$open cover of $\Pi_\alpha X_\alpha$. Then $\{\Pi_i(u_j) : j \in I\}$ is a $\nu-$open cover of X_i. By Assumption, there exists a finite subfamily $\{\Pi_i(u_j) : j = 1$ to $n\}$ such that $X_i = \bigcup \Pi_i(U_j)$.

Case 1: If $\Pi_i X_i = \bigcup_{j=1}^{n} \Pi_i(U_j)$ then $\Pi_i X_i$ is $\nu-$compact and so $\nu-$lindeloff.

Case 2: If not, there exists atmost finite $\ell_1, \ell_2, ..., \ell_n$ such that $X_{\ell_s} = \bigcup_{k=1^s}^{m} \Pi_{\ell_s}(U_{k_s})$ for each $\ell_s \in \{\ell_1, \ell_2, ..., \ell_n\}$. Therefore $\Pi_\alpha X_\alpha = \bigcup_{k=1}^{n}(U_{jk}) \cup \bigcup_{k^1=1}^{n}(U_{jk}) \cup .... \cup \bigcup_{k^s=1}^{n}(U_{jk})$. Hence $\Pi_\alpha X_\alpha$ is $\nu-$compact and so ν-lindeloff.

Theorem 3.3. *$\nu-$continuous image of a (countably) $\nu-$compact space is lindeloff.*

Proof. Let X be $\nu-$compact and $f : X \to f(X)$ is $\nu-$continuous. Let $\{U_i : i \in I\}$ be any open cover for f(X) $\Rightarrow$ each U_i is open set in f(X) $\Rightarrow$ each $f^{-1}(U_i)$ is $\nu-$open set in X $\Rightarrow$ $\{f^{-1}(U_i) : i \in I\}$ form a $\nu-$open cover for X. By $\nu-$compactness of X we have $X \subset \bigcup_{i=1}^{n} f^{-1}(U_i) \Rightarrow f(X) \subset \bigcup_{i=1}^{n} U_i \Rightarrow \{U_i : i = 1$ to $n\}$ is a finite subcover for f(X). Hence f(X) is compact and so ν-lindeloff.

Note 2. *(i) Every $\nu-$compact space is locally $\nu-$lindeloff.*

(ii) Every $\nu-$lindeloff space is locally $\nu-$lindeloff.

Theorem 3.4. *If $f : (X, \tau) \to (Y, \sigma)$ is $\nu-$irresolute, $\nu-$open and X is locally $\nu-$compact, then so Y is ν-lindeloff.*

Proof. Let $y \in Y$. Then there exists $x \in X$ such that $f(x) = y$. Since X is locally $\nu-$compact x has a $\nu-$compact neighbourhood V. Then by $\nu-$irresolute, $\nu-$open of $f, f(V)$ is a $\nu-$compact neighbourhood of y. Hence Y is $\nu-$compct and so ν-lindeloff.

Corollary 3.1. *If $f : (X, \tau) \to (Y, \sigma)$ is $\nu-$irresolute, $\nu-$open and X is $\nu-$compact, then Y is locally $\nu-$lindeloff.*

Proof. Evident from theorems 3.3 and 3.4.

Theorem 3.5. *$A \subseteq X$ be r-open. If A is locally $\nu-$compact subset of X then the subspace $(A, \tau_{/A})$ is locally $\nu-$lindeloff.*

Proof. Evident from the definition 3.4 and Theorem 3.1.

Theorem 3.6.

(i) $\nu-$closed subset of a locally $\nu-$Compact space is locally $\nu-$lindeloff.

(ii) countable product of locally $\nu-$Compact spaces is locally $\nu-$lindeloff.

(iii) countable union of locally $\nu-$Compact spaces is locally $\nu-$lindeloff.

Proof. Evident from the definition 3.4, theorem 3.2 and note 2.

The proof of the following theorems is routine.

Theorem 3.7. *If A is ν-compact subspace of X, then A is ν-lindeloff relative to X.*

Theorem 3.8. *If $f : (X, \tau) \to (Y, \sigma)$ is almost continuous, X is ν-compact and $Y = f(X)$ then Y is ν-lindeloff.*

Proof. Let $\{V_\alpha\}$ be ν-open cover of Y, then for each α there exists a regular open set A_α such that $A_\alpha \subset V_\alpha \subset \overline{A_\alpha} \Rightarrow f^{-1}(A_\alpha) \subset f^{-1}(V_\alpha) \subset f^{-1}(\overline{A_\alpha}) = \overline{(f^{-1}(A_\alpha))} \Rightarrow \{f^{-1}(V_\alpha)\}$ is $\nu-$open cover of $X \Rightarrow \{f^{-1}(V_\alpha) \cup (f^{-1}(V_\alpha)^{-0}\}$ is ν-open cover for X $\Rightarrow$ there exists $n \in N$ such that $X = \cup_{i=1}^{n}\{f^{-1}(V_{\alpha i}) \cup (f^{-1}\overline{(V_{\alpha i})})^o\} = \cup_{i=1}^{n}\{(f^{-1}\overline{(V_{\alpha i})})^o\} \Rightarrow X \subset \cup_{i=1}^{n}(f^{-1}\overline{(A_{\alpha i})})^o \Rightarrow Y \subset \cup_{i=1}^{n}(\overline{(A_{\alpha i})})^o \Rightarrow Y \subset \cup_{i=1}^{n} A_{\alpha i} \Rightarrow Y \subset \cup_{i=1}^{n} V_{\alpha i}$.

Theorem 3.9. *If f is an almost continuous open mapping of a topological space X into* *a ν-compact space Y with $f^{-1}(f(A_\alpha)) \subset \overline{(A_\alpha)}$ for each regular open set A_α of X, then X is ν-lindeloff.*

Proof. Let $\{V_\alpha\}$ be ν-open cover of X, then $\{f(V_\alpha)\}$ is semi-open cover of Y so $\{f(V_\alpha) \cup (f(V_\alpha)^{-0}\}$ is ν-open cover for $Y \Rightarrow Y = \cup_{i=1}^n\{f(V_{\alpha i}) \cup f(V_{\alpha i})^{-o}\} = \cup_{i=1}^n\{f(V_{\alpha i})^{-o}\} \subset \cup_{i=1}^n(f(A_{\alpha i})^{-o}\}$, it follows that, there exists N such that $X = \cup_{i=1}^N(f^{-1}(f(A_{\alpha i}))^{-})^o$. By lemma 2.1 and hypothesis for f, X $= \cup_{i=1}^n(f(A_{\alpha i}))^{-o} \subset \cup_{i=1}^n(f^{-1}(f(A_{\alpha i})^{-o}) \subset \cup_{i=1}^n(A_{\alpha i})^{-o} = \cup_{i=1}^n(A_{\alpha i}) \subset \cup_{i=1}^n V_{\alpha i}$. Hence X is ν-compact and so ν-lindeloff.

Remark 2. *We have the following*

(i) Every locally ν-compact space is a locally $\nu-$Lindeloff space.

(ii) locally σ-nearly-compact space is locally σ-$\nu-$compact.

(iii) locally σ-nearly-compact space is locally σ-$\nu-$lindeloff.

(iv) locally σ-$\nu-$compact space is locally σ-$\nu-$lindeloff.

Remark 3. *We have the following*

(i) Every ν-compact space is a locally $\nu-$Lindeloff space.

(ii) σ-nearly-compact space is locally σ-$\nu-$compact.

(iii) σ-nearly-compact space is locally σ-$\nu-$lindeloff.

(iv) σ-$\nu-$compact space is locally σ-$\nu-$lindeloff.

§4. Relation between $\nu-$compact spaces and Lindeloff spaces:

Lemma 4.1. *If X is ν-compact and semiregular then X is lindeloff.*

Proof. Let $\{O_i : i \in I\}$ be an open cover of X. Since X is semiregular, there is a regular open basis $B \Rightarrow$ we have ν-open cover $\{B_i^j : O_i = \cup_i B_i^j$ for each i, where $B_i^j \in B\}$. By ν-compactness of X, $X \subset \cup_{k=1}^k B_{ik}^j \Rightarrow \cup_{k=1}^k O_k$. Therefore X is compact and so lindeloff.

Lemma 4.2. *If X is ν-compact and semiregular then X is locally lindeloff[resp: locally compact].*

Corollary 4.1. *Every nearly compact and semiregular space is lindeloff.*

Proof. Let $\{O_i : i \in I\}$ be an open cover of X. Since X is semiregular, there is a regular open basis B and we have ν-open cover $\{B_i^j : O_i = \cup_i B_i^j$ for each i, where $B_i^j \in B\}$. By nearly compactness of X, $X \subset \cup_{k=1}^k B_{ik}^j \Rightarrow \cup_{k=1}^k O_k$. Therefore X is compact and so lindeloff.

Corollary 4.2. *Every nearly compact and semiregular space is locally lindeloff[resp: locally compact].*

Theorem 4.1. *If $A \subset X$ is Almost ν-regular and compact, then $\overline{A}$ is ν-lindeloff.*

Proof. Let $\{U_i\}$ be any ν-open cover of A and let $x \in A$ be any point. For $x \in A$ there exists a ν-open set U_x containing x $\Rightarrow$ by almost ν-regularity there exists an open set V_x such that $x \in V_x \subset \overline{V_x} \subset U$. For $\{V_x\}$ forms a open cover and X is compact, $X = \cup_{i=1}^n V_{xi}$. Thus $\overline{A} \subseteq \overline{(\cup_{i=1}^n V_{xi})} = \cup_{i=1}^n \overline{(V_{xi})} \subseteq \cup_{i=1}^n U_{xi}$, which implies that $\overline{A}$ is ν-compact and so ν-lindeloff.

Corollary 4.3. *If $A \subset X$ is Almost ν-regular and compact, then $\overline{A}$ is locally ν-lindeloff[resp: locally ν-compact].*

Corollary 4.4. *If $A \subset X$ is Almost regular and compact, then $\overline{A}$ is ν-lindeloff.*

Proof. Let $\{U_i\}$ be any ν-open cover of A and let $x \in A$ be any point $\Rightarrow$. For $x \in A$ there exists a ν-open set U_x containing x. By almost-regularity there exists a regular open set V_x such that $x \in V_x \subset \overline{V_x} \subset U$. For $\{V_x\}$ forms a regular open cover and X is compact, $\{V_x\}$ forms an open cover and X is compact gives $X = \cup_{i=1}^{n} V_{xi}$. Thus $A^{-} \subseteq \overline{(\cup_{i=1}^{n} V_{xi})} = \cup_{i=1}^{n} \overline{(V_{xi})} \subseteq \cup_{i=1}^{n} U_{xi}$, $\Rightarrow \overline{A}$ is ν-compact and so ν-lindeloff.

Corollary 4.5. *If $A \subset X$ is Almost regular and compact, then $\overline{A}$ is locally ν-lindeloff[resp: locally ν-compact].*

Theorem 4.2. *If $A \subset X$ is weak almost regular and nearly compact, then $\overline{A}$ is lindeloff.*

Proof. Let $\{U_i\}$ be any open cover of A and let $x \in A$ be any point, then for $x \in A$ there exists a regular-open set U_x containing x. By weak almost regularity there exists a regular-open set V_x such that $x \in V_x \subset \overline{(V_x)} \subset U$. For $\{V_x\}$ forms an open cover and X is nearly compact there exists N such that $X = \cup_{i=1}^{N} V_{xi}$. Thus $\overline{A} \subseteq \overline{(\cup_{i=1}^{N} V_{xi})} = \cup_{i=1}^{N} (\overline{(V_{xi})}) \subseteq \cup_{i=1}^{N} U_{xi}$, which implies that $\overline{A}$ is compact and so lindeloff.

Corollary 4.6. *If $A \subset X$ is weak almost regular and nearly compact, then $\overline{A}$ is lcally lindeloff[resp: locally compact].*

Corollary 4.7. *If $A \subset X$ is weak almost regular and ν-compact, then $\overline{A}$ is lindeloff.*

Proof. Let $\{U_i\}$ be any open cover of A and let $x \in A$ be any point, then for $x \in A$ there exists a regular-open set U_x containing x and by weak almost regularity there exists a regular-open set V_x such that $x \in V_x \subset \overline{V_x} \subset U$. For $\{V_x\}$ forms a open cover and X is $\nu-$compact, $X = \cup_{i=1}^{n} V_{xi}$. Thus $\overline{A} \subseteq \overline{(\cup_{i=1}^{n} V_{xi})} = \cup_{i=1}^{n} \overline{(V_{xi})} \subseteq \cup_{i=1}^{n} U_{xi}$, which implies that $\overline{A}$ is compact and so lindeloff.

Corollary 4.8. *If $A \subset X$ is weak almost regular and ν-compact, then $\overline{A}$ is locally lindeloff[resp: locally compact].*

Theorem 4.3. *Every almost ν-regular and almost compact subset A of X is ν-lindeloff.*

Proof. Let $\{U_i\}$ be any ν-open cover of A and let $x \in A$ be any point.For $i_x \in I$ $x \in U_{ix}$ there exists an open set V_x such that $x \in V_x \subset \overline{V_x} \subset U_{ix}$. Now $\{V_x\}$ forms a open cover and X is almost compact, $A \subseteq \cup_{j=1}^{n} \overline{V_{xij}}$. Thus $\{U_{xij}\}_{j=1}^{n}$ is a finite subcovering of $\{U_i\}$. Hence A is ν-compact and so ν-lindeloff.

Corollary 4.9. *Every almost ν-regular and almost compact subset A of X is locally ν-lindeloff[resp: locally ν-compact].*

Theorem 4.4. *Every weak almost regular and nearly compact subset A of X is ν-lindeloff.*

Proof. Let $\{U_i\}$ be any ν-open cover of A and let $x \in A$ be any point, then there exists $i_x \in I$ such that $x \in U_{ix}$ then there exists a regular-open set V_x such that $x \in V_x \subset \overline{V_x} \subset U_{ix}$. Now $\{V_x\}$ forms a regular-open cover and X is nearly compact, $A \subseteq \cup_{j=1}^{n} \overline{V_{xij}}$. Thus $\{U_{xij}\}_{j=1}^{n}$ is a finite subcovering of $\{U_i\}$. Hence A is ν-compact and so ν-lindeloff.

Corollary 4.10. *Every weak almost regular and nearly compact subset A of X is locally ν-lindeloff[resp: locally ν-compact].*

Corollary 4.11. *Every weak almost regular and ν-compact subset A of X is lindeloff[resp: locally lindeloff; locally compact].*

Theorem 4.5. *If in X, there exists a dense weak almost regular, nearly compact subset A of X, then X is lindeloff.*

Corollary 4.12. *(i) If in X, there exists a dense weak almost regular, nearly compact subset A of X, then X is locally lindeloff[resp: locally compact].*

Theorem 4.6. *Let A be any dense almost ν-regular subset of X such that every ν-open covering of A is a ν-open covering of X. Then X is almost compact if and only if X is ν-compact.*

§5. Relation between $\nu-$lindeloff spaces and weakly compact spaces:

Theorem 5.1. *If X is weakly compact and almost regular, then X is ν-lindeloff.*

Proof. Let $\{V_i\}$ be any ν-open cover of X. For each $x \in X$, there exists $i_x \in I$ such that $x \in V_{ix}$. Since X is almost regular, there exists a regular open set G_{ix} such that $x \in G_{ix} \subset \overline{G_{ix}} \subset V_{ix}$. This implies $x \in G_{ix} \subset \overline{G_{ix}} \subset V_{ix}$ where G_{ix} are open. Since X is weakly compact, $X = \cup_{i=1}^{n} \overline{G_{ix}}$. Thus $X = \cup_{i=1}^{n} V_{ix}$. Hence X is ν-compact and so ν-lindeloff.

