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Counter-examples to Dempster's rule of combination

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Abstract: *This chapter presents several classes of fusion problems which cannot be directly approached by the classical mathematical theory of evidence, also known as Dempster-Shafer Theory (DST), either because Shafer's model for the frame of discernment is impossible to obtain, or just because Dempster's rule of combination fails to provide coherent results (or no result at all). We present and discuss the potentiality of the DSmT combined with its classical (or hybrid) rule of combination to attack these infinite classes of fusion problems.*

5.1 Introduction

In this chapter we focus our attention on the limits of the validity of Dempster's rule of combination in Dempster-Shafer theory (DST) [5]. We provide several infinite classes of fusion problems where Dempster rule of combination fails to provide coherent results and we show how these problems can be attacked directly by the DSmT presented in previous chapters. DST and DSmT are based on a different approach for modelling the frame Θ of the problem (Shafer's model versus free-DSm, or hybrid-DSm model), on the choice of the space (classical power set 2^Θ versus hyper-power set D^Θ) on which will be defined the basic belief assignment functions $m_i(.)$ to be combined, and on the fusion rules to apply (Dempster rule versus DSm rule or hybrid DSm rule of combination).

5.2 First infinite class of counter examples

The first infinite class of counter examples for Dempster's rule of combination consists trivially in all cases for which Dempster's rule becomes mathematically not defined, i.e. one has 0/0, because of full conflicting sources. The first sub-class presented in subsection 5.2.1 corresponds to Bayesian belief functions. The subsection 5.2.2 will present counter-examples for more general conflicting sources of evidence.

5.2.1 Counter-examples for Bayesian sources

The following examples are devoted only to Bayesian sources, i.e. sources for which the focal elements of belief functions coincide only with some singletons θ_i of Θ.

5.2.1.1 Example with $\Theta = \{\theta_1, \theta_2\}$

Let's consider the frame of discernment $\Theta = \{\theta_1, \theta_2\}$, two independent experts, and the basic belief masses:

$$m_1(\theta_1) = 1 \qquad m_1(\theta_2) = 0$$

$$m_2(\theta_1) = 0 \qquad m_2(\theta_2) = 1$$

We represent these belief assignments by the mass matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Dempster's rule can not be applied because one formally gets $m(\theta_1) = 0/0$ and $m(\theta_2) = 0/0$ as well, i.e. undefined.

- The DSm rule works here because one obtains $m(\theta_1) = m(\theta_2) = 0$ and $m(\theta_1 \cap \theta_2) = 1$ (the total paradox, which it really is! if one accepts the free-DSm model). If one adopts Shafer's model and applies the hybrid DSm rule, then one gets $m_h(\theta_1 \cup \theta_2) = 1$ which makes sense in this case. The index h denotes here the mass obtained with the hybrid DSm rule to avoid confusion with result obtained with the DSm classic rule.

5.2.1.2 Example with $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$

Let's consider the frame of discernment $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$, two independent experts, and the mass matrix

$$\begin{bmatrix} 0.6 & 0 & 0.4 & 0 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

- Again, Dempster's rule can not be applied because: $\forall 1 \leq j \leq 4$, one gets $m(\theta_j) = 0/0$ (undefined!).
- But the DSm rule works because one obtains: $m(\theta_1) = m(\theta_2) = m(\theta_3) = m(\theta_4) = 0$, and $m(\theta_1 \cap \theta_2) = 0.12$, $m(\theta_1 \cap \theta_4) = 0.48$, $m(\theta_2 \cap \theta_3) = 0.08$, $m(\theta_3 \cap \theta_4) = 0.32$ (partial paradoxes/conflicts).
- Suppose now one finds out that all intersections are empty (Shafer's model), then one applies the hybrid DSm rule and one gets (index h stands here for *hybrid* rule): $m_h(\theta_1 \cup \theta_2) = 0.12$, $m_h(\theta_1 \cup \theta_4) = 0.48$, $m_h(\theta_2 \cup \theta_3) = 0.08$ and $m_h(\theta_3 \cup \theta_4) = 0.32$.

5.2.1.3 Another example with $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$

Let's consider the frame of discernment $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$, three independent experts, and the mass matrix

$$\begin{bmatrix} 0.6 & 0 & 0.4 & 0 \\ 0 & 0.2 & 0 & 0.8 \\ 0 & 0.3 & 0 & 0.7 \end{bmatrix}$$

- Again, Dempster's rule can not be applied because: $\forall 1 \leq j \leq 4$, one gets $m(\theta_j) = 0/0$ (undefined!).
- But the DSm rule works because one obtains: $m(\theta_1) = m(\theta_2) = m(\theta_3) = m(\theta_4) = 0$, and

$$m(\theta_1 \cap \theta_2) = 0.6 \cdot 0.2 \cdot 0.3 = 0.036$$

$$m(\theta_1 \cap \theta_4) = 0.6 \cdot 0.8 \cdot 0.7 = 0.336$$

$$m(\theta_2 \cap \theta_3) = 0.4 \cdot 0.2 \cdot 0.3 = 0.024$$

$$m(\theta_3 \cap \theta_4) = 0.4 \cdot 0.8 \cdot 0.7 = 0.224$$

$$m(\theta_1 \cap \theta_2 \cap \theta_4) = 0.6 \cdot 0.2 \cdot 0.7 + 0.6 \cdot 0.3 \cdot 0.8 = 0.228$$

$$m(\theta_2 \cap \theta_3 \cap \theta_4) = 0.2 \cdot 0.4 \cdot 0.7 + 0.3 \cdot 0.4 \cdot 0.8 = 0.152$$

  (partial paradoxes/conflicts) and the others equal zero. If we add all these masses, we get the sum equals to 1.

- Suppose now one finds out that all intersections are empty (Shafer's model), then one applies the hybrid DSm rule and one gets: $m_h(\theta_1 \cup \theta_2) = 0.036$, $m_h(\theta_1 \cup \theta_4) = 0.336$, $m_h(\theta_2 \cup \theta_3) = 0.024$, $m_h(\theta_3 \cup \theta_4) = 0.224$, $m_h(\theta_1 \cup \theta_2 \cup \theta_4) = 0.228$, $m_h(\theta_2 \cup \theta_3 \cup \theta_4) = 0.152$.

