Additions to A Class of Position-Dependent Mass Liénard Differential Equations via a General Nonlocal Transformation

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Abstract

This work aims to present some specific examples of the generalized mixed Liénard differential equation and position-dependent mass Liénard equation depicted in A Class of Position-Dependent Mass Liénard Differential Equations via a General Nonlocal Transformation.

1. Consider, first, the generalized mixed Liénard nonlinear differential equation [1]

$$\ddot{x}+\left(l\frac{g'(x)}{g(x)}-\gamma\varphi'(x)\right)\dot{x}^2+\mu\dot{x}\exp(\gamma\varphi(x))+\frac{\omega^2\exp(2\gamma\varphi(x))\int g(x)^l dx}{g(x)^l}=0 \tag{1}$$

Let $g(x)=1$. Then equation (1) reduces to

$$\ddot{x}-\gamma\varphi'(x)\dot{x}^2+\mu\dot{x}\exp(\gamma\varphi(x))+\omega^2 x\exp(2\gamma\varphi(x))=0 \tag{2}$$

For $\varphi(x)=\ln(f(x))$, one can get

$$\ddot{x}-\gamma\frac{f'(x)}{f(x)}\dot{x}^2+\mu\dot{x}f(x)^\gamma+\omega^2 xf(x)^{2\gamma}=0 \tag{3}$$

Substituting $f(x)=\frac{1}{\sqrt{1+\lambda x^2}}$, $\gamma=1$, and $\mu=0$, into (3), gives

$$\ddot{x}+\frac{\lambda x}{1+\lambda x^2}\dot{x}^2+\frac{\omega^2 x}{1+\lambda x^2}=0 \tag{4}$$

Equation (4) is the so called equation of a particle moving on a rotating parabola [2]. Usually the solution of (4) is well known to be expressed in terms of elliptic integral of the second kind. The present theory shows that the solution of (4) may be, in principle, expressed in terms of sinusoidal function by linearization, taking into consideration the nonlocal transformation which shows its connection with the linear harmonic oscillator equation [1].

Putting now $f(x)=\frac{1}{\sqrt{1-\lambda x^2}}$, $\gamma=1$, and $\mu=0$, into the equation (3), yields immediately

$$\ddot{x}-\frac{\lambda x}{1-\lambda x^2}\dot{x}^2+\frac{\omega^2 x}{1-\lambda x^2}=0 \tag{5}$$

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One may see that equation (5) is a quadratic Liénard-type nonlinear differential equation expressed with quadratic damping. To be more precise, equation (5) consists of an inverted version of the Mathews-Lakshmanan oscillator equation. The solution of equation (5) may also in principle be written in terms of sinusoidal function.

2. Consider now the generalized position-dependent mass equation [1]

$$\ddot{x}+(l\frac{g'(x)}{g(x)}-\gamma\frac{f'(x)}{f(x)})\dot{x}^2+\mu\dot{x}f(x)^{\gamma}+\frac{\omega^2 f(x)^{2\gamma}\int g(x)^l dx}{g(x)^l}=0 \tag{6}$$

Setting $l=1$, $\gamma=0$, and $\mu=0$, leads to

$$\ddot{x}+\frac{g'(x)}{g(x)}\dot{x}^2+\frac{\omega^2\int g(x)dx}{g(x)}=0 \tag{7}$$

For $g(x)=x^m$, equation (7) becomes the well known quadratic Liénard equation

$$\ddot{x}+m\frac{\dot{x}^2}{x}+\frac{\omega^2 x}{m+1}=0 \tag{8}$$

known also as an integrable Painlevé equation by quadratures. Setting $g(x)=\frac{1}{(\alpha^2+x^2)^{3/2}}$, yields as quadratic Liénard equation

$$\ddot{x}-\frac{3x}{\alpha^2+x^2}\dot{x}^2+\frac{\omega^2 x}{\alpha^2}(\alpha^2+x^2)=0 \tag{9}$$

The generalized quadratic Liénard equation (7) may also be generated from the variable transformation

$$\dot{u}=\dot{x}g(x)\exp(f(x)) \tag{10}$$

with $\ddot{u}+\omega^2 u=0$, under a consistent relationship between $f(x)$, and $g(x)$.

3. An additional specific example of the generalized position-dependent mass Liénard nonlinear differential equation (6) may be, for $f(x)=g(x)=x$, and $l=\gamma$, obtained as the generalized modified Emden-type nonlinear oscillator differential equation

$$\ddot{x}+\mu x^l\dot{x}+\frac{\omega^2}{1+l}x^{2l+1}=0 \tag{11}$$

known to have the ability in the case $\mu=0$, to generate the famous second-order Duffing nonlinear oscillator differential equations

$$\ddot{x}+\frac{\omega^2}{2}x^3=0 \tag{12}$$

for $l=1$, and for $l=2$

$$\ddot{x} + \frac{\omega^2}{3} x^5 = 0 \tag{13}$$

Finally it should be noted that the generalized position-dependent mass Liénard differential equation (6) may generate also the well known shifted Mathews-Lakshmanan nonlinear oscillator equations and Morse-type nonlinear oscillator equation and several other nonlinear position-dependent mass oscillator equations.

References

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