From note 2 we have the following theorem and we state without proof.

Theorem 5.2. *If X is weakly compact and almost regular, then X is locally ν-lindeloff[resp: locally ν-compact].*

Conclusion

In this paper we studied about relation between $\nu-$compact space and Lindelof and $\nu-$Lindelof and weakly Lindelof spaces and coverning properties of weak and strong forms of $\nu-$continuous maps.

References

[1] S. Balasubramanian, P. A. S. Vyjayanthi and C. Sandhya, $\nu-$compact spaces, *Sci. Magna,* 5(1) (2009), pp. 78–82.

[2] S. Balasubramanian, C. Sandhya and P. A. S. Vyjayanthi, Note on regularity and $\nu-$compactness, *International Journal of Contemporary Sciences,* 5 (2010), pp. 777–784.

[3] S. Balasubramanian, P. A. S. Vyjayanthi and C. Sandhya, Note on regularity and $\nu-$Lindeloffness, *Sci. Magna,* 6(2) (2010), pp. 58–66.

[4] C. Sandhya, The role of $\nu-$open sets in General topology, Ph.D Thesis, Dravidian University, A.P-India(2011).

[5] P. Aruna Swathi Vyjayanthi, The role of $\nu-$closed sets and its Related Topological Axioms, Ph.D Thesis, Dravidian University, A.P-India.

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Somewhat $\nu g-$closed mappings

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Abstract In this paper we discuss new type of closed mappings called Somewhat $\nu g-$closed mappings, its properties and interrelation with other Somewhat closed mappings are studied.

Keywords Somewhat closed mappings, Somewhat $\nu g-$closed mappings, Almost Somewhat $\nu g-$closed mappings.
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§1. Introduction

Mappings plays a vital role in the theory of General Topology, Functional Analysis and other related subjects. Out of which closed maps plays an impotant role. In this way many mathematicians introduced differnt version of closed mappings. In 1978, Long and Herrington used almost closedness using Singhal. In 1983 El-Deeb et. al defined preclosed maps. In 1986, Greenwood and Reilly used $\alpha-$closed maps. In 1990 Asit Kumar sen and P. Bhattacharya further studied properties of preclosed maps. In 2014 S. Balasubramanian introduced Somewhat closed functions, in the same year S. Balasubramanian, C. Sandhya and P. A. S. Vyjayanthi studied Somewhat ν-Closed functions, S. Balasubramanian, C. Sandhya and M. D. S. Saikumar studied somewhat rg-closed mappings and S. Balasubramanian and Ch. Chaitanya studied somewhat αg-closed mappings. Recently in the year 2015 S. Balasubramanian introduced and studied Somewhat *-closed functions[where * = r-; semi-; pre-; α-; β-; rα-; b-; γ-]and Somewhat $^\#$-closed functions[where $^\#$ = g-; sg-; gs-; pg-; gp-; βg-; gβ-; rαg-; rgα-].

Inspired with these developments, the author of the present paper further introduce Somewhat $\nu g-$closed mappings, almost Somewhat $\nu g-$closed mappings and Somewhat M-$\nu g-$closed mappings. Moreover basic properties and relationship with other types of such mappings are studied. Throughout the paper a space X means a topological space (X,τ).

§2. Preliminaries

Definition 2.1. *A function f is said to be*

(i) somewhat continuous[resp: somewhat b-continuous] if for $U \in \sigma$ and $f^{-1}(U) \neq \phi$, there exists an open[resp: b-open] set V in X such that $V \neq \phi$ and $V \subset f^{-1}(U)$.

(ii) somewhat open[resp: somewhat b-open] provided that if $U \in \tau$ and $U \neq \phi$, then there exists a proper open[resp: b-open] set V in Y such that $V \neq \phi$ and $V \subset f(U)$.

(iii) somewhat closed[resp: somewhat ν-closed; somewhat rg-closed; somewhat g-closed] provided that if $U \in C(\tau)$ and $U \neq \phi$, then there exists a non-empty proper closed[resp: ν-closed; rg-closed; g-closed] set V in Y such that $f(U) \subset V$.

(iv) somewhat *-closed[where * = r-; semi-; pre-; α-; β-; rα-; b-; γ-] provided that if $U \in C(\tau)$ and $U \neq \phi$, then there exists a non-empty proper $*$-closed set V in Y such that $f(U) \subset V$.

(v) somewhat $^\#$-closed[where $^\#$ = g-; sg-; gs-; pg-; gp-; βg-; gβ-; rαg-; rgα-] provided that if $U \in C(\tau)$ and $U \neq \phi$, then there exists a non-empty proper $\#$ closed set V in Y such that $f(U) \subset V$.

(vi) somewhat αg-closed provided that if $U \in C(\tau)$ and $U \neq \phi$, then there exists a non-empty proper αg-closed set V in Y such that $f(U) \subset V$.

Definition 2.2. *If X is a set and τ and σ are topologies on X, then τ is said to be equivalent [resp: b-equivalent] to σ provided if $U \in \tau$ and $U \neq \phi$, then there is an open[resp: b-open] set V in X such that $V \neq \phi$ and $V \subset U$ and if $U \in \sigma$ and $U \neq \phi$, then there is an open[resp: b-open] set V in (X, τ) such that $V \neq \phi$ and $U \supset V$.*

Definition 2.3. *$A \subset X$ is said to be dense in X if there is no proper closed set C in X such that $M \subset C \subset X$.*

§3. SOMEWHAT $\nu g-$CLOSED MAPS:

Definition 3.1. *A function f is said to be somewhat $\nu g-$closed provided that if U closed in X and $U \neq \phi$, then $\exists$ proper $V \in \nu GC(Y)$ and $V \neq \phi$ such that $f(U) \subset V$.*

Example 1. *Let $X = \{a, b, c\}, \tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b, c\}, X\}$. f defined by $f(a) = a, f(b) = c$ and $f(c) = b$ is somewhat $\nu g-$closed and somewhat closed.*

Example 2. *Let $X = \{a, b, c\}, \tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{a\}, X\}$. f defined by $f(a) = a, f(b) = c$ and $f(c) = a$ is somewhat $\nu g-$closed but not somewhat closed.*

Theorem 3.1. *Let f be a closed function and g somewhat $\nu g-$closed. Then $g \circ f$ is somewhat $\nu g-$closed.*

Theorem 3.2. *For a bijective function f, the following are equivalent:*

(i) f is somewhat $\nu g-$closed.

(ii) If C is a open subset of X, such that $f(C) \neq Y$, then there is a $\nu g-$open subset D of Y such that $D \neq Y$ and $D \subset f(C)$.

Proof. (i)$\Rightarrow$(ii): Let C be any open subset of X such that $f(C) \neq Y$. Then $X - C$ is closed in X and $X - C \neq \phi$. Since f is somewhat $\nu g-$closed, there exists a $\nu g-$closed set $V \neq \phi$ in Y such that $V \supset f(X - C)$. Put $D = Y - V$. Clearly D is $\nu g-$open in Y and we claim $D \neq Y$. If D = Y, then $V = \phi$, which is a contradiction. Since $V \supset f(X - C), D = Y - V \subset (Y - f(X - C)) = f(C)$.
(ii)$\Rightarrow$(i): Let U be any nonempty closed subset of X. Then $C = X - U$ is a open set in X and $f(X - U) = f(C) = Y - f(U)$ implies $f(C) \neq Y$. Therefore, by (ii), there is a $\nu g-$open set D of Y such that $D \neq Y$ and $f(C) \subset D$. Clearly $V = Y - D$ is a $\nu g-$closed set and $V \neq \phi$. Also, $V = Y - D \supset Y - f(C) = Y - f(X - U) = f(U)$.

Theorem 3.3. *The following statements are equivalent:*

(i) f is somewhat $\nu g-$closed.

(ii) If A is a $\nu g-$dense subset of Y, then $f^{-1}(A)$ is a dense subset of X.

Proof. (i)⇒(ii): Suppose A is a $\nu g-$dense set in Y. If $f^{-1}(A)$ is not dense in X, then there exists a closed set B in X such that $f^{-1}(A) \subset B \subset X$. Since f is somewhat $\nu g-$closed and $X - B$ is open, there exists a nonempty $\nu g-$closed set C in Y such that $C \subset f(X - B)$. Therefore, $C \subset f(X - B) \subset f(f^{-1}(Y - A)) \subset Y - A$. That is, $A \subset Y - C \subset Y$. Now, $Y - C$ is a $\nu g-$closed set and $A \subset Y - C \subset Y$. This implies that A is not a $\nu g-$dense set in Y , which is a contradiction. Therefore, $f^{-1}(A)$ is a dense set in X.

(ii)⇒(i): Suppose A is a nonempty open subset of X. We want to show that $\nu g(f(A))^o \neq \phi$. Suppose $\nu g(f(A))^o = \phi$. Then, $\nu g\overline{(f(A))} = Y$. Therefore, by (ii), $f^{-1}(Y - f(A))$ is dense in X. But $f^{-1}(Y - f(A)) \subset X - A$. Now, $X - A$ is closed. Therefore, $f^{-1}(Y - f(A)) \subset X - A$ gives $X = \overline{(f^{-1}(Y - f(A)))} \subset X - A$. This implies that $A = \phi$, which is contrary to $A \neq \phi$. Therefore, $\nu g(f(A))^o \neq \phi$. Hence f is somewhat $\nu g-$closed.

Theorem 3.4. *Let f be somewhat $\nu g-$closed and A be any r-closed subset of X. Then $f_{|A} : (A; \tau_{|A}) \to (Y, \sigma)$ is somewhat $\nu g-$closed.*

Proof. Let U is closed in $\tau_{|A}$ such that $U \neq \phi$. Since U is r-closed in A and A is closed in X, U is r-closed in X and since f is somewhat $\nu g-$closed, $\exists V \in \nu GC(Y)$, $\ni f(U) \subset V$. Thus, for any closed set U of A with $U \neq \phi$, $\exists V \in \nu GC(Y) \ni f(U) \subset V$ which implies $f_{|A}$ is somewhat $\nu g-$closed.

Theorem 3.5. *Let f be a function and $X = A \cup B$, where $A, B \in RO(X)$. If the restriction functions $f_{|A}$ and $f_{|B}$ are somewhat $\nu g-$closed, then f is somewhat $\nu g-$closed.*

Proof. Let U be any closed subset of X such that $U \neq \phi$. Since $X = A \cup B$, either $A \cap U \neq \phi$ or $B \cap U \neq \phi$ or both $A \cap U \neq \phi$ and $B \cap U \neq \phi$. Since U is closed in X, U is closed in both A and B.

Case (i): If $A \cap U \neq \phi$, where $U \cap A$ is closed in A. Since $f_{|A}$ is somewhat $\nu g-$closed, $\exists V \in \nu GC(Y) \ni f(U \cap A) \subset f(U) \subset V$, which implies that f is somewhat $\nu g-$closed.

Case (ii): If $B \cap U \neq \phi$, where $U \cap B$ is closed in B. Since $f_{|B}$ is somewhat $\nu g-$closed, $\exists V \in \nu GC(Y) \ni f(U \cap B) \subset f(U) \subset V$, which implies that f is somewhat $\nu g-$closed.

Case (iii): If both $A \cap U \neq \phi$ and $B \cap U \neq \phi$. Then by case (i) and (ii) f is somewhat $\nu g-$closed.

Remark 1. *Two topologies τ and σ for X are said to be $\nu g-$equivalent if and only if the identity function $f : (X, \tau) \to (Y, \sigma)$ is somewhat $\nu g-$closed in both directions.*

Theorem 3.6. *Let $f : (X, \tau) \to (Y, \sigma)$ be a somewhat open function. Let τ^* and σ^* be topologies for X and Y, respectively such that τ^* is equivalent to τ and σ^* is $\nu g-$equivalent to σ. Then $f : (X; \tau^*) \to (Y; \sigma^*)$ is somewhat $\nu g-$closed.*

Note 1. *From the definition 2.1 we have the following implication diagram among the following somewhat closed mappings* *(i)* $swt.g.closed \quad swt.gs.closed$

$\downarrow \quad \downarrow \quad \nwarrow$

$swt.rg\alpha.closed \rightarrow swt.rg.closed \rightarrow \mathbf{swt.\nu g.closed} \leftarrow swt.sg.closed \leftarrow swt.\beta g.closed$

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$

$\nearrow swt.r\alpha.closed \rightarrow swt.\nu.closed \searrow \quad \uparrow \quad \uparrow$

$swt.r.closed \rightarrow swt.\pi.closed \rightarrow swt.closed \rightarrow swt.\alpha.closed \rightarrow swt.s.closed \rightarrow swt.\beta.closed$

$\swarrow \quad \downarrow \searrow \quad \searrow$

$swt.\pi g.closed \quad swt.p.closed \rightarrow swt.\omega.closed \not\rightarrow swt.g\alpha.closed$

$\searrow \quad \searrow$

$swt.gpr.closed \leftarrow swt.gp.closed \leftarrow swt.pg.closed \quad swt.r\omega.closed$ *None is reversible.*

(ii) If $\nu GC(Y) = RC(Y)$, then the reverse relations hold for all slightly closed maps.

$swt.g.closed \quad swt.gs.closed$

$\updownarrow \quad \updownarrow \quad \searrow\nwarrow$

$swt.rg\alpha.closed \leftrightarrow swt.rg.closed \leftrightarrow \mathbf{swt.\nu g.closed} \leftrightarrow swt.sg.closed \leftrightarrow swt.\beta g.closed$

$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$

$\swarrow\nearrow swt.r\alpha.closed \leftrightarrow swt.\nu.closed \nwarrow\searrow \quad \updownarrow \quad \updownarrow$

$swt.r.closed \leftrightarrow swt.\pi.closed \leftrightarrow swt.closed \leftrightarrow swt.\alpha.closed \leftrightarrow swt.s.closed \leftrightarrow swt.\beta.closed$

From above note 1 we have the following.

Theorem 3.7. *Let f be a closed function and g swt-[swt-r-; swt-rα-; swt-ν-; swt-α-; swt-semi-; swt-β-; swt-π-]closed. Then $g \circ f$ is somewhat $\nu g-$closed.*

Theorem 3.8. *Let f be a closed function and g swt-g-[swt-rg-; swt-rgα-; swt-sg-; swt-gs-; swt-βg-]closed. Then $g \circ f$ is somewhat $\nu g-$closed.*

§4. ALMOST SOMEWHAT $\nu g-$CLOSED MAPS:

Definition 4.1. *A function f is said to be almost somewhat-$\nu g-$closed provided that if $U \in RC(\tau)$ and $U \neq \phi$, then $\exists$ proper $V \in \nu GC(Y)$ and $V \neq \phi$ such that $f(U) \subset V$.*

Example 3. *Let $X = \{a, b, c\}, \tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{a\}, X\}$. f defined by $f(a) = a, f(b) = c$ and $f(c) = a$ is almost somewhat $\nu g-$closed and but not somewhat closed and almost somewhat closed.*

Theorem 4.2. *For a bijective function f, the following are equivalent:*

(i) f is almost somewhat-$\nu g-$closed.