5.2.1.4 More general

Let's consider the frame of discernment $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$, with $n \geq 2$, and k experts, for $k \geq 2$. Let $\mathbf{M} = [a_{ij}]$, $1 \leq i \leq k$, $1 \leq j \leq n$, be the mass matrix with k rows and n columns. If each column of the mass matrix contains at least a zero, then Dempster's rule can not be applied because one obtains for all $1 \leq j \leq n$, $m(\theta_j) = 0/0$ which is undefined! The degree of conflict is 1. However, one can use the classical DSm rule and one obtains: for all $1 \leq j \leq n$, $m(\theta_j) = 0$, and also partial paradoxes/conflicts: $\forall 1 \leq v_s \leq n$, $1 \leq s \leq w$, and $2 \leq w \leq k$, $m(\theta_{v_1} \cap \theta_{v_2} \cap \ldots \cap \theta_{v_w}) = \sum (a_{1t_1}) \cdot (a_{2t_2}) \cdot \ldots \cdot (a_{kt_k})$, where the set $T = \{t_1, t_2, \ldots, t_k\}$ is equal to the set $V = \{v_1, v_2, \ldots, v_w\}$ but the order may be different and the elements in the set T could be repeated; we mean from set V one obtains set T if one repeats some elements of V; therefore: summation $\sum$ is done upon all possible combinations of elements from columns v_1, v_2, $\ldots$, v_w such that at least one element one takes from each of these columns v_1, v_2, $\ldots$, v_w and also such that from each row one takes one element only; the product $(a_{1t_1}) \cdot (a_{2t_2}) \cdot \ldots \cdot (a_{kt_k})$ contains one element only from each row 1, 2, $\ldots$, k respectively, and one or more elements from each of the columns v_1, v_2, $\ldots$, v_w respectively.

5.2.2 Counter-examples for more general sources

We present in this section two numerical examples involving general (i.e. non Bayesian) sources where Dempster's rule cannot be applied.

5.2.2.1 Example with $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$

Let's consider $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$, two independent experts, and the mass matrix:

	θ_1	θ_2	θ_3	θ_4	$\theta_1 \cup \theta_2$
$m_1(.)$	0.4	0.5	0	0	0.1
$m_2(.)$	0	0	0.3	0.7	0

Dempster's rule cannot apply here because one gets 0/0 for all $m(\theta_i)$, $1 \leq i \leq 4$, but the DSm rules (classical or hybrid) work.

Using the DSm classical rule: $m(\theta_1 \cap \theta_3) = 0.12$, $m(\theta_1 \cap \theta_4) = 0.28$, $m(\theta_2 \cap \theta_3) = 0.15$, $m(\theta_2 \cap \theta_4) = 0.35$, $m(\theta_3 \cap (\theta_1 \cup \theta_2)) = 0.03$, $m(\theta_4 \cap (\theta_1 \cup \theta_2)) = 0.07$.

Suppose now one finds out that one has a Shafer model; then one uses the hybrid DSm rule (denoted here with index h): $m_h(\theta_1 \cup \theta_3) = 0.12$, $m_h(\theta_1 \cup \theta_4) = 0.28$, $m_h(\theta_2 \cup \theta_3) = 0.15$, $m_h(\theta_2 \cup \theta_4) = 0.35$, $m_h(\theta_3 \cup \theta_1 \cup \theta_2) = 0.03$, $m_h(\theta_4 \cup \theta_1 \cup \theta_2) = 0.07$.

5.2.2.2 Another example with $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$

Let's consider $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$, three independent experts, and the mass matrix:

	θ_1	θ_2	θ_3	θ_4	$\theta_1 \cup \theta_2$	$\theta_3 \cup \theta_4$
$m_1(.)$	0.4	0.5	0	0	0.1	0
$m_2(.)$	0	0	0.3	0.6	0	0.1
$m_3(.)$	0.8	0	0	0	0.2	0

Dempster's rule cannot apply here because one gets 0/0 for all $m(\theta_i)$, $1 \leq i \leq 4$, but the DSm rules (classical or hybrid) work.

Using the DSm classical rule, one gets:

$$m(\theta_1) = m(\theta_2) = m(\theta_3) = m(\theta_4) = 0$$
$$m(\theta_1 \cup \theta_2) = m(\theta_3 \cup \theta_4) = 0$$
$$m(\theta_1 \cap \theta_3) = 0.096$$
$$m(\theta_1 \cap \theta_3 \cap (\theta_1 \cup \theta_2)) = m(\theta_1 \cap \theta_3) = 0.024$$
$$m(\theta_1 \cap \theta_4) = 0.192$$
$$m(\theta_1 \cap \theta_4 \cap (\theta_1 \cup \theta_2)) = m(\theta_1 \cap \theta_4) = 0.048$$
$$m(\theta_1 \cap (\theta_3 \cup \theta_4)) = 0.032$$
$$m(\theta_1 \cap (\theta_3 \cup \theta_4) \cap (\theta_1 \cup \theta_2)) = m(\theta_1 \cap (\theta_3 \cup \theta_4)) = 0.008$$
$$m(\theta_2 \cap \theta_3 \cap \theta_1) = 0.120$$
$$m(\theta_2 \cap \theta_3 \cap (\theta_1 \cup \theta_2)) = m(\theta_2 \cap \theta_3) = 0.030$$
$$m(\theta_2 \cap \theta_4 \cap \theta_1) = 0.240$$
$$m(\theta_2 \cap \theta_4 \cap (\theta_1 \cup \theta_2)) = m(\theta_2 \cap \theta_4) = 0.060$$
$$m(\theta_2 \cap (\theta_3 \cup \theta_4) \cap \theta_1) = m((\theta_1 \cap \theta_2) \cap (\theta_3 \cup \theta_4)) = 0.040$$
$$m(\theta_2 \cap (\theta_3 \cup \theta_4) \cap (\theta_1 \cup \theta_2)) = m(\theta_2 \cap (\theta_3 \cup \theta_4)) = 0.010$$
$$m((\theta_1 \cup \theta_2) \cap \theta_3 \cap \theta_1) = m(\theta_1 \cap \theta_3) = 0.024$$
$$m((\theta_1 \cup \theta_2) \cap \theta_3) = 0.006$$
$$m((\theta_1 \cup \theta_2) \cap \theta_4 \cap \theta_1) = m(\theta_1 \cap \theta_4) = 0.048$$
$$m((\theta_1 \cup \theta_2) \cap \theta_4) = 0.012$$
$$m((\theta_1 \cup \theta_2) \cap (\theta_3 \cup \theta_4) \cap \theta_1) = m(\theta_1 \cap (\theta_3 \cup \theta_4)) = 0.008$$
$$m((\theta_1 \cup \theta_2) \cap (\theta_3 \cup \theta_4)) = 0.002$$

After cumulating, one finally gets with DSm classic rule:

$$m(\theta_1 \cap \theta_3) = 0.096 + 0.024 + 0.024 = 0.144$$
$$m(\theta_1 \cap \theta_4) = 0.192 + 0.048 + 0.048 = 0.288$$
$$m(\theta_2 \cap \theta_3) = 0.030$$
$$m(\theta_2 \cap \theta_4) = 0.060$$
$$m(\theta_1 \cap \theta_2 \cap \theta_3) = 0.120$$
$$m(\theta_1 \cap \theta_2 \cap \theta_4) = 0.240$$
$$m((\theta_1 \cup \theta_2) \cap \theta_3) = 0.006$$
$$m((\theta_1 \cup \theta_2) \cap \theta_4) = 0.012$$
$$m(\theta_1 \cap (\theta_3 \cup \theta_4)) = 0.032 + 0.008 + 0.008 = 0.048$$
$$m(\theta_1 \cap \theta_2 \cap (\theta_3 \cup \theta_4)) = 0.040$$
$$m(\theta_2 \cap (\theta_3 \cup \theta_4)) = 0.010$$
$$m((\theta_1 \cup \theta_2) \cap (\theta_3 \cup \theta_4)) = 0.002$$

Suppose now, one finds out that all intersections are empty. Using the hybrid DSm rule one gets:

$$m_h(\theta_1 \cup \theta_3) = 0.144$$
$$m_h(\theta_1 \cup \theta_4) = 0.288$$
$$m_h(\theta_2 \cup \theta_3) = 0.030$$
$$m_h(\theta_2 \cup \theta_4) = 0.060$$
$$m_h(\theta_1 \cup \theta_2 \cup \theta_3) = 0.120 + 0.006 = 0.126$$
$$m_h(\theta_1 \cup \theta_2 \cup \theta_4) = 0.240 + 0.012 = 0.252$$
$$m_h(\theta_1 \cup \theta_3 \cup \theta_4) = 0.048$$
$$m_h(\theta_2 \cup \theta_3 \cup \theta_4) = 0.010$$
$$m_h(\theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4) = 0.040 + 0.002 = 0.042$$

5.2.2.3 More general

Let's consider the frame of discernment $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$, with $n \geq 2$, and k experts, for $k \geq 2$, and the mass matrix $\mathbf{M}$ with k rows and $n+u$ columns, where $u \geq 1$, corresponding to θ_1, θ_2, ..., θ_n, and u uncertainties $\theta_{i_1} \cup \ldots \cup \theta_{i_s}$, ..., $\theta_{j_1} \cup \ldots \cup \theta_{j_t}$ respectively.

If the following conditions occur:

- each column contains at least one zero;
- all uncertainties are different from the total ignorance $\theta_1 \cup \ldots \cup \theta_n$ (i.e., they are partial ignorances);
- the partial uncertainties are disjoint two by two;
- for each non-null uncertainty column c_j, $n+1 \leq j \leq n+u$, of the form say $\theta_{p_1} \cup \ldots \cup \theta_{p_w}$, there exists a row such that all its elements on columns p_1, ..., p_w, and c_j are zero.

then Dempster's rule of combination cannot apply for such infinite class of fusion problems because one gets $0/0$ for all $m(\theta_i)$, $1 \leq i \leq n$. The DSm rules (classical or hybrid) work for such infinite class of examples.

5.3 Second infinite class of counter examples

This second class of counter-examples generalizes the famous Zadeh example given in [7, 8].

5.3.1 Zadeh's example

Two doctors examine a patient and agree that it suffers from either meningitis (M), contusion (C) or brain tumor (T). Thus $\Theta = \{M, C, T\}$. Assume that the doctors agree in their low expectation of a tumor, but disagree in likely cause and provide the following diagnosis

$$m_1(M) = 0.99 \qquad m_1(T) = 0.01 \qquad \text{and} \qquad m_2(C) = 0.99 \qquad m_2(T) = 0.01$$

If we combine the two basic belief functions using Dempster's rule of combination, one gets the unexpected final conclusion

$$m(T) = \frac{0.0001}{1 - 0.0099 - 0.0099 - 0.9801} = 1$$

which means that the patient suffers with certainty from brain tumor !!!. This unexpected result arises from the fact that the two bodies of evidence (doctors) agree that the patient most likely does not suffer from tumor but are in almost full contradiction for the other causes of the disease. This very simple but interesting example shows the limitations of the practical use of the DST for automated reasoning.

This example has been examined in literature by several authors to explain the anomaly of the result of Dempster's rule of combination in such case. Due to the high degree of conflict arising in such extreme case, willingly pointed out by Zadeh to show the weakness of this rule, it is often argued that in such case the result of Dempster's rule must not be taken directly without checking the level of the conflict between sources of evidence. This is trivially true but there is no theoretical way to decide *beforehand* if one can trust or not the result of such rule of combination, especially in complex systems involving many sources and many hypotheses. This is one of its major drawback. The issue consists generally in choosing rather somewhat arbitrarily or heuristically some threshold value on the degree of conflict between sources to accept or reject the result of the fusion [9]. Such approach can't be solidly justified from theoretical analysis. Assuming such threshold is set to a given value, say 0.70 for instance, is it acceptable to reject the fusion result if the conflict appears to be 0.7001 and accept it when the conflict becomes 0.6999? What to do when the decision about the fusion result is rejected and one has no assessment on the reliability of the sources or when the sources have the same reliability/confidence but an important decision has to be taken anyway? There is no theoretical solid justification which can reasonably support such kind of approaches commonly used in practice up to now.