(ii) If C is a r-open subset of X, such that $f(C) \neq Y$, then there is a $\nu g-$open subset D of Y such that $D \neq Y$ and $D \subset f(C)$.

Proof. (i)$\Rightarrow$(ii): Let C be any r-open subset of X such that $f(C) \neq Y$. Then $X - C$ is r-closed in X and $X - C \neq \phi$. Since f is somewhat $\nu g-$closed, there exists a $\nu g-$closed set $V \neq \phi$ in Y such that $V \supset f(X - C)$. Put $D = Y - V$. Clearly D is $\nu g-$open in Y and we claim $D \neq Y$. If D = Y, then $V = \phi$, which is a contradiction. Since $V \supset f(X - C), D = Y - V \subset (Y - f(X - C)) = f(C)$.

(ii)$\Rightarrow$(i): Let U be any nonempty r-closed subset of X. Then $C = X - U$ is an r-open set in X and $f(X - U) = f(C) = Y - f(U)$ implies $f(C) \neq Y$. Therefore, by (ii), there is a $\nu g-$open set D of Y such that $D \neq Y$ and $f(C) \subset D$. Clearly $V = Y - D$ is a $\nu g-$closed set and $V \neq \phi$. Also, $V = Y - D \supset Y - f(C) = Y - f(X - U) = f(U)$.

Theorem 4.3. *The following statements are equivalent:*

(i) f is almost somewhat-$\nu g-$closed.

(ii)If A is a $\nu g-$dense subset of Y, then $f^{-1}(A)$ is a $\nu g-$dense subset of X.

Proof. (i)$\Rightarrow$(ii): Suppose A is a $\nu g-$dense set in Y. If $f^{-1}(A)$ is not $\nu g-$dense in X, then there exists a $\nu g-$closed set B in X such that $f^{-1}(A) \subset B \subset X$. Since f is somewhat $\nu g-$closed and $X - B$ is $\nu g-$closed, there exists a nonempty $\nu g-$closed set C in Y such that $C \subset f(X - B)$. Therefore, $C \subset f(X - B) \subset f(f^{-1}(Y - A)) \subset Y - A$. That is, $A \subset Y - C \subset Y$. Now, $Y - C$ is a $\nu g-$closed set and $A \subset Y - C \subset Y$. This implies that A is not a $\nu g-$dense set in Y, which is a contradiction. Therefore, $f^{-1}(A)$ is a $\nu g-$dense set in X.

(ii)$\Rightarrow$(i): Suppose A is a nonempty $\nu g-$closed subset of X. We want to show that $\nu g(f(A))^o \neq \phi$. Suppose $\nu g(f(A))^o = \phi$. Then, $\nu g\overline{(f(A))} = Y$. Therefore, by (ii), $f^{-1}(Y - f(A))$ is $\nu g-$dense in X. But $f^{-1}(Y - f(A)) \subset X - A$. Now, $X - A$ is $\nu g-$closed. Therefore, $f^{-1}(Y - f(A)) \subset X - A$ gives $X = \overline{(f^{-1}(Y - f(A)))} \subset X - A$. This implies that $A = \phi$, which is contrary to $A \neq \phi$. Therefore, $\nu g(f(A))^o \neq \phi$. Hence f is almost somewhat-$\nu g-$closed.

Theorem 4.4. *Let f be almost somewhat-$\nu g-$closed and A be any r-closed subset of X. Then $f_{|A} : (A; \tau_{|A}) \to (Y, \sigma)$ is almost somewhat-$\nu g-$closed.*

Proof. Let $U \in RC(\tau_{|A})$ such that $U \neq \phi$. Since U is r-closed in A and A is closed in X, U is r-closed in X and since f is almost somewhat-$\nu g-$closed, $\exists V \in \nu GC(Y)$, $\ni f(U) \subset V$. Thus, for any $U \in RC(A)$ with $U \neq \phi$, $\exists V \in \nu GC(Y) \ni f(U) \subset V$ which implies $f_{|A}$ is almost somewhat-$\nu g-$closed.

Theorem 4.5. *Let f be a function and $X = A \cup B$, where $A, B \in RO(X)$. If the restriction functions $f_{|A}$ and $f_{|B}$ are almost somewhat-$\nu g-$closed, then f is almost somewhat-$\nu g-$closed.*

Proof. Let $U \in RC(X) \ni U \neq \phi$. Since $X = A \cup B$, either $A \cap U \neq \phi$ or $B \cap U \neq \phi$ or both $A \cap U \neq \phi$ and $B \cap U \neq \phi$. $U \in RC(X)$, $U \in RC(A)$ and $U \in RC(B)$.

Case (i): If $A \cap U \neq \phi$, where $U \cap A \in RC(A)$. Since $f_{|A}$ is almost somewhat-$\nu g-$closed, $\exists V \in \nu GC(Y) \ni f(U \cap A) \subset f(U) \subset V$, which implies that f is almost somewhat-$\nu g-$closed.

Case (ii): If $B \cap U \neq \phi$, where $U \cap B \in RC(B)$. Since $f_{|B}$ is almost somewhat-$\nu g-$closed, $\exists V \in \nu GC(Y) \ni f(U \cap B) \subset f(U) \subset V$, which implies that f is almost somewhat-$\nu g-$closed.

Case (iii): If both $A \cap U \neq \phi$ and $B \cap U \neq \phi$. Then by case (i) and (ii) f is almost somewhat-$\nu g-$closed.

Remark 2. *Two topologies τ and σ for X are said to be $\nu g-$equivalent if and only if the identity function $f : (X, \tau) \to (Y, \sigma)$ is almost somewhat-$\nu g-$closed in both directions.*

Theorem 4.6. *Let $f : (X, \tau) \to (Y, \sigma)$ be a somewhat open function. Let τ^* and σ^* be topologies for X and Y, respectively such that τ^* is equivalent to τ and σ^* is $\nu g-$equivalent to σ. Then $f : (X; \tau^*) \to (Y; \sigma^*)$ is almost somewhat-$\nu g-$closed.*

Note 2. *From the definition 2.1 we have the following implication diagram among the following almost somewhat closed mappings*

$$
\begin{array}{c}
\text{(i) } al.swt.g.closed \quad al.swt.gs.closed \\
\downarrow \qquad \downarrow \quad \nwarrow \\
al.swt.rg\alpha.closed \rightarrow al.swt.rg.closed \rightarrow \mathbf{al.swt.}\nu g\mathbf{.closed} \leftarrow al.swt.sg.closed \leftarrow al.swt.\beta g.closed \\
\uparrow \qquad \uparrow \qquad \uparrow \qquad \qquad \uparrow \qquad \uparrow \\
\nearrow al.swt.r\alpha.closed \rightarrow al.swt.\nu.closed \searrow \qquad \uparrow \qquad \uparrow \\
al.swt.r.closed \rightarrow al.swt.\pi.closed \rightarrow al.swt.closed \rightarrow al.swt.\alpha.closed \rightarrow al.swt.s.closed \rightarrow al.swt.\beta.closed \\
\swarrow \qquad \downarrow \searrow \qquad \searrow \\
al.swt.\pi g.closed \quad al.swt.p.closed \rightarrow al.swt.\omega.closed \not\leftrightarrow al.swt.g\alpha.closed \\
\searrow \qquad \searrow \\
al.swt.gpr.closed \leftarrow al.swt.gp.closed \leftarrow al.swt.pg.closed \qquad al.swt.r\omega.closed \text{ None is reversible.}
\end{array}
$$

$$
\begin{array}{c}
\text{(ii) } swt.g.closed \quad swt.gs.closed \\
\downarrow \qquad \downarrow \quad \nwarrow \\
swt.rg\alpha.closed \rightarrow swt.rg.closed \rightarrow \mathbf{al.swt.}\nu g\mathbf{.closed} \leftarrow swt.sg.closed \leftarrow swt.\beta g.closed \\
\uparrow \qquad \uparrow \qquad \uparrow \qquad \qquad \uparrow \qquad \uparrow \\
\nearrow swt.r\alpha.closed \rightarrow swt.\nu.closed \searrow \qquad \uparrow \qquad \uparrow \\
swt.r.closed \rightarrow swt.\pi.closed \rightarrow swt.closed \rightarrow swt.\alpha.closed \rightarrow swt.s.closed \rightarrow swt.\beta.closed \\
\swarrow \qquad \downarrow \searrow \qquad \searrow \\
swt.\pi g.closed \quad swt.p.closed \rightarrow swt.\omega.closed \not\leftrightarrow swt.g\alpha.closed \\
\searrow \qquad \searrow \\
swt.gpr.closed \leftarrow swt.gp.closed \leftarrow swt.pg.closed \qquad swt.r\omega.closed \text{ None is reversible.}
\end{array}
$$

(iii) If $\nu GC(Y) = RC(Y)$, then the reverse relations hold for all almost slightly closed maps.

$$
\begin{array}{c}
al.swt.g.closed \quad al.swt.gs.closed \\
\updownarrow \qquad \updownarrow \quad \searrow\nwarrow \\
al.swt.rg\alpha.closed \leftrightarrow al.swt.rg.closed \leftrightarrow \mathbf{al.swt.}\nu g\mathbf{.closed} \leftrightarrow al.swt.sg.closed \leftrightarrow al.swt.\beta g.closed \\
\updownarrow \qquad \updownarrow \qquad \updownarrow \qquad \qquad \updownarrow \qquad \updownarrow \\
\swarrow\nearrow al.swt.r\alpha.closed \leftrightarrow al.swt.\nu.closed \nwarrow\searrow \qquad \updownarrow \qquad \updownarrow \\
al.swt.r.closed \leftrightarrow al.swt.\pi.closed \leftrightarrow al.swt.closed \leftrightarrow al.swt.\alpha.closed \leftrightarrow al.swt.s.closed \leftrightarrow al.swt.\beta.closed
\end{array}
$$

From above note 2 we have the following.

Theorem 4.7. *Let f be a closed function and g al-swt-[al-swt-r-; al-swt-rα-; al-swt-ν-; al-swt-α-; al-swt-semi-; al-swt-β-; al-swt-π-]closed. Then $g \circ f$ is almost somewhat $\nu g-$closed.*

Corollary 4.1. *Let f be a closed function and g swt-[swt-r-; swt-rα-; swt-ν-; swt-α-; swt-semi-; swt-β-; swt-π-]closed. Then $g \circ f$ is almost somewhat $\nu g-$closed.*

Theorem 4.8. *Let f be a closed function and g al-swt-g-[al-swt-rg-; al-swt-rgα-; al-swt-sg-; al-swt-gs-; al-swt-βg-]closed. Then $g \circ f$ is almost somewhat $\nu g-$closed.*

Corollary 4.2. *Let f be a closed function and g swt-g-[swt-rg-; swt-rgα-; swt-sg-; swt-gs-; swt-βg-]closed. Then $g \circ f$ is almost somewhat $\nu g-$closed.*

§5. SOMEWHAT M-$\nu g-$CLOSED MAPS:

Definition 5.1. *A function f is said to be somewhat M-$\nu g-$closed provided that if $U \in \nu GC(\tau)$ and $U \neq \phi$, then $\exists$ proper $V \in \nu GC(Y)$ and $V \neq \phi$ such that $f(U) \subset V$.*

Example 4. *Let $X = \{a, b, c\}, \tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{a\}, X\}$. f defined by $f(a) = a, f(b) = c$ and $f(c) = a$ is somewhat M-$\nu g-$closed.*

Theorem 5.1. *Let f be a closed function and g somewhat M-$\nu g-$closed. Then $g \circ f$ is somewhat M-$\nu g-$closed.*

Theorem 5.2. *For a bijective function f, the following are equivalent:*

(i) f is somewhat M-$\nu g-$closed.

(ii) If C is a $\nu g-$open subset of X, such that $f(C) \neq Y$, then there is a $\nu g-$open subset D of Y such that $D \neq Y$ and $D \subset f(C)$.

Proof. (i)$\Rightarrow$(ii): Let C be any $\nu g-$open subset of X such that $f(C) \neq Y$. Then $X - C$ is $\nu g-$closed in X and $X - C \neq \phi$. Since f is somewhat $\nu g-$closed, there exists a $\nu g-$closed set $V \neq \phi$ in Y such that $V \supset f(X - C)$. Put $D = Y - V$. Clearly D is $\nu g-$open in Y and we claim $D \neq Y$. If D = Y, then $V = \phi$, which is a contradiction. Since $V \supset f(X - C), D = Y - V \subset (Y - f(X - C)) = f(C)$.

(ii)$\Rightarrow$(i): Let U be any nonempty $\nu g-$closed subset of X. Then $C = X - U$ is a $\nu g-$open set in X and $f(X - U) = f(C) = Y - f(U)$ implies $f(C) \neq Y$. Therefore, by (ii), there is a $\nu g-$open set D of Y such that $D \neq Y$ and $f(C) \subset D$. Clearly $V = Y - D$ is a $\nu g-$closed set and $V \neq \phi$. Also, $V = Y - D \supset Y - f(C) = Y - f(X - U) = f(U)$.

Theorem 5.3. *The following statements are equivalent:*

(i) f is somewhat M-$\nu g-$closed.

(ii)If A is a $\nu g-$dense subset of Y, then $f^{-1}(A)$ is a $\nu g-$dense subset of X.

Proof. (i)$\Rightarrow$(ii): Suppose A is a $\nu g-$dense set in Y. If $f^{-1}(A)$ is not $\nu g-$dense in X, then there exists a $\nu g-$closed set B in X such that $f^{-1}(A) \subset B \subset X$. Since f is somewhat $\nu g-$closed and $X - B$ is $\nu g-$closed, there exists a nonempty $\nu g-$closed set C in Y such that $C \subset f(X - B)$. Therefore, $C \subset f(X - B) \subset f(f^{-1}(Y - A)) \subset Y - A$. That is, $A \subset Y - C \subset Y$. Now, $Y - C$ is a $\nu g-$closed set and $A \subset Y - C \subset Y$. This implies that A is not a $\nu g-$dense set in Y , which is a contradiction. Therefore, $f^{-1}(A)$ is a $\nu g-$dense set in X.

(ii)$\Rightarrow$(i): Suppose A is a nonempty $\nu g-$closed subset of X. We want to show that $\nu g(f(A))^o \neq \phi$. Suppose $\nu g(f(A))^o = \phi$. Then, $\nu g\overline{(f(A))} = Y$. Therefore, by (ii), $f^{-1}(Y - f(A))$ is $\nu g-$dense in X. But $f^{-1}(Y - f(A)) \subset X - A$. Now, $X - A$ is $\nu g-$closed. Therefore, $f^{-1}(Y - f(A)) \subset X - A$ gives $X = \overline{(f^{-1}(Y - f(A)))} \subset X - A$. This implies that $A = \phi$, which is contrary to $A \neq \phi$. Therefore, $\nu g(f(A))^o \neq \phi$. Hence f is somewhat M-$\nu g-$closed.