The two major explanations of this problem found in literature are mainly based, either on the fact that problem arises from the closed-world assumption of Shafer's model Θ and it is suggested to work rather with an open-world model, and/or the fact that sources of evidence are not reliable. These explanations although being admissible are not necessarily the only correct (sufficient) explanations. Note that the open-world assumption can always be easily relaxed advantageously by introducing a new hypothesis, say θ_0 in the initial frame $\Theta = \{\theta_1, \dots, \theta_n\}$ in order to close it. θ_0 will then represent all possible alternatives (although remaining unknown) of initial hypotheses $\theta_1, \dots \theta_n$. This idea has been already proposed by Yager in [6] through his hedging solution. Upon our analysis, it is not necessary to adopt/follow the open-world model neither to admit the assumption about the reliability of the sources to find a justification in this counter-intuitive result. Actually, both sources can have the same reliability and Shafer's model can be accepted for the combination of the two reports by using another rule of combination. This is exactly the purpose of the hybrid DSm rule of combination. Of course when one has some prior information on the reliability of sources, one has to take them into account properly by some discounting methods. The discounting techniques can also apply in the DSmT framework and there is no incompatibility to mix both (i.e. discounting techniques with DSm rules of combinations) when necessary (when there is strong reason to justify doing it, i.e. when one has prior reliable information on reliability of the sources). The discounting techniques must never been used as an artificial ad-hoc mechanism to update Dempster's result once problem has arisen. We strongly disagree with the idea that all problems with Dempster's rule can be solved *beforehand* by discounting techniques. This can help obviously to improve the assessment of belief function to be combined when used properly and fairly, but this does not fundamentally solve the inherent problem of Dempster's rule itself when conflict remains high.

The problem comes from the fact that both sources provide essentially their belief with respect only to their own limited knowledge and experience. It is also possible in some cases, that sources of information even don't have the same interpretation of concepts included in the frame of the problem. Such kind of situation frequently appears for example in debates on TV, on radio or in most of the meetings where important decision/approval have to be drawn and when the sources don't share the same opinion. This is what happens daily in real life and one has to deal with such conflicting situations anyway. In other words, the sources do not speak about the same events or even they do, they there is a possibility that they do not share the same interpretation of the events. This has already been pointed out by Dubois and Prade in [3] (p. 256). In Zadeh's controversy example, it is possible that the first doctor is expert mainly in meningitis and in brain tumor while the second doctor is expert mainly in cerebral contusion and in brain tumor. Because of their limited knowledges and experiences, both doctors can also have also the same reliability. If they have been asked to give their reports only on $\Theta = \{M, C, T\}$ (but not on an extended frame), their reports have to be taken with same weight and the combination has to be done anyway when one has no solid reason to reject one report with respect to the other one; the result of the Demsper's rule still remains very questionable. No rational brain surgeon would take the decision for a brain intervention (i.e. a risky tumor ablation) based on Dempster's rule result, neither the family of the patient. Therefore upon our analysis, the two previous explanations given in literature (although being possible and admissible in some cases) are not necessary and sufficient to explain the source of the anomaly. Several alternatives to Dempster's rule to circumvent this anomaly have been proposed in literature mainly through the works of R. Yager [6], D. Dubois and H. Prade [2] already reported in chapter 1 or by Daniel in [1]. The DSmT offers just a new issue for solving also such controversy example as it will be shown. In summary, some extreme caution on the degree of conflict of the sources must always be taken before taking a final decision based on Dempster's rule of combination, especially when vital wagers are involved.

If we now adopt the free-DSm model, i.e. we replace the initial Shafer model by accepting the possibility of non null intersections between hypotheses M, C and T and by working directly on hyper-power set D^Θ then one gets directly and easily the following result with the classical DSm rule of combination:

$$m(M \cap C) = 0.9801 \qquad m(M \cap T) = 0.0099 \qquad m(C \cap T) = 0.0099 \qquad m(T) = 0.0001$$

which makes sense when working with such a new model. Obviously same result can be obtained (the proof is left here to the reader) when working with Dempster's rule based on the following refined frame Θ_{ref} defined with basic belief functions on power set $2^{\Theta_{ref}}$:

$$\Theta_{ref} = \{\theta_1 = M \cap C \cap T, \theta_2 = M \cap C \cap \bar{T}, \theta_3 = M \cap \bar{C} \cap T, \theta_4 = \bar{M} \cap C \cap T,$$
$$\theta_5 = M \cap \bar{C} \cap \bar{T}, \theta_6 = \bar{M} \cap C \cap \bar{T}, \theta_7 = \bar{M} \cap \bar{C} \cap T\}$$

where $\bar{T}$,$\bar{C}$ and $\bar{M}$ denote respectively the complement of T, C and M.

The equality of both results (i.e. by the classical DSm rule based on the free-DSm model and by Dempster's rule based on the refined frame) is just normal since the normalization factor $1 - k$ of Dempster's rule in this case reduces to 1 because of the new choice of the new model. Based on this remark, one could then try to argue that DSmT (together with its DSm classical rule for free-DSm model) is superfluous. Such claim is obviously wrong for the two following reasons: it is unecessary to work with a bigger space (keeping in mind that $|D^\Theta| < |2^{\Theta_{ref}}|$) to get the result (the DSm rule offers just a direct and more convenient issue to get the result), but also because in some fusion problems involving vague/-continuous concepts, the refinement is just impossible to obtain and we are unfortunately forced to deal with ambiguous concepts/hypotheses (see [4] for details and justification).

If one has no doubt on the reliability of both Doctors (or no way to assess it) and if one is absolutely sure that the true origin of the suffering of the patient lies only in the frame $\Theta = \{M, C, T\}$ and we consider these origins as truly exclusive, then one has to work with the initial frame of discernment Θ satisfying Shafer's model. As previously shown, Dempster's rule fails to provide a reasonable and acceptable conclusion in such high conflicting case. However, this case can be easily handled by the hybrid DSm rule of combination. The hybrid DSm rule applies now because Shafer's model is nothing but a particular hybrid model including all exclusivity constraints between hypotheses of the frame Θ (see chapter 4 for details). One then gets with the hybrid DSm rule for this simple case (more general and complex examples have been already presented in chapter 4), after the proper mass transfer of all sources of the conflicts:

$$m(M \cup C) = 0.9801 \qquad m(M \cup T) = 0.0099 \qquad m(C \cup T) = 0.0099 \qquad m(T) = 0.0001$$

This result is not surprising and makes perfectly sense with common intuition actually since it provides a coherent and reasonable solution to the problem. It shows clearly that a brain intervention for ablation of an hypothetical tumor is not recommended, but preferentially a better examination of the patient focused on Meningitis or Contusion as possible source of the suffering. The consequence of the results of Dempster's rule and the hybrid DSm rule is therefore totally different.

5.3.2 Generalization with $\Theta = \{\theta_1, \theta_2, \theta_3\}$

Let's consider $0 < \epsilon_1, \epsilon_2 < 1$ be two very tiny positive numbers (close to zero), the frame of discernment be $\Theta = \{\theta_1, \theta_2, \theta_3\}$, have two experts (independent sources of evidence s_1 and s_2) giving the belief masses

$$m_1(\theta_1) = 1 - \epsilon_1 \quad m_1(\theta_2) = 0 \quad m_1(\theta_3) = \epsilon_1$$

$$m_2(\theta_1) = 0 \quad m_2(\theta_2) = 1 - \epsilon_2 \quad m_2(\theta_3) = \epsilon_2$$

From now on, we prefer to use matrices to describe the masses, i.e.

$$\begin{bmatrix} 1-\epsilon_1 & 0 & \epsilon_1 \\ 0 & 1-\epsilon_2 & \epsilon_2 \end{bmatrix}$$

- Using Dempster's rule of combination, one gets

  $$m(\theta_3) = \frac{(\epsilon_1\epsilon_2)}{(1-\epsilon_1)\cdot 0 + 0\cdot(1-\epsilon_2) + \epsilon_1\epsilon_2} = 1$$

  which is absurd (or at least counter-intuitive). Note that whatever positive values for ϵ_1, ϵ_2 are, Dempster's rule of combination provides always the same result (one) which is abnormal. The only acceptable and correct result obtained by Dempster's rule is really obtained only in the trivial case when $\epsilon_1 = \epsilon_2 = 1$, i.e. when both sources agree in θ_3 with certainty which is obvious.

- Using the DSm rule of combination based on free-DSm model, one gets $m(\theta_3) = \epsilon_1\epsilon_2$, $m(\theta_1 \cap \theta_2) = (1-\epsilon_1)(1-\epsilon_2)$, $m(\theta_1 \cap \theta_3) = (1-\epsilon_1)\epsilon_2$, $m(\theta_2 \cap \theta_3) = (1-\epsilon_2)\epsilon_1$ and the others are zero which appears more reliable/trustable.

- Going back to Shafer's model and using the hybrid DSm rule of combination, one gets $m(\theta_3) = \epsilon_1\epsilon_2$, $m(\theta_1 \cup \theta_2) = (1-\epsilon_1)(1-\epsilon_2)$, $m(\theta_1 \cup \theta_3) = (1-\epsilon_1)\epsilon_2$, $m(\theta_2 \cup \theta_3) = (1-\epsilon_2)\epsilon_1$ and the others are zero.

Note that in the special case when $\epsilon_1 = \epsilon_2 = 1/2$, one has

$$m_1(\theta_1) = 1/2 \quad m_1(\theta_2) = 0 \quad m_1(\theta_3) = 1/2 \qquad \text{and} \qquad m_2(\theta_1) = 0 \quad m_2(\theta_2) = 1/2 \quad m_2(\theta_3) = 1/2$$

Dempster's rule of combinations still yields $m(\theta_3) = 1$ while the hybrid DSm rule based on the same Shafer's model yields now $m(\theta_3) = 1/4$, $m(\theta_1 \cup \theta_2) = 1/4$, $m(\theta_1 \cup \theta_3) = 1/4$, $m(\theta_2 \cup \theta_3) = 1/4$ which is normal.

5.3.3 Generalization with $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$

Let's consider $0 < \epsilon_1, \epsilon_2, \epsilon_3 < 1$ be three very tiny positive numbers, the frame of discernment be $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$, have two experts giving the mass matrix

$$\begin{bmatrix} 1-\epsilon_1-\epsilon_2 & 0 & \epsilon_1 & \epsilon_2 \\ 0 & 1-\epsilon_3 & 0 & \epsilon_3 \end{bmatrix}$$

Again using Dempster's rule of combination, one gets $m(\theta_4) = 1$ which is absurd while using the DSm rule of combination based on free-DSm model, one gets $m(\theta_4) = \epsilon_2\epsilon_3$ which is reliable. Using the DSm classical rule: $m(\theta_1 \cap \theta_2) = (1-\epsilon_1-\epsilon_2)(1-\epsilon_3)$, $m(\theta_1 \cap \theta_4) = (1-\epsilon_1-\epsilon_3)\epsilon_3$, $m(\theta_3 \cap \theta_2) = \epsilon_1(1-\epsilon_3)$, $m(\theta_3 \cap \theta_4) = \epsilon_1\epsilon_3$, $m(\theta_4) = \epsilon_2\epsilon_3$. Suppose one finds out that all intersections are empty, then one applies the hybrid DSm rule: $m_h(\theta_1 \cup \theta_2) = (1-\epsilon_1-\epsilon_2)(1-\epsilon_3)$, $m_h(\theta_1 \cup \theta_4) = (1-\epsilon_1-\epsilon_3)\epsilon_3$, $m_h(\theta_3 \cup \theta_2) = \epsilon_1(1-\epsilon_3)$, $m_h(\theta_3 \cup \theta_4) = \epsilon_1\epsilon_3$, $m_h(\theta_4) = \epsilon_2\epsilon_3$.

5.3.4 More general

Let's consider $0 < \epsilon_1, \ldots, \epsilon_n < 1$ be very tiny positive numbers, the frame of discernment be $\Theta = \{\theta_1, \ldots, \theta_n, \theta_{n+1}\}$, have two experts giving the mass matrix

$$\begin{bmatrix} 1-S_1^p & 0 & \epsilon_1 & 0 & \epsilon_2 & \ldots & 0 & \epsilon_p \\ 0 & 1-S_{p+1}^n & 0 & \epsilon_{p+1} & 0 & \ldots & \epsilon_{n-1} & \epsilon_n \end{bmatrix}$$

where $1 \leq p \leq n$ and $S_1^p \triangleq \sum_{i=1}^p \epsilon_i$ and $S_{p+1}^n \triangleq \sum_{i=p+1}^n \epsilon_i$. Again using Dempster's rule of combination, one gets $m(\theta_{n+1}) = 1$ which is absurd while using the DSm rule of combination based on free-DSm model, one gets $m(\theta_{n+1}) = \epsilon_p\epsilon_n$ which is reliable. This example is similar to the previous one, but generalized.