Theorem 5.4. *Let f be somewhat M-$\nu g-$closed and A be any r-closed subset of X. Then $f_{|A} : (A; \tau_{|A}) \rightarrow (Y, \sigma)$ is somewhat M-$\nu g-$closed.*

Proof. Let $U \in \nu GC(\tau_{|A})$ such that $U \neq \phi$. Since U is r-closed in A and A is closed in X, U is r-closed in X and since f is somewhat M-$\nu g-$closed, $\exists V \in \nu GC(Y), \ni f(U) \subset V$. Thus, for any $U \in \nu GC(A)$ with $U \neq \phi$, $\exists V \in \nu GC(Y) \ni f(U) \subset V$ which implies $f_{|A}$ is somewhat M-$\nu g-$closed.

Theorem 5.5. *Let f be a function and $X = A \cup B$, where $A, B \in RO(X)$. If the restriction functions $f_{|A}$ and $f_{|B}$ are somewhat M-$\nu g-$closed, then f is somewhat M-$\nu g-$closed.*

Proof. Let $U \in \nu GC(X) \ni U \neq \phi$. Since $X = A \cup B$, either $A \cap U \neq \phi$ or $B \cap U \neq \phi$ or both $A \cap U \neq \phi$ and $B \cap U \neq \phi$. Since $U \in \nu GC(X)$, $U \in \nu GC(A)$ and $U \in \nu GC(B)$.

Case (i): If $A \cap U \neq \phi$, where $U \cap A \in RC(A)$. Since $f_{|A}$ is somewhat M-$\nu g-$closed, $\exists V \in \nu GC(Y) \ni f(U \cap A) \subset f(U) \subset V$, which implies that f is somewhat M-$\nu g-$closed.

Case (ii): If $B \cap U \neq \phi$, where $U \cap B \in RC(B)$. Since $f_{|B}$ is somewhat M-$\nu g-$closed, $\exists V \in \nu GC(Y) \ni f(U \cap B) \subset f(U) \subset V$, which implies that f is somewhat M-$\nu g-$closed.

Case (iii): If both $A \cap U \neq \phi$ and $B \cap U \neq \phi$. Then by case (i) and (ii) f is somewhat M-$\nu g-$closed.

Remark 3. *Two topologies τ and σ for X are said to be $\nu g-$equivalent if and only if the identity function $f\colon (X, \tau) \to (Y, \sigma)$ is somewhat M-$\nu g-$closed in both directions.*

Theorem 5.6. *Let $f\colon (X, \tau) \to (Y, \sigma)$ be a somewhat closed function. Let τ^* and σ^* be topologies for X and Y, respectively such that τ^* is equivalent to τ and σ^* is $\nu g-$equivalent to σ. Then $f\colon (X; \tau^*) \to (Y; \sigma^*)$ is somewhat M-$\nu g-$closed.*

Conclusion

In this paper we studied basic properties of somewhat $\nu g-$closed; almost somewhat $\nu g-$closed and somewhat M-$\nu g-$closed maps. Interrelation among different types of such functions are studied.

References

[1] D. Andrijevic, On b-open sets, *Math. Vesnik,* 48 (1996), pp. 59–63.

[2] S. Balasubramanian, Somewhat closed functions, *International Journal of Mathematical Archive,* 5(10) (2014), pp. 77-80.

[3] S. Balasubramanian, C. Sandhya and P. A. S. Vyjayanthi, Somewhat ν-Closed functions, *Proc. National Workshop on Latest developments in Algebra and its applications*, T. J. P. S. College, Guntur (2014), pp. 188-193.

[4] S. Balasubramanian, C. Sandhya and M. D. S. Saikumar, Somewhat rg-closed mappings, U.G.C APSCHE sponsored Nat. Seminar in Latest developments in Mathematics and its Applications, Acharya Nagarjuna University, Guntur, Andhrapradesh on Dec 22-23, 2014. (Abstract)

[5] S. Balasubramanian and Ch. Chaitanya, Somewhat αg-closed mappings, *International journal of Advanced scientific and Technical Research*, 4(6) (2014), pp. 493–498.

[6] S. Balasubramanian, Somewhat *-closed functions, *International Journal of Mathematical Archive,* 6(6) (2015), pp. 176–181.

[7] S. Balasubramanian, Somewhat $^\#$-closed functions, *International Journal of Research in Pure Algebra,* 5(6) (2015), pp. 88–93.

[8] K. R. Gentry and H. B. Hoyle, Somewhat continuous functions, *Czechslovak Math. J.,* 21 (1971), pp. 5–12.

[9] T. Noiri and N. Rajesh, Somewhat b-continuous functions, *J. Adv. Res. in Pure Math.,* 3(3) (2011), pp. 1–7.

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The mean value of $\widetilde{P}(n)$ over cube-full numbers

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Abstract Let $n > 1$ be an integer, $\widetilde{P}(n)$ is the exponential divisor function. In this paper, we shall investigate the mean value of $\widetilde{P}(n)$ over cube-full integers, that is,

$$\sum_{\substack{n\leq x \\ n\ is\ cube-full}} \widetilde{P}(n) = \sum_{n\leq x} \widetilde{P}(n) f_3(n).$$

where $f_3(n)$ is the characteristic function of cube-full integers, i.e.

$$f_3(n) = \begin{cases} 1, & n \text{ is cube-full;} \\ 0, & \text{otherwise.} \end{cases}$$

Keywords Divisor problem, Dirichlet convolution method, Mean value.

1. Introduction and preliminaries

An integer $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ is called $k - full$ number if all the exponents $a_1 \geq k$, $a_2 \geq k, \cdots, a_r \geq k$. When $k = 3$, n is called $cube - full$ integer. Let $f_3(n)$ be the characteristic function of $cube - full$ integers, i.e.

$$f_3(n) = \begin{cases} 1, & n \text{ is cube-full;} \\ 0, & \text{otherwise.} \end{cases}$$

In 1982, M.V.Subbarao[1] gave the definition of the exponential divisor, i.e. $n > 1$ is an integer and $n = \prod_{i=1}^{r} p_i^{a_i}$, $d = \prod_{i=1}^{r} p_i^{c_i}$, if $c_i \mid a_i$, $i = 1, 2, \cdots, r$, then d is an exponential divisor of n. We denote $d \mid_e n$. Two integers $n, m > 1$ have common exponential divisors if they have the same prime factors and in this case. i.e. for $n = \prod_{i=1}^{r} p_i^{a_i}$, $m = \prod_{i=1}^{r} p_i^{b_i}, a_i, b_i \geq 1$ $(1 \leq i \leq r)$, the greatest common exponential divisor of n and m is $(n, m)_e = \prod_{i=1}^{r} p_i^{(a_i, b_i)}$. Here $(1, 1)_e = 1$ by convention and $(1, m)_e$ does not exist for $m > 1$.

The integers $n, m > 1$ are called exponentially coprime, if they have the same prime factors and $(a_i, b_i) = 1$ for every $1 \leq i \leq r$, with the notation of above. In this case $(n, m)_e = k(n) =$

1This work is supported by Natural Science Foundation of China (Grant No:11001154), and Natural Science Foundation of Shandong Province(No: ZR2015AM010).

$k(m)$. 1 and $m>1$ are not exponentially coprime. Let $\widetilde{P}(n)=\sum\limits_{1\le j\le n}(j,n)_e$. Obviously $\widetilde{P}(n)$ is multiplicative and for every prime p, $\widetilde{P}(p^\alpha)=\sum\limits_{1\le c\le\alpha}p^{(c,\alpha)}=\sum\limits_{d|\alpha}p^d\varphi(\alpha/d)$, here $\widetilde{P}(p^2)=p+p^2$, $\widetilde{P}(p^3)=2p+p^3$, $\widetilde{P}(p^4)=2p+p^2+p^4$, $\widetilde{P}(p^5)=4p+p^5$, $\widetilde{P}(p^6)=2p+2p^2+p^3+p^6$, $\widetilde{P}(p^7)=6p+p^7$, $\widetilde{P}(p^8)=4p+2p^2+p^4+p^8$.

Many authors have investigated the properties of the exponential divisor function $\widetilde{P}(n)$. Recently L. Toth[2] proved the following result:

$$\sum_{n\le x}\widetilde{P}(n)=cx^2+O(x\log^{5/3}x),$$

where the constant c is given by $c=\frac{1}{2}\prod\limits_{P}(1+\sum\limits_{\alpha=2}^{\infty}\widetilde{P}(p^\alpha-p\widetilde{P}(p^{\alpha-1}))/p^{2\alpha})$. For k=2, S. Li[3] proved that

$$\begin{aligned}\sum_{\substack{n\le x\\ n\ is\ square-full}}\widetilde{P}(n) &= \frac{1}{3}\frac{\zeta(\frac{3}{2})H(\frac{1}{2})}{\zeta(3)}x^{3/2}+\frac{1}{4}\frac{\zeta(\frac{2}{3})H(\frac{1}{3})}{\zeta(2)}x^{4/3}\\ &\quad +O(x^{7/6}\exp(-D(\log x)^{3/5}(\log\log x)^{-1/5})),\end{aligned}$$

where $H(s):=\sum\limits_{n=1}^{\infty}\frac{h(n)}{n^s}=\prod\limits_{p}\left(1-\frac{1}{p^{9s}}+\cdots\right)$, which is absolutely convergent for $\Re s>\frac{1}{9}+\epsilon$.

The aim of this short paper is to establish the following asymptotic formula for the mean value of the function $\widetilde{P}(n)$ over cube-full numbers.

Theorem. *For some $D>0$,*

$$\sum_{\substack{n\le x\\ n\ is\ cube-full}}\widetilde{P}(n)=c_1x^{4/3}+c_2x^{5/4}+c_3x^{6/5}+O(x^{9/8}\exp(-D(\log x)^{3/5}(\log\log x)^{-1/5})),$$

where c_1, c_2, c_3 are computable constants.

2. Some Preliminary lemmas

In order to prove our theorem, we need the following lemmas.

Lemma 2.1. *Let $1\le a<b<c$, $\triangle(a,b,c;x)$ is defined by*

$$D(a,b,c;x)=\sum_{1\le k\le x}d(a,b,c:k)=\zeta(\frac{b}{a})\zeta(\frac{c}{a})x^{\frac{1}{a}}+\zeta(\frac{a}{b})\zeta(\frac{c}{b})x^{\frac{1}{b}}+\zeta(\frac{a}{c})\zeta(\frac{b}{c})x^{\frac{1}{c}}+\Delta(a,b,c;x),$$

we have

$$\triangle(3,4,5;x)\ll x^{\frac{22}{177}}\log^3 x.$$

Proof. see A.Ivić[4].

Lemma 2.2. *Let $\widetilde{P}^*(n)=\frac{\widetilde{P}(n)}{n}$, $\Re s>1$, we have*

$$\sum_{\substack{n=1\\ n\ is\ cube-full}}^{\infty}\frac{\widetilde{P}^*(n)}{n^s}=\frac{\zeta(3s)\zeta(4s)\zeta(5s)}{\zeta(8s)}G(s),$$

where the Dirichlet series $G(s) := \sum\limits_{n=1}^{\infty} \frac{g(n)}{n^s}$ *is absolutely convergent for* $\Re s > \frac{1}{9} + \epsilon$.

Proof.
$$\sum_{\substack{n=1 \\ n \text{ is cube-full}}}^{\infty} \frac{\widetilde{P}^*(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\widetilde{P}^*(n) f_3(n)}{n^s}$$
$$= \prod_P \left(1 + \frac{\widetilde{P}^*(p^3) f_3(p^3)}{p^{3s}} + \frac{\widetilde{P}^*(p^4) f_3(p^4)}{p^{4s}} + \frac{\widetilde{P}^*(p^5) f_3(p^5)}{p^{5s}} + \cdots + \frac{\widetilde{P}^*(p^r) f_3(p^r)}{p^{rs}}\right)$$
$$= \prod_p \left(1 + \frac{1}{p^{3s}} + \frac{1}{p^{4s}} + \frac{1}{p^{5s}} + \frac{2}{p^{2+3s}} + \frac{1}{p^{2+4s}} + \frac{2}{p^{3+4s}} + \frac{4}{p^{4+5s}} \cdots\right)$$
$$= \zeta(3s) \prod_p \left(1 + \frac{1}{p^{4s}} + \frac{1}{p^{5s}} + \frac{2}{p^{2+3s}} + \frac{1}{p^{2+4s}} - \frac{1}{p^{2+6s}} - \frac{1}{p^{2+7s}} \cdots\right)$$
$$= \zeta(3s)\zeta(4s) \prod_p \left(1 + \frac{1}{p^{5s}} - \frac{1}{p^{8s}} - \frac{1}{p^{9s}} + \frac{2}{p^{2+3s}} - \frac{1}{p^{2+7s}} \cdots\right)$$
$$= \frac{\zeta(3s)\zeta(4s)\zeta(5s)}{\zeta(8s)} \prod_p \left(1 - \frac{1}{p^{9s}} + \cdots\right)$$
$$= \frac{\zeta(3s)\zeta(4s)\zeta(5s)}{\zeta(8s)} G(s),$$

where $G(s) := \prod\limits_p \left(1 - \frac{1}{p^{9s}} + \cdots\right)$ is absolutely convergent for $\Re s > \frac{1}{9} + \epsilon$.

Lemma 2.3. *Let* $f(n)$ *be an arithmetical function for which*

$$\sum_{n \le x} f(n) = \sum_{j=1}^{l} x^{a_j} P_j(\log x) + O(x^a), \sum_{n \le x} |f(n)| = O(x^{a_1} \log^r x).$$

where $a_1 \ge a_2 \ge \cdots \ge a_l > 1/c > a \ge 0,\ r \ge 0,\ P_1(t), \cdots, P_l(t)$ *are polynomials in* t *of degrees not exceeding* r*, and* $c \ge 1$ *and* $b \ge 1$ *are fixed integers. Suppose for* $\Re s > 1$ *that*

$$\sum_{n=1}^{\infty} \frac{\mu_b(n)}{n^s} = \frac{1}{\zeta^b(s)}.$$

If $h(n) = \sum\limits_{d^c | n} \mu_b(d) f(n/d^c)$*, then*

$$\sum_{n \le x} h(n) = \sum_{j=1}^{l} x^{a_j} R_j(\log x) + E_c(x).$$

where $R_1(t), \cdots, R_l(t)$ *are polynomials in* t *of degrees not exceeding* r*, and for some* $D > 0$

$$E_c(x) \ll x^{1/c} \exp((-D(\log x)^{3/5} (\log\log x)^{-1/5})).$$

Proof. If $b = 1$, Lemma 2.3 is Theorem 14.2 of Ivić[4].

3. Proof of the Theorem

Now we prove our theorem.