5.3.5 Even more general

Let's consider $0 < \epsilon_1, \ldots, \epsilon_n < 1$ be very tiny positive numbers (close to zero), the frame of discernment be $\Theta = \{\theta_1, \ldots, \theta_n, \theta_{n+1}\}$, have $k \geq 2$ experts giving the mass matrix of k rows and $n+1$ columns such that:

- one column, say column j, is $(\epsilon_{j_1}, \epsilon_{j_2}, \ldots, \epsilon_{j_k})'$ (transposed vector), where $1 \leq j \leq n+1$ where $\{\epsilon_{j_1}, \epsilon_{j_2}, \ldots, \epsilon_{j_k}\}$ is included in $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$;
- and each column (except column j) contains at least one element equals to zero.

Then Dempster's rule of combination gives $m(\theta_j) = 1$ which is absurd, while the classical DSm rule gives $m(\theta_j) = \epsilon_{j_1} \cdot \epsilon_{j_2} \cdot \ldots \cdot \epsilon_{j_k} \neq 0$ which is reliable.

Actually, we need to set restrictions only for ϵ_{j_1}, ϵ_{j_2}, …, and ϵ_{j_k} to be very tiny positive numbers, not for all ϵ_1, ϵ_2, …, ϵ_n (the others can be anything in the interval $[0, 1)$ such that the sum of elements on each row be equal 1).

5.4 Third infinite class of counter examples

This third class of counter-examples deals with belief functions committing a non null mass to some uncertainties.

5.4.1 Example with $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$

Let's consider $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$, two independent experts, and the mass matrix:

	θ_1	θ_2	θ_3	θ_4	$\theta_3 \cup \theta_4$
$m_1(.)$	0.99	0	0	0	0.01
$m_2(.)$	0	0.98	0	0	0.02

If one applies Dempster's rule, one gets

$$m(\theta_3 \cup \theta_4) = \frac{(0.01 \cdot 0.02)}{(0+0+0+0+0.01 \cdot 0.02)} = 1$$

(total ignorance), which doesn't bring any information to the fusion. This example looks similar to Zadeh's example, but is different because it is referring to uncertainty (not to contradictory) result. Using the DSm classical rule: $m(\theta_1 \cap \theta_2) = 0.9702$, $m(\theta_1 \cap (\theta_3 \cup \theta_4)) = 0.0198$, $m(\theta_2 \cap (\theta_3 \cup \theta_4)) = 0.0098$, $m(\theta_3 \cup \theta_4) = 0.0002$. Suppose now one finds out that all intersections are empty (i.e. one adopts Shafer's model). Using the hybrid DSm rule one gets: $m_h(\theta_1 \cup \theta_2) = 0.9702$, $m_h(\theta_1 \cup \theta_3 \cup \theta_4) = 0.0198$, $m_h(\theta_2 \cup \theta_3 \cup \theta_4) = 0.0098$, $m_h(\theta_3 \cup \theta_4) = 0.0002$.

5.4.2 Example with $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}$

Let's consider $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4, , \theta_5\}$, three independent experts, and the mass matrix:

	θ_1	θ_2	θ_3	θ_4	θ_5	$\theta_4 \cup \theta_5$
$m_1(.)$	0.99	0	0	0	0	0.01
$m_2(.)$	0	0.98	0.01	0	0	0.01
$m_3(.)$	0.01	0.01	0.97	0	0	0.01

- If one applies Dempster's rule, one gets

$$m(\theta_4 \cup \theta_5) = \frac{(0.01 \cdot 0.01 \cdot 0.01)}{(0+0+0+0+0.01 \cdot 0.01 \cdot 0.01)} = 1$$

  (total ignorance), which doesn't bring any information to the fusion.

- Using the DSm classical rule one gets:

$$m(\theta_1 \cap \theta_2) = 0.99 \cdot 0.98 \cdot 0.01 + 0.99 \cdot 0.98 \cdot 0.01 = 0.019404$$

$$m(\theta_1 \cap \theta_3) = 0.99 \cdot 0.01 \cdot 0.01 + 0.99 \cdot 0.01 \cdot 0.97 = 0.009702$$

$$m(\theta_1 \cap \theta_2 \cap \theta_3) = 0.99 \cdot 0.98 \cdot 0.97 + 0.99 \cdot 0.01 \cdot 0.01 = 0.941193$$

$$m(\theta_1 \cap \theta_3 \cap (\theta_4 \cup \theta_5)) = 0.99 \cdot 0.01 \cdot 0.01 + 0.99 \cdot 0.01 \cdot 0.97 + 0.01 \cdot 0.01 \cdot 0.01 = 0.009703$$

$$m(\theta_1 \cap (\theta_4 \cup \theta_5)) = 0.99 \cdot 0.01 \cdot 0.01 + 0.99 \cdot 0.01 \cdot 0.01 + 0.01 \cdot 0.01 \cdot 0.01 = 0.000199$$

$$m((\theta_4 \cup \theta_5) \cap \theta_2 \cap \theta_1) = 0.01 \cdot 0.98 \cdot 0.01 + 0.99 \cdot 0.01 \cdot 0.01 + 0.99 \cdot 0.98 \cdot 0.01 = 0.009899$$

$$m((\theta_4 \cup \theta_5) \cap \theta_2) = 0.01 \cdot 0.98 \cdot 0.01 + 0.01 \cdot 0.98 \cdot 0.01 + 0.01 \cdot 0.01 \cdot 0.01 = 0.000197$$

$$m((\theta_4 \cup \theta_5) \cap \theta_2 \cap \theta_3) = 0.01 \cdot 0.98 \cdot 0.97 + 0.01 \cdot 0.01 \cdot 0.01 = 0.009507$$

$$m((\theta_4 \cup \theta_5) \cap \theta_3) = 0.01 \cdot 0.01 \cdot 0.97 + 0.01 \cdot 0.01 \cdot 0.01 + 0.01 \cdot 0.01 \cdot 0.97 = 0.000195$$

$$m(\theta_4 \cup \theta_5) = 0.01 \cdot 0.01 \cdot 0.01 = 0.000001$$

The sum of all masses is 1.

- Suppose now one finds out that all intersections are empty (Shafer's model), then one uses the hybrid DSm rule and one gets:

$$\begin{array}{ll}
m_h(\theta_1 \cup \theta_2) = 0.019404 & m_h(\theta_1 \cup \theta_3) = 0.009702 \\
m_h(\theta_1 \cup \theta_2 \cup \theta_3) = 0.941193 & m_h(\theta_1 \cup \theta_3 \cup \theta_4 \cup \theta_5) = 0.009703 \\
m_h(\theta_1 \cup \theta_4 \cup \theta_5) = 0.000199 & m_h(\theta_4 \cup \theta_5 \cup \theta_2 \cup \theta_1) = 0.009899 \\
m_h(\theta_4 \cup \theta_5 \cup \theta_2) = 0.000197 & m_h(\theta_4 \cup \theta_5 \cup \theta_2 \cup \theta_3) = 0.009507 \\
m_h(\theta_4 \cup \theta_5 \cup \theta_3) = 0.000195 & m_h(\theta_4 \cup \theta_5) = 0.000001
\end{array}$$

The sum of all masses is 1.

5.4.3 More general

Let $\Theta = \{\theta_1, \dots, \theta_n\}$, where $n \geq 2$, k independent experts, $k \geq 2$, and the mass matrix $\mathbf{M}$ of k rows and $n+1$ columns, corresponding to θ_1, θ_2, $\dots$, θ_n, and one uncertainty (different from the total uncertainty $\theta_1 \cup \theta_2 \cup \dots \cup \theta_n$) say $\theta_{i_1} \cup \dots \cup \theta_{i_s}$ respectively. If the following conditions occur:

- each column contains at least one zero, except the last column (of uncertainties) which has only non-null elements, $0 < \epsilon_1, \epsilon_2, \dots, \epsilon_k < 1$, very tiny numbers (close to zero);
- the columns corresponding to the elements $\theta_{i_1}, \dots, \theta_{i_s}$ are null (all their elements are equal to zero).

If one applies Dempster's rule, one gets $m(\theta_{i_1} \cup \dots \cup \theta_{i_s}) = 1$ (total ignorance), which doesn't bring any information to the fusion.

5.4.4 Even more general

One can extend the previous case even more, considering to u uncertainty columns, $u \geq 1$ as follows.

Let $\Theta = \{\theta_1, \dots, \theta_n\}$, where $n \geq 2$, k independent experts, $k \geq 2$, and the mass matrix $\mathbf{M}$ of k rows and $n+u$ columns, corresponding to θ_1, θ_2, $\dots$, θ_n, and u uncertainty columns (different from the total uncertainty $\theta_1 \cup \theta_2 \cup \dots \cup \theta_n$) respectively. If the following conditions occur:

- each column contains at least one zero, except one column among the last u uncertainty ones which has only non-null elements $0 < \epsilon_1, \epsilon_2, \dots, \epsilon_k < 1$, very tiny numbers (close to zero);

- the columns corresponding to all elements $\theta_{i_1}, \ldots, \theta_{i_s}, \ldots, \theta_{r_1}, \ldots, \theta_{r_s}$ (of course, these elements should not be all $\theta_1, \theta_2, \ldots, \theta_n$, but only a part of them) that occur in all uncertainties are null (i.e., all their elements are equal to zero).

If one applies Dempster's rule, one gets $m(\theta_{i_1} \cup \ldots \cup \theta_{i_s}) = 1$ (total ignorance), which doesn't bring any information to the fusion.

5.5 Fourth infinite class of counter examples

This infinite class of counter-examples concerns Dempster's rule of conditioning defined as [5] :

$$\forall B \in 2^\Theta, \quad m(B|A) = \frac{\sum_{X,Y \in 2^\Theta, (X\cap Y)=B} m(X)m_A(Y)}{1 - \sum_{X,Y \in 2^\Theta, (X\cap Y)=\emptyset} m(X)m_A(Y)}$$

where $m(.)$ is any proper basic belief function defined over 2^Θ and $m_A(.)$ is a particular belief function defined by choosing $m_A(A) = 1$ for any $A \in 2^\Theta$ with $A \neq \emptyset$.

5.5.1 Example with $\Theta = \{\theta_1, \ldots, \theta_6\}$

Let's consider $\Theta = \{\theta_1, \ldots, \theta_6\}$, one expert and a certain body of evidence over θ_2, with the mass matrix:

	θ_1	θ_2	θ_3	$\theta_4 \cup \theta_5$	$\theta_5 \cup \theta_6$
$m_1(.)$	0.3	0	0.4	0.2	0.1
$m_{\theta_2}(.)$	0	1	0	0	0

- Using Dempster's rule of conditioning, one gets: $m(.|\theta_2) = 0/0$ for all the masses.
- Using the DSm classical rule, one gets:

$$m(\theta_1 \cap \theta_2|\theta_2) = 0.3 \qquad m(\theta_2 \cap \theta_3|\theta_2) = 0.4 \qquad m(\theta_2 \cap (\theta_4 \cup \theta_5)|\theta_2) = 0.2 \qquad m(\theta_2 \cap (\theta_5 \cup \theta_6)|\theta_2) = 0.1$$

- If now, one finds out that all intersections are empty (we adopt Shafer's model), then using the hybrid DSm rule, one gets:

$$m_h(\theta_1 \cup \theta_2|\theta_2) = 0.3 \qquad m_h(\theta_2 \cup \theta_3|\theta_2) = 0.4 \qquad m_h(\theta_2 \cup \theta_4 \cup \theta_5|\theta_2) = 0.2 \qquad m_h(\theta_2 \cup \theta_5 \cup \theta_6|\theta_2) = 0.1$$

5.5.2 Another example with $\Theta = \{\theta_1, \ldots, \theta_6\}$

Let's change the previous counter-example and use now the following mass matrix:

	θ_1	θ_2	θ_3	$\theta_4 \cup \theta_5$	$\theta_5 \cup \theta_6$
$m_1(.)$	1	0	0	0	0
$m_{\theta_2}(.)$	0	1	0	0	0

- Using Dempster's rule of conditioning, one gets: $m(.|\theta_2) = 0/0$ for all the masses.
- Using the DSm classical rule, one gets: $m(\theta_1 \cap \theta_2|\theta_2) = 1$, and others 0.
- If now, one finds out that all intersections are empty (we adopt Shafer's model), then using the hybrid DSm rule, one gets: $m_h(\theta_1 \cup \theta_2|\theta_2) = 1$, and others 0.