Let

$$\zeta(3s)\zeta(4s)\zeta(5s)G(s) = \sum_{n=1}^{\infty} \frac{\sum\limits_{n=ml} d(3,4,5;m)g(l)}{n^s} := \sum_{n=1}^{\infty} \frac{h(n)}{n^s} (\Re s > 1)$$

where $h(n) = \sum\limits_{n=ml} d(3,4,5;m)g(l)$.

By the formula(14.44) of Ivić[4]and Lemma 2.1, we can get

$$\begin{aligned}\sum_{n\le x} d(3,4,5;n) &= \zeta(\frac{4}{3})\zeta(\frac{5}{3})x^{1/3} + \zeta(\frac{3}{4})\zeta(\frac{5}{4})x^{1/4} + \zeta(\frac{3}{5})\zeta(\frac{4}{5})x^{1/5} + \Delta(3,4,5;x) \\ &= \zeta(\frac{4}{3})\zeta(\frac{5}{3})x^{1/3} + \zeta(\frac{3}{4})\zeta(\frac{5}{4})x^{1/4} + \zeta(\frac{3}{5})\zeta(\frac{4}{5})x^{1/5} + O(x^{\frac{22}{177}}). \qquad (1)\end{aligned}$$

Then from (1) and Abel integration formula we have the relation

$$\begin{aligned}\sum_{n\le x} h(n) &= \sum_{ml\le x} d(3,4,5;m)g(l) = \sum_{l\le x} g(l) \sum_{m\le x/l} d(3,4,5;m) \\ &= \sum_{l\le x} g(l)\big[\zeta(\frac{4}{3})\zeta(\frac{5}{3})(\frac{x}{l})^{1/3} + \zeta(\frac{3}{4})\zeta(\frac{5}{4})(\frac{x}{l})^{1/4} + \zeta(\frac{3}{5})\zeta(\frac{4}{5})(\frac{x}{l})^{1/5} + O((\frac{x}{l})^{\frac{22}{177}})\big] \\ &= \zeta(\frac{4}{3})\zeta(\frac{5}{3})x^{1/3}\sum_{l\le x}\frac{g(l)}{l^{1/3}} + \zeta(\frac{3}{4})\zeta(\frac{5}{4})x^{1/4}\sum_{l\le x}\frac{g(l)}{l^{1/4}} + \zeta(\frac{3}{5})\zeta(\frac{4}{5})x^{1/5}\sum_{l\le x}\frac{g(l)}{l^{1/5}} \\ &\quad + O(x^{\frac{22}{177}}\sum_{l\le x}\frac{|g(l)|}{l^{22/177}}) \\ &= \zeta(\frac{4}{3})\zeta(\frac{5}{3})x^{1/3}\sum_{l=1}^{\infty}\frac{g(l)}{l^{1/3}} + \zeta(\frac{3}{4})\zeta(\frac{5}{4})x^{1/4}\sum_{l=1}^{\infty}\frac{g(l)}{l^{1/4}} + \zeta(\frac{3}{5})\zeta(\frac{4}{5})x^{1/5}\sum_{l=1}^{\infty}\frac{g(l)}{l^{1/5}} + O(x^{1/3}\sum_{l>x}\frac{|g(l)|}{l^{1/3}}) \\ &\quad + O(x^{1/4}\sum_{l>x}\frac{|g(l)|}{l^{1/4}}) + O(x^{1/5}\sum_{l>x}\frac{|g(l)|}{l^{1/5}}) \\ &\quad + O(x^{\frac{22}{177}}\sum_{l\le x}\frac{|g(l)|}{l^{22/177}}).\end{aligned}$$

Because $G(s)$ is absolutely convergent for $\sigma > \frac{1}{9} + \epsilon$, so we have $M(l) := \sum\limits_{t\le l} |g(t)| \ll l^{1/9+\epsilon}$. According to Abel's summation formula, we have the following estimate

$$\begin{aligned}x^{\frac{1}{3}}\sum_{l>x}\frac{|g(l)|}{l^{1/3}} &= x^{\frac{1}{3}}\int_x^{\infty} l^{-\frac{1}{3}} d\big(M(l)\big) \\ &= x^{\frac{1}{3}}\big(l^{-\frac{1}{3}}M(l)\big)\Big|_x^{\infty} + x^{\frac{1}{3}}\int_x^{\infty} M(l)d\big(l^{-\frac{1}{3}}\big) \\ &\ll x^{\frac{1}{9}}.\end{aligned}$$

Similarly, we have

$$x^{\frac{1}{4}}\sum_{l>x}\frac{|g(l)|}{l^{1/4}} \ll x^{\frac{1}{9}}, x^{\frac{1}{5}}\sum_{l>x}\frac{|g(l)|}{l^{1/5}} \ll x^{\frac{1}{9}}, x^{\frac{22}{177}}\sum_{l\le x}\frac{|g(l)|}{l^{22/177}} \ll x^{\frac{22}{177}}.$$

So

$$\sum_{n\leq x} h(n) = \zeta(\frac{4}{3})\zeta(\frac{5}{3})G(\frac{1}{3})x^{\frac{1}{3}} + \zeta(\frac{3}{4})\zeta(\frac{5}{4})G(\frac{1}{4})x^{\frac{1}{4}} + \zeta(\frac{3}{5})\zeta(\frac{4}{5})G(\frac{1}{5})x^{\frac{1}{5}} + O(x^{\frac{1}{9}}).$$

By Lemma 2.3 and Perron's formula, we get

$$\sum_{\substack{n\leq x \\ n\ is\ cube-full}} \widetilde{P}^*(n) = \frac{\zeta(\frac{4}{3})\zeta(\frac{5}{3})G(\frac{1}{3})}{\zeta(\frac{8}{3})}x^{\frac{1}{3}} + \frac{\zeta(\frac{3}{4})\zeta(\frac{5}{4})G(\frac{1}{4})}{\zeta(2)}x^{\frac{1}{4}} + \frac{\zeta(\frac{3}{5})\zeta(\frac{4}{5})G(\frac{1}{5})}{\zeta(\frac{8}{5})}x^{\frac{1}{5}}$$

$$+O(x^{1/8}\exp(-D(\log x)^{3/5}(\log\log x)^{-1/5})).$$

From the definitions of $\widetilde{P}^*(n)$ and Abel's summation formula, we can easily get

$$\begin{aligned}\sum_{\substack{n\leq x \\ n\ is\ cube-full}} \widetilde{P}(n) &= \sum_{\substack{n\leq x \\ n\ is\ cube-full}} \widetilde{P}^*(n)n = \int_1^x td\Big(\sum_{\substack{n\leq t \\ n\ is\ cube-full}} \widetilde{P}^*(n)\Big) \\ &= \frac{1}{4}\frac{\zeta(\frac{4}{3})\zeta(\frac{5}{3})G(\frac{1}{3})}{\zeta(\frac{8}{3})}x^{\frac{4}{3}} + \frac{1}{5}\frac{\zeta(\frac{3}{4})\zeta(\frac{5}{4})G(\frac{1}{4})}{\zeta(2)}x^{\frac{5}{4}} + \frac{1}{6}\frac{\zeta(\frac{3}{5})\zeta(\frac{4}{5})G(\frac{1}{5})}{\zeta(\frac{8}{5})}x^{\frac{6}{5}} \\ &+O(x^{9/8}\exp(-D(\log x)^{3/5}(\log\log x)^{-1/5})).\end{aligned}$$

where $D > 0$.

Now our theorem is proved.

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References

[1] M. V. Subbarao, On some arithmetic convolutions in the theory of arithmetic functions, Lecture Notes in Mathematics, Vol. 251, Springer, 1972, pp. 247–271.

[2] L. Tóth, On certain arithmetical functions involving exponential divisors, *Ann. Univ. Sci. Budapest Sect. Comput.,* 24 (2004), pp. 285–294.

[3] S. Li and L. Dong, The mean value of $\widetilde{P}(n)$ over square-full numbers, *Sci. Magna,* 10(2) (2014), pp. 99–103.

[4] A.Ivić, The Riemann Zeta-function, John Wiley & Sons, 1985.

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Gracefulness and one modulo N gracefulness of $L_n \bigotimes S_m$

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Abstract A function f is called a graceful labelling of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, \ldots, q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. A graph G is said to be one modulo N graceful (where N is a positive integer) if there is a function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-1), N(q-1)+1\}$ in such a way that (i) ϕ is $1-1$ (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$where $\phi^*(uv)=|\phi(u) - \phi(v)|$. In this paper we prove that the graph $L_n \bigotimes S_m$ is graceful and one modulo N graceful where $L_n \bigotimes S_m$ is the graph obtained from the ladder L_n by identifying one vertex of the ladder L_n with any vertex of the star S_m other than the centre of S_m.

Keywords Graceful, modulo N graceful, ladder, star.
2000 Mathematics Subject Classification: 05C78.

§1. Introduction

S. W. Golomb[2] introduced graceful labelling. Odd gracefulness is introduced by R. B. Gnanajothi[1]. C. Sekar[7] intoduced one modulo three graceful labelling. V. Ramachandran and C. Sekar[5] introduced the concept of one modulo N graceful where N is a positive integer. In the case $N = 2$, the labelling is odd graceful and in the case $N = 1$ the labelling is graceful. In this paper we prove that the graph $L_n \bigotimes S_m$ for all positive integers n and m is one modulo N graceful for any positive integer N.

§2. Main results

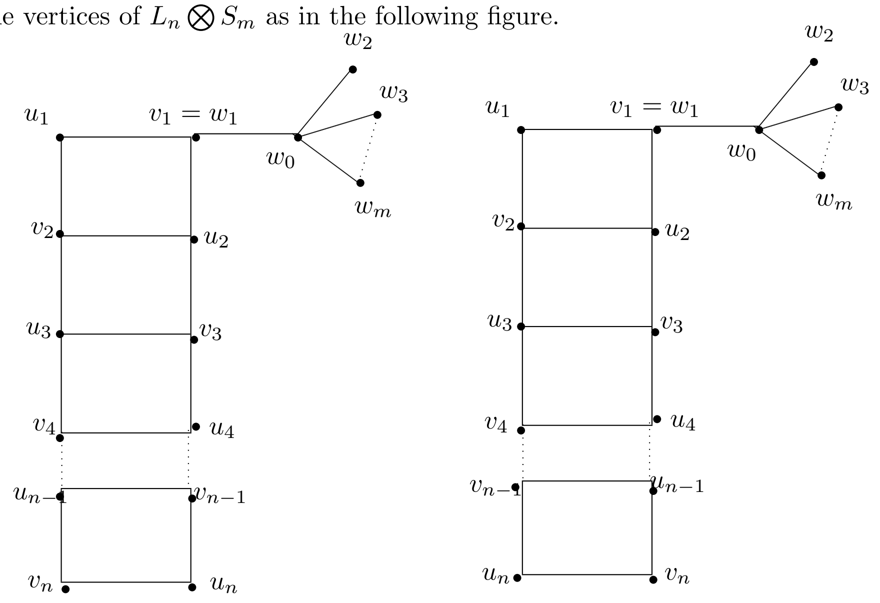
Definition 2.1. *A graph G is said to be one modulo N graceful (where N is a positive integer) if there is a function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-1), N(q-1)+1\}$in such a way that (i) ϕ is 1-1 (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$where $\phi^*(uv)=|\phi(u) - \phi(v)|$.*

Definition 2.2. *$L_n \bigotimes S_m$ is the graph obtained from the ladder L_n by identifying one vertex of the ladder L_n with any vertex of the star S_m other than the centre of S_m.*

Theorem 2.1. *$L_n \bigotimes S_m$ is one modulo N graceful for all positive integers n and m where N is any positive integer.*

Proof. Case (i) When a vertex of the star S_m other than the centre of S_m is identified with a vertex of degree two of the ladder L_n.

We label the vertices of $L_n \bigotimes S_m$ as in the following figure.



n is even $\qquad$ n is odd

Let V be the set of all vertices of $L_n \bigotimes S_m$ and E be the set of all edges of $L_n \bigotimes S_m$. Clearly $L_n \bigotimes S_m$ has $2n + m$ vertices and $3n + m - 2$ edges. $V = \{u_1, u_2, \ldots u_n, w_1 = v_1, v_2, \ldots, v_n, w_0, w_2, w_3, \ldots, w_m\}$,

$$E = \begin{cases} u_i v_i, & \text{for} \quad i = 1, 2, \ldots, n, \\ u_i v_{i+1} & \text{for} \quad i = 1, 2, \ldots, n-1, \\ v_i u_{i+1} & \text{for} \quad i = 1, 2, \ldots, n-1, \\ w_0 w_i & \text{for} \quad i = 1, 2, \ldots, m, \\ v_1 w_0. & \end{cases}$$

Define the following functions:

$\phi : V \to \{0, 1, 2, 3, 4, \ldots, N(3n + m - 3) + 1\}$ as follows:

$\phi(w_0) = 0$,

$\phi(w_i) = 3N(n-1) + N_i + 1$ for $i = 1, 2, 3, \ldots, m$,

$\phi(u_i) = N_i$ for $i = 1, 2, 3, \ldots, n$,

$\phi(v_i) = 3N_n - 2N_i + 1$ for $i = 1, 2, 3, \ldots, n$.

From the definition of ϕ it is clear that $\{\phi(w_0)\} \cup \{\phi(w_i), i = 1, 2, \ldots, m\} \cup \{\phi(u_i), i = 1, 2, \ldots, n\} \cup \{\phi(v_i), i = 1, 2, \ldots, n\} = \{0\} \cup \{N(3n - 2) + 1, N(3n - 1) + 1, \ldots, N(3n + m - 3) + 1\} \cup \{N, 2N, \ldots, Nn\} \cup \{N(3n - 2) + 1, N(3n - 4) + 1, \ldots, Nn + 1\}$.

Thus it is clear that the vertices have distinct labels. Therefore ϕ is 1-1.

We compute the edge labelling in the following sequence.

$$\phi^*(v_n v_{n-1}) = \mid \phi(v_n) - \phi(v_{n-1}) \mid = \begin{cases} N+1, & \text{if } n \text{ is odd,} \\ 2N+1, & \text{if } n \text{ is even.} \end{cases}$$

$$\phi^*(u_{n-1} u_n) = \mid \phi(u_{n-1}) - \phi(u_n) \mid = \begin{cases} 2N+1, & \text{if } n \text{ is odd,} \\ N+1, & \text{if } n \text{ is even.} \end{cases}$$

For $i = 1, 2, 3, \ldots, m$, $\phi^*(w_i w_0) = \mid \phi(w_i) - \phi(w_0) \mid = N(3n-3+i)+1$.

For $i = 1, 2, 3, \ldots, n$, $\phi^*(v_i u_i) = \mid \phi(v_i) - \phi(u_i) \mid = 3N(n-i)+1$.

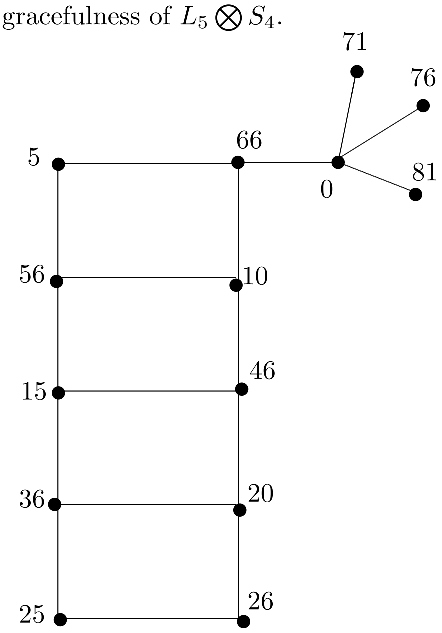
For $i = 1, 2, 3, \ldots, n-2$, $\phi^*(v_{i+1} v_i) = \mid \phi(v_{i+1}) - \phi(v_i) \mid = N(3n-3i-2)+1$.

For $i = 1, 2, 3, \ldots, n-2$, $\phi^*(u_i u_{i+1}) = \mid \phi(u_i) - \phi(u_{i+1}) \mid = N(3n-3i-1)+1$.

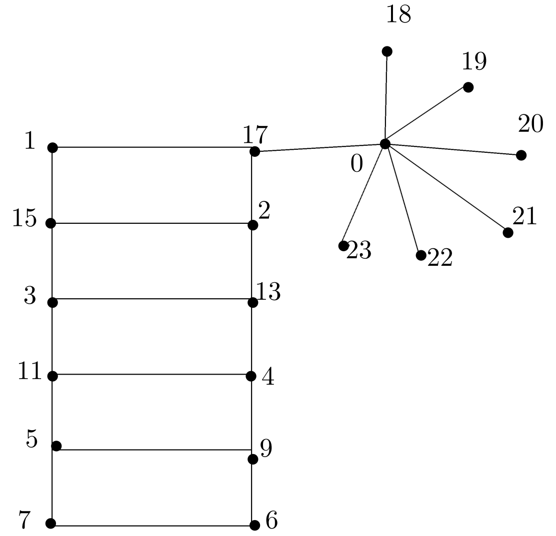
This shows that the edges have the distinct labels $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$.

It is clear from the above labelling that the function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-1), N(q-1)+1\}$ is in such a way that (i) ϕ is 1-1 (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$ where $\phi^*(uv)=|\phi(u)-\phi(v)|$. Hence $L_n \bigotimes S_m$ is one modulo N graceful.

Example 2.1. One modulo 5 gracefulness of $L_5 \bigotimes S_4$.



Example 2.2. Gracefulness of $L_6 \bigotimes S_7$.

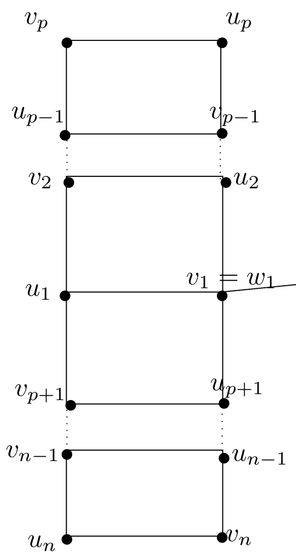


Case (ii) When a vertex of the star S_m other than the centre of S_m is identified with a vertex of degree three of the ladder L_n.

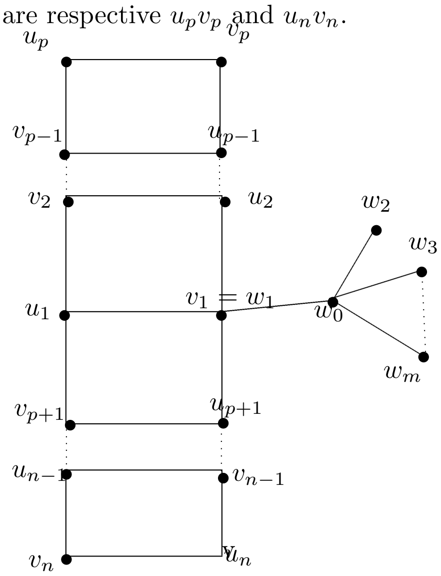
Subcase (ii) (a) Assume $N > 1$. We label the vertices as follows when $p > \frac{n}{2}$ if n is even and when $p \geq \frac{n+1}{2}$ if n is odd.

Note 2.1. 1. If p is even and n is odd then top and bottom edges are respective $v_p u_p$ and $v_n u_n$

2. If p is odd and n is odd then the top and bottom edges are respective $u_p v_p$ and $u_n v_n$.



p is even and n is even



p is odd and n is even

Case (i) p is even.

Define the function:

$\phi : V \rightarrow$\{0,1,2,3,4,...,N(3n+m-3)+1\} as follows:

$\phi(w_0) = 0,$

$\phi(w_i) = 3N(n-1) + Ni + 1$ for $i = 2, 3, \ldots, m,$

$$\phi(u_i) = \begin{cases} 2N, & \text{if i=1 ,} \\ N(2i-3), & \text{if i=2,3,4,...,p ,} \\ 2N(i-p) + 2N, & \text{if i=p+1, p+2, ..., n.} \end{cases}$$

$$\phi(v_i) = \begin{cases} 3Nn - 2N + 1, & \text{if i=1 ,} \\ 3Np - 2N + 1 - Ni, & \text{if i=2, 4, 6, ..., p,} \\ 3Np - 4N + 1 - Ni, & \text{if i=3, 5, 7, ..., p-1,} \\ (3Nn - 2N + 1) - N(i-p), & \text{if i=p+1, p+2, ..., n.} \end{cases}$$

From the definition of ϕ it is clear that $\{\phi(w_0)\} \cup \{\phi(w_i), i = 1, 2, \ldots, m\} \cup \{\phi(u_i), i = 1, 2, \ldots, n\} \cup \{\phi(v_i), i = 1, 2, \ldots, n\} = \{0\} \cup \{ N(3n-1)+1, N(3n)+1, \ldots, N(3n+m-3)+1\} \cup \{2N, N, 3N, \ldots, N(2p-3), 4N, 6N, \ldots, 2N(1+n-p)\} \cup \{N(3n-2)+1, N(3n-3)+1, N(3n-4)+1, \ldots, N(2n-2+p)+1, N(3n-4)+1, N(3n-6)+1, \ldots, 2N(p-1)+1, N(3p-7)+1, N(3p-9)+1 \ldots, N(2p-3)+1\}$.

Thus it is clear that the vertices have distinct labels. Therefore ϕ is $1-1$.

We compute the edge labelling in the following sequence:

$\phi^*(v_1u_1) =\mid \phi(v_1) - \phi(u_1) \mid = 3Nn - 4N + 1.$

$\phi^*(v_2u_1) =\mid \phi(v_2) - \phi(u_1) \mid = 3N(p-2) + 1.$

$\phi^*(v_1u_2) =\mid \phi(v_1) - \phi(u_2) \mid = 3N(n-1) + 1.$

$\phi^*(v_1w_0) =\mid \phi(v_1) - \phi(w_0) \mid = 3Nn - 2N + 1.$

$\phi^*(v_{p+1}u_1) =\mid \phi(v_{p+1}) - \phi(u_1) \mid = N(3n-5) + 1.$

$\phi^*(v_1u_{p+1}) =\mid \phi(v_1) - \phi(u_{p+1}) \mid = 3N(n-2) + 1.$

For $i = 2, 3, \ldots, m,$

$\phi^*(w_iw_0) =\mid \phi(w_i) - \phi(w_0) \mid = 3N(n-1) + Ni + 1.$

For $i = p+1, p+2, \ldots, n,$

$\phi^*(v_iu_i) =\mid \phi(v_i) - \phi(u_i) \mid = N(3n - 4 - 3i + 3p) + 1.$

For $i = 2, 4, \ldots, p,$

$\phi^*(v_iu_i) =\mid \phi(v_i) - \phi(u_i) \mid = N(3p + 1 - 3i) + 1.$

For $i = 3, 5, \ldots, p-1,$

$\phi^*(v_iu_i) =\mid \phi(v_i) - \phi(u_i) \mid = N(3p - 3i - 1) + 1.$

For $i = 2, 4, \ldots, p-2,$

$\phi^*(v_{i+1}u_i) =\mid \phi(v_{i+1}) - \phi(u_i) \mid = N(3p - 3i - 2) + 1.$

For $i = 3, 5, \ldots, p-1,$

$\phi^*(v_{i+1}u_i) =\mid \phi(v_{i+1}) - \phi(u_i) \mid = N(3p - 3i) + 1.$

For $i = 2, 4, \ldots, p-2,$

$\phi^*(v_iu_{i+1}) =\mid \phi(v_i) - \phi(u_{i+1}) \mid = N(3p - 3i - 1) + 1.$

For $i = 3, 5, \ldots, p-1,$

$\phi^*(v_iu_{i+1}) =\mid \phi(v_i) - \phi(u_{i+1}) \mid = 3N(p - i - 1) + 1.$

For $i = p+1, p+2, \ldots, n-1,$

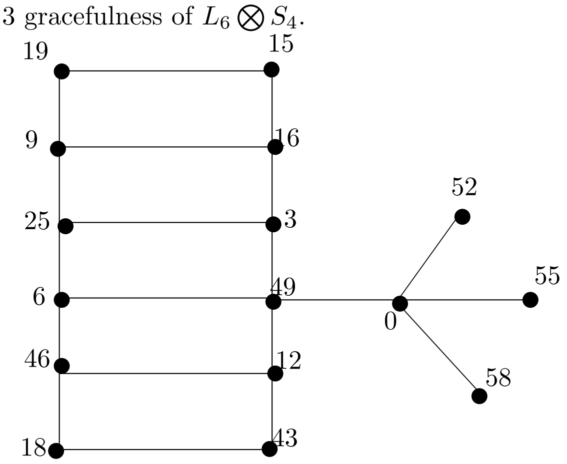
$\phi^*(v_i u_{i+1}) = \mid \phi(v_i) - \phi(u_{i+1}) \mid = 3N(n-2-i+p)+1.$

$\phi^*(v_{i+1} u_i) = \mid \phi(v_{i+1}) - \phi(u_i) \mid = N(3n-5-3i+3p)+1.$

This shows that the edges have the distinct labels $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$.

It is clear from the above labelling that the function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-1), N(q-1)+1\}$ is in such a way that (i) ϕ is 1-1 (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$ where $\phi^*(uv)$=$|\phi(u) - \phi(v)|$. Hence $L_n \bigotimes S_m$ is one modulo N graceful for p is even.

Example 2.3. One modulo 3 gracefulness of $L_6 \bigotimes S_4$.



Case (ii) p is odd.

Define the function:

$\phi : V \to \{0, 1, 2, 3, 4, \ldots, N(3n+m-3)+1\}$ as follows:

$\phi(w_0) = 0.$

$\phi(w_i) = 3(n-1)+i+1$, for $i = 2, 3, \ldots, m$.

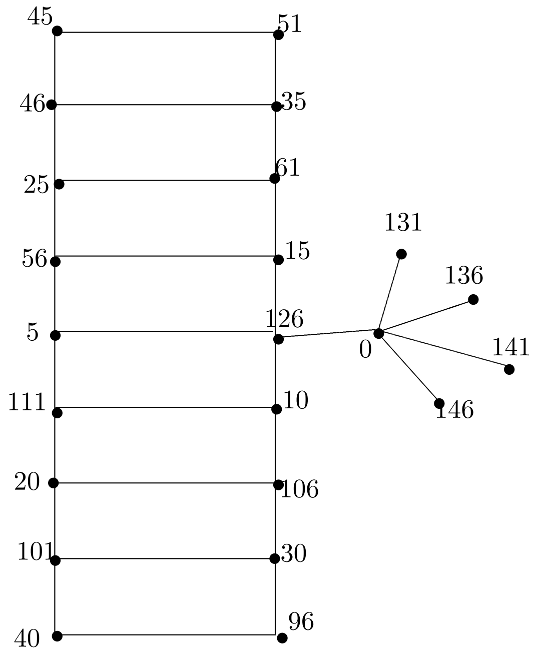
$$\phi(u_i) = \begin{cases} N, & \text{if i=1,} \\ N(2i-1), & \text{if i=2, 3, \ldots, p,} \\ 2N+2N(i-[p+1]), & \text{if i=p+1, p+2, \ldots, n.} \end{cases}$$

$$\phi(v_i) = \begin{cases} 3Nn-2N+1, & \text{if i=1,} \\ (3Np-4N+1)-N(i-2), & \text{if i=2, 4, \ldots, p,} \\ (3Np-3N+1)-N(i-3), & \text{if i=3, 5, \ldots, p-1,} \\ (3Nn-5N+1)-N(i-[p+1], & \text{if i=p+1, p+2, \ldots, n.} \end{cases}$$

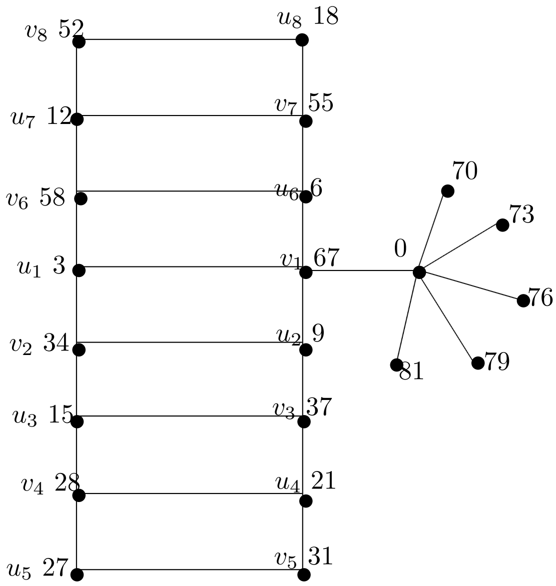
The proof is similar to the proof in case (i).

Clearly ϕ defines a one modulo N graceful labelling of $L_n \bigotimes S_m$ p is odd.

Example 2.4. One modulo 5 gracefulness of $L_9 \bigotimes S_5$.



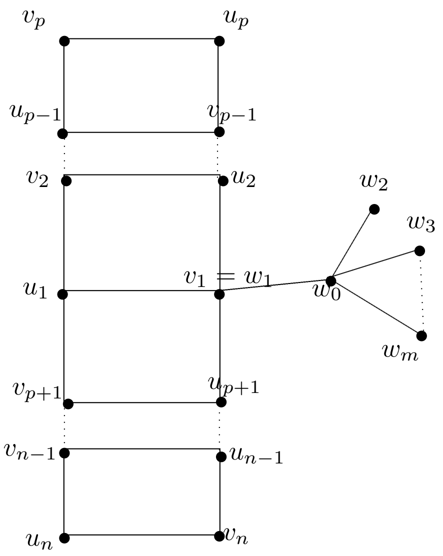
Example 2.5. One modulo 3 gracefulness of $L_8 \bigotimes S_6$.



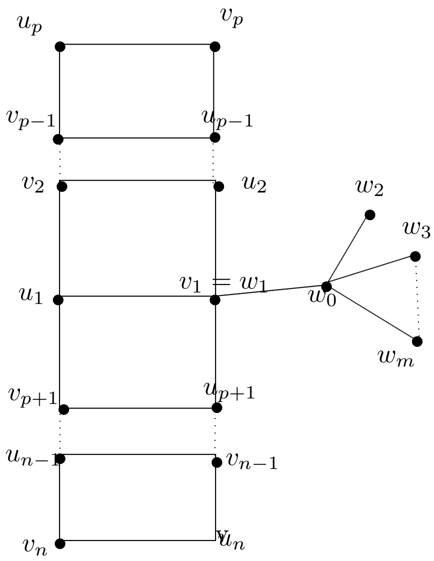
Subcase (ii) (b) Assume $N = 1$. We label the vertices as follows when $p > \frac{n}{2}$ if n is even and when $p \geq \frac{n+1}{2}$ if n is odd.