5.5.3 Generalization

Let $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$, where $n \geq 2$, and two basic belief functions/masses $m_1(.)$ and $m_2(.)$ such that there exist $1 \leq (i \neq j) \leq n$, where $m_1(\theta_i) = m_2(\theta_j) = 1$, and 0 otherwise. Then Dempster's rule of conditioning can not be applied because one gets division by zero.

5.5.4 Example with $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ and ignorance

Let's consider $\Theta = \{\theta_1, \theta_2, \theta_1, \theta_2\}$, one expert and a certain ignorant body of evidence over $\theta_3 \cup \theta_4$, with the mass matrix:

	θ_1	θ_2	$\theta_3 \cup \theta_4$
$m_1(.)$	0.3	0.7	0
$m_{\theta_3 \cup \theta_4}(.)$	0	0	1

- Using Dempster's rule of conditioning, one gets 0/0 for all masses $m(.|\theta_3 \cup \theta_4)$.
- Using the classical DSm rule, one gets: $m(\theta_1 \cap (\theta_3 \cup \theta_4)|\theta_3 \cup \theta_4) = 0.3$, $m(\theta_2 \cap (\theta_3 \cup \theta_4)|\theta_3 \cup \theta_4) = 0.7$ and others 0.
- If now one finds out that all intersections are empty (Shafer's model), using the hybrid DSm rule, one gets $m(\theta_1 \cup \theta_3 \cup \theta_4|\theta_3 \cup \theta_4) = 0.3$, $m(\theta_2 \cup \theta_3 \cup \theta_4|\theta_3 \cup \theta_4) = 0.7$ and others 0.

5.5.5 Generalization

Let $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n, \theta_{n+1}, \ldots, \theta_{n+m}\}$, for $n \geq 2$ and $m \geq 2$. Let's consider the mass $m_1(.)$, which is a row of its values assigned for $\theta_1, \theta_2, \ldots, \theta_n$, and some unions among the elements $\theta_{n+1}, \ldots, \theta_{n+m}$ such that all unions are disjoint with each other. If the second mass $m_A(.)$ is a conditional mass, where A belongs to $\{\theta_1, \theta_2, \ldots, \theta_n\}$ or unions among $\theta_{n+1}, \ldots, \theta_{n+m}$, such that $m_1(A) = 0$, then Dempster's rule of conditioning can not be applied because one gets division by zero, which is undefined. [We did not consider any intersection of θ_i because Dempster's rule of conditioning doesn't accept paradoxes]. But the DSm rule of conditioning does work here as well.

5.5.6 Example with a paradoxical source

A counter-example with a paradox (intersection) over a non-refinable frame, where Dempster's rule of conditioning can not be applied because Dempster-Shafer theory does not accept paradoxist/conflicting information between elementary elements θ_i of the frame Θ:

Let's consider the frame of discernment $\Theta = \{\theta_1, \theta_2\}$, one expert and a certain body of evidence over θ_2, with the mass matrix:

	θ_1	θ_2	$\theta_1 \cap \theta_2$	$\theta_1 \cup \theta_2$
$m_1(.)$	0.2	0.1	0.4	0.3
$m_{\theta_2}(.)$	0	1	0	0

Using the DSm rule of conditioning, one gets

$$m(\theta_1|\theta_2) = 0 \qquad m(\theta_2|\theta_2) = 0.1 + 0.3 = 0.4 \qquad m(\theta_1 \cap \theta_2|\theta_2) = 0.2 + 0.4 = 0.6 \qquad m(\theta_1 \cup \theta_2|\theta_2) = 0$$

and the sum of fusion results is equal to 1.

Suppose now one finds out that all intersections are empty. Using the hybrid DSm rule when $\theta_1 \cap \theta_2 = \emptyset$, one has:

$$\begin{aligned}
m_h(\theta_1 \cap \theta_2|\theta_2) &= 0\\
m_h(\theta_1|\theta_2) &= m(\theta_1|\theta_2) + [m_1(\theta_1)m_2(\theta_1 \cap \theta_2) + m_2(\theta_1)m_1(\theta_1 \cap \theta_2)] = 0\\
m_h(\theta_2|\theta_2) &= m(\theta_2|\theta_2) + [m_1(\theta_2)m_2(\theta_1 \cap \theta_2) + m_2(\theta_2)m_1(\theta_1 \cap \theta_2)] = 0.4 + 0.1(0) + 1(0.4) = 0.8\\
m_h(\theta_1 \cup \theta_2|\theta_2) &= m(\theta_1 \cup \theta_2|\theta_2) + [m_1(\theta_1)m_2(\theta_2) + m_2(\theta_1)m_1(\theta_2)]\\
&\quad + [m_1(\theta_1 \cap \theta_2)m_2(\theta_1 \cup \theta_2) + m_2(\theta_1 \cap \theta_2)m_1(\theta_1 \cup \theta_2)] + [m_1(\theta_1 \cap \theta_2)m_2(\theta_1 \cap \theta_2)]\\
&= 0 + [0.2(1) + 0(0.1)] + [0.4(0) + 0(0.3)] + [0.4(0)]\\
&= 0.2 + [0] + [0] + [0] = 0.2
\end{aligned}$$

5.6 Conclusion

Several infinite classes of counter-examples to Dempster's rule of combination have been presented in this chapter for didactic purposes to show the limitations of this rule in the DST framework. These infinite classes of fusion problems bring the necessity of generalizing the DST to a more flexible theory which permits the combination of any kind of sources of information with any degree of conflict and working on any frame with exclusive or non-exclusive elements. The DSmT with the hybrid DSm rule of combination proposes a new issue to satisfy these requirements based on a new mathematical framework.

5.7 References

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