Note 2.2. 1. If p is even and n is odd then top and bottom edges are respective $v_pu_p$and v_nu_n.

2. If p is odd and n is odd then the top and bottom edges are respective $u_pv_p$and u_nv_n.



p is even and n is even



p is odd and n is even

Case (i) p is even.

Define the function:

$\phi : V \rightarrow \{0, 1, 2, 3, 4, \ldots, (3n + m - 2)\}$ as follows:

$\phi(w_0) = 0$.

$\phi(w_i) = 3(n - 1) + i + 1$, for $i = 2, 3, \ldots, m$.

$$\phi(u_i) = \begin{cases} 2, & \text{if i=1,} \\ 1, & \text{if i=2,} \\ (3p-2)-(i-3), & \text{if i=3, 5, } \ldots \text{, p-1,} \\ (3p+1)-(i-4), & \text{if i=4, 6, } \ldots \text{, p,} \\ 2(i-p)+2, & \text{if i=p+1, p+2, } \ldots \text{, n.} \end{cases}$$

$$\phi(v_i) = \begin{cases} 3n-1, & \text{if i=1,} \\ 2i-1, & \text{if i=2, 3, 4, } \ldots \text{, p,} \\ 3n+p-1-i, & \text{if i=p+1, p+2, } \ldots \text{, n.} \end{cases}$$

From the definition of ϕ it is clear that $\{\phi(w_0)\} \cup \{\phi(w_i), i = 2, 3, \ldots, m\} \cup \{\phi(u_i), i = 1, 2, \ldots, n\} \cup \{\phi(v_i), i = 1, 2, \ldots, n\} = \{0\} \cup \{\ 3n, 3n+1, \ldots, 3n+m-2\} \cup \{1, 2, 3p-2, 3p-4 \ldots, 2p+2, 3p+1, 3p-1, \ldots, 2p+5, 4, 6, \ldots, 2(n-p+1)\} \cup \{3n-1, 3, 5, 2p-1, 3n-2, 3n-3, \ldots, 2n-1+p\}$.

Thus it is clear that the vertices have distinct labels. Therefore ϕ is 1-1.

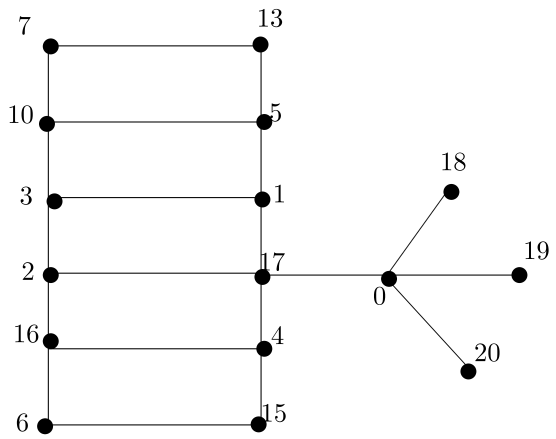
We compute the edge labelling in the following sequence:

$\phi^*(v_1u_1) =\mid \phi(v_1) - \phi(u_1) \mid = 3n - 3$.

$\phi^*(v_2u_2) =| \phi(v_2) - \phi(u_2) | = 2.$
$\phi^*(v_3u_2) =| \phi(v_3) - \phi(u_2) | = 4.$
$\phi^*(v_2u_1) =| \phi(v_2) - \phi(u_1) | = 1.$
$\phi^*(v_1u_2) =| \phi(v_1) - \phi(u_2) | = 3n - 2.$
$\phi^*(v_1w_0) =| \phi(v_1) - \phi(w_0) | = 3n - 1.$
$\phi^*(v_{p+1}u_1) =| \phi(v_{p+1}) - \phi(u_1) | = 3n - 4.$
$\phi^*(v_1u_{p+1}) =| \phi(v_1) - \phi(u_{p+1}) | = 3n - 5.$
For $i = 2, 3, \ldots, m$, $\phi^*(w_iw_0) =| \phi(w_i) - \phi(w_0) | = 3(n-1) + i + 1.$
For $i = p+1, p+2, \ldots, n$, $\phi^*(v_iu_i) =| \phi(v_i) - \phi(u_i) | = 3(n - 1 - i + p).$
For $i = 4, 6 \ldots, p$, $\phi^*(v_iu_i) =| \phi(v_i) - \phi(u_i) | = 3p - 3i + 6.$
For $i = 3, 5, \ldots, p-1$,
$\phi^*(u_iv_i) =| \phi(u_i) - \phi(v_i) | = 3p - 3i + 2.$
For $i = 4, 6, \ldots, p$, $\phi^*(u_iv_{i-1}) =| \phi(u_i) - \phi(v_{i-1}) | = 3p - 3i + 8.$
For $i = 3, 5, \ldots, p-1$, $\phi^*(u_iv_{i-1}) =| \phi(u_i) - \phi(v_{i-1}) | = 3p - 3i + 4.$
For $i = 4, 6, \ldots, p-2$, $\phi^*(u_iv_{i+1}) =| \phi(u_i) - \phi(v_{i+1}) | = 3p - 3i + 5.$
For $i = 3, 5, \ldots, p-1$, $\phi^*(u_iv_{i+1}) =| \phi(u_i) - \phi(v_{i+1}) | = 3p - 3i.$
For $i = p+1, p+2, \ldots, n-1$, $\phi^*(v_iu_{i+1}) =| \phi(v_i) - \phi(u_{i+1}) | = 3(n - i + p) - 5$, $\phi^*(v_{i+1}u_i) =| \phi(v_{i+1}) - \phi(u_i) | = 3(n - i + p) - 4.$

This shows that the edges have the distinct labels $\{1, 2, \ldots, q\}$. Clearly ϕ defines a graceful labelling of $L_n \bigotimes S_m$ for p is even.

Example 2.6. Gracefulness of $L_6 \bigotimes S_4$.



Case (ii) p is odd.

Define the function:

$\phi : V \to \{0, 1, 2, 3, 4, \ldots, (3n + m - 2) + 1\}$ as follows:

$\phi(w_0) = 0.$

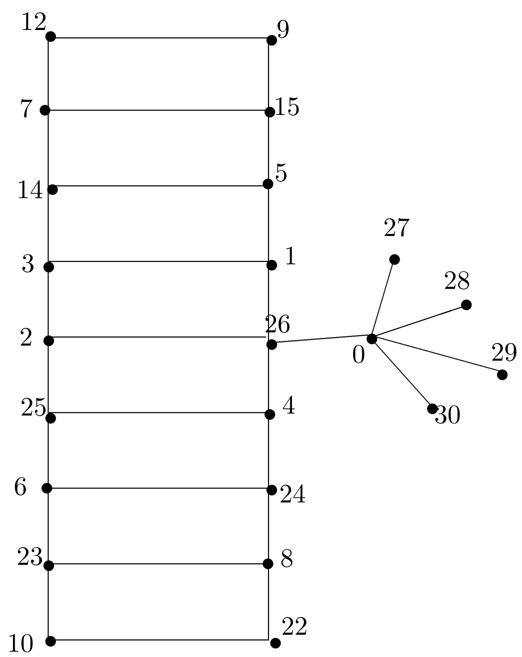
$\phi(w_i) = 3(n-1) + i + 1$, for $i = 2, 3, \ldots, m.$

$$\phi(u_i) = \begin{cases} 2, & \text{if i=1,} \\ 1, & \text{if i=2,} \\ (3p-1)-(i-3), & \text{if i=3, 5, \ldots, p-1,} \\ 3p-(i-4), & \text{if i=4, 6, \ldots, p,} \\ 2(i-p)+2, & \text{if i=p+1, p+2, \ldots, n.} \end{cases}$$

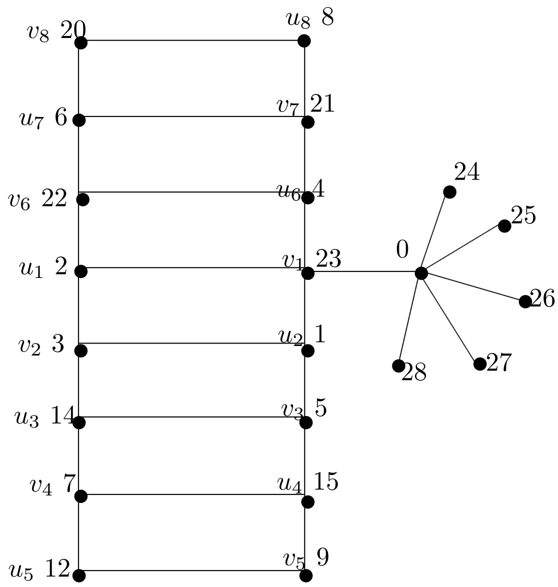
$$\phi(v_i) = \begin{cases} 3n-1, & \text{if i=1,} \\ 2i-1, & \text{if i=2,3,4,\ldots,p,} \\ 3n+p-1-i, & \text{if i=p+1, p+2, \ldots, n.} \end{cases}$$

The proof is similar to the proof in case (i). Clearly ϕ defines a graceful labelling of $L_n \bigotimes S_m$ for p is odd.

Example 2.7. Gracefulness of $L_9 \bigotimes S_5$.



Example 2.8. Gracefulness of $L_8 \bigotimes S_6$.



References

[1] R. B. Gnanajothi, Topics in graph theory, Ph. D. Thesis, Madurai Kamaraj University, 1991.

[2] S. W. Golomb, How to number a graph in Graph theory and computing R. C. Read ed., Academic press, New york, 1972, 23-27.

[3] Joseph A. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics,* 18 (2011), #DS6.

[4] K. M. Kathiresan and S. Amutha, Arbitrary supersubdivisions of stars are graceful, *Indian J. Pure Appl. Math.,* 35 (2004), pp. 81–84.

[5] V. Ramachandran and C. Sekar, One modulo N gracefulness of Acyclic graphs, *Ultra Scientist of Physical Sciences*, 25 (2013), pp. 417–424.

[6] A. Rosa, On certain valuations of the vertices of a graph, Theory of graphs, (International Symposium, Rome July 1966) Gordom and Breach, N. Y and Dunod paris, (1967), 349-355.

[7] C. Sekar, Studies in graph theory, Ph. D. Thesis, Madurai Kamaraj University, 2002.

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A study on prime one-sided ideals, socles of entire Dirichlet series as a gamma ring

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Abstract The present paper deals with the study of the set of all Entire Dirichlet series X which forms a Γ-ring and establish results on socles for this set. Also the results on prime one-sided ideals are studied for the set X.

Keywords Dirichlet series, Gamma ring, simple Gamma ring, prime Gamma ring, socles of a Gamma ring.
2000 Mathematics Subject Classification: 30B50, 17D20, 17C20.

§1. Introduction

Let

$$f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \qquad s = \sigma + it, \quad (\sigma,\ t \in \mathbb{R}). \tag{1}$$

If $a_n's$ belong to $\mathbb{C}$ and $\lambda_n's \in \mathbb{R}$ which satisfy the condition $0 < \lambda_1 < \lambda_2 < \lambda_3 \ldots < \lambda_n \ldots; \lambda_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\log |a_n|}{\lambda_n} = -\infty, \tag{2}$$

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = K < \infty, \tag{3}$$

then from [7] the Dirichlet series (1) represents an entire function.

Let X denote the set of all entire functions. Let Γ be the set of series (1) for which $(ne)^{c_1\lambda_n e}(n!)^{c_2}|a_n|$ is bounded where $c_1, c_2 \geq 0$ and c_1, c_2 are simultaneously not zero. Then by [7] every element of Γ represents an entire function. The norm in Γ is defined as follows

$$\|f\| = \sup_{n\geq 1} (ne)^{c_1\lambda_n e}(n!)^{c_2}|a_n|. \tag{4}$$

In [4] Nobusawa generalized the Wedderburn-Artin Theorem for simple and semi-simple Γ-rings. Barnes in [8] obtained analogues of the classical Noether-Lasker theorems concerning primary representations of ideals for Γ-rings. Booth and Groenewald in [2] discussed one-to-one correspondence between the prime left ideals of the gamma ring and the right operator ring. Also they discussed the bi-ideals and quasi-ideals of a gamma ring.

The purpose of the present paper is to consider the set of all Entire Dirichlet series which forms a Γ-ring and establish various results on prime one-sided ideals and socles for this set X.

§2. Basic results

Following definitions are required to prove the main results. For all notions relevant to ring theory, refer [1] and [3].

Definition 2.1. *Let M and Γ be two additive abelian groups. If there exists a mapping $M\ X\,\Gamma\ X\,M \to M$ such that for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ the conditions*

(1.a) $(x + y)\alpha z = x\alpha z + y\alpha z$,

(1.b) $x(\alpha + \beta)z = x\alpha z + x\beta z$,

(1.c) $x\alpha(y + z) = x\alpha y + x\alpha z$,

(1.d) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

are satisfied then M is a Γ-ring.

An additive subgroup I of M is a left (right) ideal of M if $M\Gamma I \subset I$ ($I\Gamma M \subset I$). If I is both a left and a right ideal of M then I is a two- sided ideal or simply an ideal of M. For all other concepts we refer [5] and [6].

Definition 2.2. *Let M be a Γ-ring. An ideal P of M is prime if for all pairs of ideals S and T of M, $S\Gamma T \subseteq P$ implies $S \subseteq P$ or $T \subseteq P$. A Γ-ring M is prime if the zero ideal is prime.*

Definition 2.3. *An ideal Q of M is semi-prime if for any ideal U, $U\Gamma U \subseteq Q$ implies $U \subseteq Q$. A Γ-ring M is semi-prime if the zero ideal is semi-prime.*

Definition 2.4. *A one-sided ideal P of X is called prime if for $a(s), b(s) \in X$, $a(s) \cdot \Gamma \cdot X \cdot \Gamma \cdot b(s) \subseteq P$ implies $a(s) \in P$ or $b(s) \in P$.*

Definition 2.5. *A set $\{I_\alpha : \alpha \in A\}$ of minimal left ideals of X is said to be independent if $I_\alpha \bigcap \sum_{\beta\neq\alpha} I_\beta = 0$ for all $\alpha \in A$.*

Now let $a(s), b(s) \in X$ and $\alpha(s) \in \Gamma$ such that

$$a(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s},\ b(s) = \sum_{n=1}^{\infty} b_n e^{\lambda_n s},\ \alpha(s) = \sum_{n=1}^{\infty} \alpha_n e^{\lambda_n s}. \tag{5}$$

The binary operations that is addition and scalar multiplication in X X Γ X X is defined as-

$$a(s) + \alpha(s) + b(s) = \sum_{n=1}^{\infty}(a_n + \alpha_n + b_n)e^{\lambda_n s},$$

$$a(s) \cdot \alpha(s) \cdot b(s) = \sum_{n=1}^{\infty}(a_n \cdot \alpha_n \cdot b_n)\ e^{\lambda_n s}.$$

Clearly the set X forms a Γ-ring. Also X is a prime and a semi-prime Γ-ring as $\{0\}$ ideal is prime and semi-prime respectively.

Let G be a free abelian group generated by the set of all ordered pairs $\{\alpha(s),\ a(s)\}$ where $a(s) \in X$ and $\alpha(s) \in \Gamma$.

Let T be a subgroup of elements $\sum_i m_i\{\alpha_i(s),\ a_i(s)\} \in G$ where m_i are integers such that

$$\sum_i m_i.[x(s).\alpha_i(s).a_i(s)] = 0 \text{ for all } x(s) \in X.$$

Denote by R' the factor group G/T and by $[\alpha(s), a(s)]$ the coset $\{\alpha(s), a(s)\} + T$. Clearly every element in R' can be expressed as a finite sum $\sum_i [\alpha_i(s), a_i(s)]$.

Also for all $a_1(s), a_2(s) \in X$ and $\beta(s) \in \Gamma$,

$$[\alpha(s), a_1(s)] + [\beta(s), a_1(s)] = [\alpha(s) + \beta(s), a_1(s)],$$

$$[\alpha(s), a_1(s)] + [\alpha(s), a_2(s)] = [\alpha(s), a_1(s) + a_2(s)].$$

Define multiplication in R' by

$$\sum_i [\alpha_i(s), a_i(s)].\sum_j [\beta_j(s), b_j(s)] = \sum_{i,j} [\alpha_i(s), a_i(s).\beta_j(s).b_j(s)].$$

Then R' forms a ring. Furthermore X is a right R'-module with the definition

$$x(s).\sum_i [\alpha_i(s), a_i(s)] = \sum_i [x(s).\alpha_i(s).a_i(s)] \text{ for all } x(s) \in X.$$

We call the ring R' the right operator ring of Γ-ring X. Similarly one can define the left operator ring L' of X. Every minimal left ideal of a Γ-ring X is of the form $X.\alpha(s).e(s)$ where $e(s) = \sum_{n=1}^{\infty} e_n e^{\lambda_n s} \in I$ and $e_n.\alpha_n.e_n = e_n$.

§3. Main results

In this section main results are proved.

Lemma 3.1. *Let (ϕ, i) be a homomorphism of a Γ-ring X onto a Γ-ring Y with kernel K then if I is an ideal in X then $I\phi$ is an ideal in Y.*

Theorem 3.1. *If P is an ideal of the Γ-ring X then the Γ-residue class ring X/P is a prime Γ-ring if and only if P is a prime ideal in X.*

Proof. Let X/P be a prime Γ-ring. Let A and B be ideals of X such that $A\Gamma B \subseteq P$. Let (ϕ, i) be a natural homomorphism from X onto X/P. By Lemma 3.1, $A\phi$ and $B\phi$ are ideals of X/P such that $(A\phi)\Gamma(B\phi) = (0)$ implies $A\phi = (0)$ or $B\phi = (0)$. This implies $A \subseteq P$ or $B \subseteq P$. Thus P is a prime ideal in X. Conversely let P be a prime ideal in X. Each ideal of X/P is of the form A/P where A is an ideal in X. Thus we assume that A/P, B/P be ideals of X/P such that $(A/P)\Gamma(B/P) = (0)$. Now $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. This implies $A = P$ or $B = P$. Thus $A/P = (0)$ or $B/P = (0)$. Hence the theorem.

Barnes in [8] characterized $P(X)$ as the intersection of all prime ideals of X. Again $\{0\}$ is a semi-prime ideal of X if and only if $P(X) = (0)$ or $\{0\}$ is a semi-prime ideal of X if and only if X contains no non-zero strongly nilpotent right (left) ideal. Thus one can say that a Γ-ring X has zero prime radical if and only if it contains no strongly nilpotent ideal.

Socles of Γ-rings

The sum $S_l\,(S_r)$ of all minimal left (right) ideals of X is called left (right) socle of X. It is clear that if X has no minimal left (right) ideals then the left (right) socle of X is 0.

Lemma 3.2. *Let X be a Γ-ring. If I is a minimal left ideal of X then for each $\alpha(s) \in \Gamma$ and each $a(s) \in X$, $I.\alpha(s).a(s)$ is either zero or a minimal left ideal of X.*

Proof. Let $I.\alpha(s).a(s) \neq 0$ and J be a non-zero left ideal of X contained in $I.\alpha(s).a(s)$ then there exists $x(s) \in I$ with $0 \neq x(s).\alpha(s).a(s) \in J$. Let $H = \{z(s) \in I | z(s).\alpha(s).a(s) \in J\}$. Therefore H is a non-zero left ideal of X contained in I. Minimality of I implies $H = I$ which implies $I.\alpha(s).a(s) \subseteq J$ which further implies $I.\alpha(s).a(s) = J$. Thus $I.\alpha(s).a(s)$ is a minimal left ideal of X and this completes the proof.

Theorem 3.2. *If X is a Γ-ring then the left socle and the right socle of X are ideals of X.*

Proof. By symmetry one needs to only prove that the left socle S_l of X is an ideal of X. It is clear that S_l is a left ideal of X. We need to prove that S_l is a right ideal of X. Assume that $\alpha(s) \in \Gamma$, $a(s) \in X$, $f(s) \in S_l$ and $f(s) \in I_1 + I_2 + ... + I_n$ where I_i are minimal left ideals of X. Then

$$f(s).\alpha(s).a(s) \in I_1.\alpha(s).a(s) + I_2.\alpha(s).a(s) + ... + I_n.\alpha(s).a(s).$$

By Lemma 3.2, $I_i.\alpha(s).a(s)$ is either zero or a minimal left ideal of X. Hence $f(s).\alpha(s).a(s) \in S_l$. This implies S_l is a right ideal of X.

This completes the proof of the theorem.

Theorem 3.3. *If X is a simple Γ-ring having minimal left ideals then X is a direct sum of minimal left ideals.*

Proof. A left socle is defined as the sum of minimal left ideals. Since X is a simple Γ-ring therefore left socle of X is X itself. Consider a family A of all independent sets of minimal left ideals of X. The family A is partially ordered by inclusion. By Zorn's lemma one can obtain a maximal independent set in A say $\{I_\alpha : \alpha \in B\}$. By the maximality of this set $I \bigcap \sum\limits_{\alpha \in B} I_\alpha = I$ for each minimal left ideal I of X and $I \subseteq \sum\limits_{\alpha \in B} I_\alpha$. Thus $X = \sum\limits_{\alpha \in B} I_\alpha$ (direct sum). Hence the theorem.

Theorem 3.4. *Let X be a Γ-ring. If X has no non-zero strongly nilpotent ideals then the left socle S_l and the right socle S_r of X coincide.*

Proof. A Γ-ring X without non-zero strongly nilpotent ideals has minimal left ideals if and only if it has minimal right ideals. Moreover, every minimal left ideal is of the form $X.\alpha(s).e(s)$ where $e_n.\alpha_n.e_n = e_n$. Also, $X.\alpha(s).e(s)$ is a minimal left ideal if and only if $e(s).\alpha(s).X$ is a minimal right ideal of X. Let $S_l = \sum\limits_i X.\alpha_i(s).e_i(s)$ where $X.\alpha_i(s).e_i(s)$ are minimal left ideals of X and $e_{n_i}.\alpha_{n_i}.e_{n_i} = e_{n_i}$. Since $e_i(s).\alpha_i(s).X$ are minimal right ideals of X, therefore, $\sum\limits_i e_i(s).\alpha_i(s).X \subseteq S_r$. But S_r is an ideal of X which implies $X.\alpha_i(s).e_i(s) \subseteq S_r$. This further implies $S_l \subseteq S_r$. Similarly $S_r \subseteq S_l$. Thus $S_l = S_r$.

This completes the proof of the theorem.

Theorem 3.5. *Let P be a left ideal of a Γ-ring X. Then the following are equivalent*

(1) *P is prime.*

(2) *I, J are left ideals of X, $I\Gamma J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.*

Proof. (1) implies (2)

Let I, J be left ideals of X such that $I, J \nsubseteq P$. Let $x(s) \in I, y(s) \in J$ such that $x(s), y(s)$ doesnot belong to P. Then there exists $c(s) \in X$, $\alpha(s), \beta(s) \in \Gamma$ such that $x(s).\alpha(s).c(s).\beta(s).y(s)$ doesnot belong to P. Since $x(s).\alpha(s).c(s).\beta(s).y(s) \in I\Gamma J$. This implies $I\Gamma J \nsubseteq P$.

(2) implies (1)

Let $a(s).\Gamma.X.\Gamma.b(s) \subseteq P$. Then $(X.\Gamma.a(s)).\Gamma.(X.\Gamma.b(s)) \subseteq P$. Since $X.\Gamma.a(s)$ and $X.\Gamma.b(s)$ are left ideals of X this implies $X.\Gamma.a(s) \subseteq P$ or $X.\Gamma.b(s) \subseteq P$. Suppose $X.\Gamma.a(s) \subseteq P$. Let I be the left ideal of X generated by $a(s)$. Then $I\Gamma I \subseteq X.\Gamma.a(s) \subseteq P$ which implies $I \subseteq P$ which further implies $a(s) \in P$. Similarly if $X.\Gamma.b(s) \subseteq P$ implies $b(s) \in P$. Hence the theorem.

We now establish the relationships between prime one-sided ideals of X and R'.

Theorem 3.6. *Let P be a prime left (right) ideal of R' then P^* is a prime left (right) ideal of X.*

Proof. Since P is a left(right) ideal of R', P^* is a left (right) ideal of X. Let $x(s)$, $y(s)$ doesnot belong to P^*. Then there exists $\alpha(s), \beta(s) \in \Gamma$ such that $[\alpha(s), x(s)], [\beta(s), y(s)]$ doesnot belong to P. Since P is prime there exists $r(s) \in R'$ such that

$$[\alpha(s), x(s)].r(s).[\beta(s), y(s)] \text{ doesnot belong to } P,$$

that is there exists $\gamma(s) \in \Gamma$ and $z(s) \in X$ such that

$$[\alpha(s), x(s)].[\gamma(s), z(s)].[\beta(s), y(s)] \text{ doesnot belong to } P,$$

which implies

$$[\alpha(s), x(s).\gamma(s).z(s).\beta(s).y(s)] \text{ doesnot belong to } P,$$

which further implies

$$x(s).\gamma(s).z(s).\beta(s).y(s) \text{ doesnot belong to } P^*.$$

Thus P^* is prime in X which completes the proof of the theorem.

Theorem 3.7. *Let Q be a prime left(right) ideal of X. Then $Q^{*'}$ is a prime left(right) ideal of R'.*

Proof. Since Q is a left (right) ideal of X, $Q^{*'}$ is a left(right) ideal of R'.

Suppose $x(s), y(s)$ doesnot belong to $Q^{*'}$ then there exists $a(s), b(s) \in X$ such that $a(s).x(s), b(s).y(s)$ doesnot belong to Q. Since Q is a prime one-sided ideal of X, there exists $c(s) \in X$, $\alpha(s), \beta(s) \in \Gamma$ such that

$$\{a(s).x(s)\}.\alpha(s).c(s).\beta(s).\{b(s).y(s)\} \text{ doesnot belong to } Q,$$

which implies

$$a(s).\{x(s).[\alpha(s), c(s)].[\beta(s).b(s)].y(s)\} \text{ doesnot belong to } Q,$$

which further implies

$$x(s).[\alpha(s), c(s)].[\beta(s).b(s)].y(s) \text{ doesnot belong to } Q^{*'}.$$

Thus

$$x(s).R'.y(s)\,\text{doesnot belong to}\,Q^{*'}.$$

Hence $Q^{*'}$ is prime in R'. Hence the theorem.

Theorem 3.8. *The mapping $P \to P^*$ defines a one-to-one correspondence between the sets of prime left ideals of R' and X.*

Proof. Let P be a prime left ideal of R'. By Theorem 3.6, P^* is a prime left ideal of X. It is easily verified that $(P^*)^{*'} = \{r(s) \in R' | R'.r(s) \subseteq P\}$. Since P is the left ideal of R' implies $P \subseteq (P^*)^{*'}$. Now let $x(s) \in (P^*)^{*'}$. This implies $R'.x(s) \subseteq P$ and hence $x(s).R'.x(s) \subseteq P$. Since P is prime in R' one gets $x(s) \in P$ which implies $(P^*)^{*'} \subseteq P$. Thus $P = (P^*)^{*'}$.

Suppose now that Q is a prime left ideal of X. By Theorem 3.7, $Q^{*'}$ is a prime left ideal of R'. Now $(Q^{*'})^* = \{a(s) \in X | X.\Gamma.a(s) \subseteq Q\}$. Since Q is a left ideal of X implies $Q \subseteq (Q^{*'})^*$. Now let $x(s) \in (Q^{*'})^*$ which implies $X.\Gamma.x(s) \subseteq Q$. Hence $x(s).\Gamma.X.\Gamma.x(s) \subseteq Q$ further implies $x(s) \in Q$. Thus $(Q^{*'})^* = Q$. This completes the proof.

Corollary 3.1. *Let $P(X)$ be the prime radical of X. Then $P(X)$ is the intersection of the prime left ideals of X.*

Proof. Let $P(R')$ denote the prime radical of R'. Then $P(R')$ is the intersection of the prime left ideals of R'. Moreover $P(R')^* = P(X)$ in [9]. Hence

$$\begin{aligned} P(X) &= \left(\bigcap\{I|\, I\,\text{is a prime left ideal of}\, R'\}\right)^* \\ &= \bigcap\{I|\, I \text{ is a prime left ideal of } R'\}^* \\ &= \{J|\, J \text{ is a prime left ideal of } X\} \text{ (By Theorem 3.8).} \end{aligned}$$

References

[1] F. A. Szasz, Radicals of rings, Wiley, Chichester, 1981.

[2] G. L. Booth and N. J. Groenewald, On prime one-sided ideals, bi-ideals and quasi-ideals of a gamma ring, *J. Austral. Math. Soc.,* 53(A) (1992), pp. 55-63.

[3] N. Jacobson, Structure of rings, Amer. Math. Soc. Colloquium Publ. 37 Providence, 1964.

[4] N. Nobusawa, On a generalization of the ring theory, *Osaka J. Math.,* 1 (1964), pp. 81–89.

[5] S. Kyuno, On prime gamma rings, *Pacific J. Math.,* 75 (1978), pp. 185–190.

[6] S. Kyuno, Prime ideals in gamma rings, *Pacific J. Math.,* 98 (1982), pp. 375–379.

[7] S. Mandelbrojt, Dirichlet Series: principles and methods, D. Riedel Pub. Co., Dordrecht, Holland, 1972.

[8] W. E. Barnes, On the Γ-rings of Nobusawa, *Pacific J. Math.,* 18 (1966), pp. 411–422.

[9] W. E. Coppage and J. Luh, Radicals of gamma rings, *J. Math. Soc. Japan,* 23 (1971), pp. 40-52.

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