Generally Covariant Quantum Theory: Gravitons.

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Abstract

We finalize the project initiated in [1, 2, 3, 4, 7, 8, 9, 10] by studying graviton theory in our setting. Given the results in [1, 3, 9, 10], there is not so much left to accomplish and we start by deepening our understanding of some points left open in [1, 10]. Perturbative finiteness of the theory follows ad-verbatim from the analysis in [3, 9] and we do not bother here about writing it down explicitly. Rather, our aim is to provide for a couple of *new* physical and mathematical insights regarding the genesis of the structure of the quantal graviton theory.

1 Introduction.

This paper is the culmination of a series [1, 2, 3, 4, 7, 8, 9, 10] of papers by this author about how to obtain a well defined covariant quantum theory in *any* time-orientable curved spacetime background. This is a very remarkable thing to say and, indeed, the construction relies upon many fine points standard quantum field theory does not take into account. In [1], we prepared the ground for defining a covariant quantum theory for particles of spin $0, \frac{1}{2}, 1$ on any background whereas in [3, 4] we have defined the interacting theory for bosonic spin-0 particles and later on showed that the theory is well defined, in any case perturbatively finite. Thereafter, [9, 10] we extended those results to quantum electrodynamics and non-abelian gauge theory in general, thereby including particles of spin-$\frac{1}{2}, 1$. In this paper, we perform the extension to gravitons which is somewhat exceptional since one of the regularization parameters, as we will show soon, is *expected* to have a lower bound determined by the Planck length and the details of the geometry. Indeed, the introduction of a dynamical length scale appears to put lower bounds on the "friction" associated to the creation and annihilation process and clears out some mathematical unwarranted assumption made in [11]. The idea of this paper is the same than the one in [10]: we provide for an alternative, covariant, derivation of graviton theory in our setting starting from a couple of physical principles it has to obey. We will distantiate ourselves from a perturbation analysis of the Einstein-Hilbert action *assuming* infinitesimal perturbations $h_{\alpha\beta}$ and analycity of a series expansion

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which is, later on, evaluated outside its domain of convergence [11] when calculating the path integral. At least, this is the case in the latter approach but *not* so in our philosophy of regulated propagators with two friction parameters $\mu, \kappa > 0$ of dimension length^2 and length^{-1} respectively where, roughly speaking, $\frac{\alpha}{\mu}$ forms an upper bound [3] on the absolute value of the propagator where the dimensionless number α depends, amongst others, on some characteristics of the geometry. Therefore, one needs that $\alpha l_p^2 < \mu$ for the *modified* perturbation series to converge. Hence, friction on the creation and annihilation process of gravitons, as well as on the propagation process, is going to make the theory well defined given some additional assumption on supression of scattering processes with many gravitons. In conventional approaches, it is *not* clear what the meaning of diffeomorphism invariance is at quantal level. Indeed, in a path integral language, one should show that the *primary* "measure" is diffeomorphism invariant and no such "measure", in contrast to non-abelian gauge theory, has ever been constructed. Most physicists take diffeomorphism invariance to mean "diffeomorphism invariance of the action" but then, in a path integral language, one should gauge fix the action in order to make it well defined. This will involve ghosts and terms which are *not* diffeomorphism invariant in the primary sense. In contrast to some gauge choices, I prefer to maintain a manifestly covariant formulation and one which *is* diffeomorphism invariant in another sense; those terms, which are covariant in the second sense but not in the first should couple to the ghosts.

I wished I could argue even more directly, just as it was the case in [10] for the correct structure of non-abelian gauge theory at the quantal level. In principle, Einstein's theory delivers only one interaction "vertex" which is then regarded as an infinite series of interaction vertices because we have to invert $g_{\mu\nu} + l_p h_{\mu\nu}$, something I have to do [11] too with that difference that the operation is going to be *made* well defined, in contrast to that reference, where the expansion diverges for large perturbations in the path integral. What do I mean by this precisely? If you write out the interaction vertices coming from a series expansion of the Einstein-Hilbert action for $g + l_p h$, then the only things which matter are vacuum expectation values

$$T\langle hhhh\nabla h\nabla h\rangle Z$$

where h has to appear an even number of times as does the Levi-Civita connection ∇ of g; the latter appears, moreover, at most twice. Z is a tensor composed out of g and the Riemann tensor of g which ensures that the identity is a scalar and T is, as usual, the time ordering product. Any such "vertex" can be written down as a closed graph made out of an even number of vertices with two legs and zero or two vertices with three legs. The number of contractions associated to it is given by $(2n)!$ where $2n$ is the number of h symbols or vertices of the "vertex" graph. So, when evaluating convergence of the perturbation series, we have to replace $hhhh\nabla h\nabla hZ$ by $(2n)!C^n$ where

$$||T\langle hh\rangle||, ||T\langle h\nabla h\rangle||, ||T\langle\nabla h\nabla h\rangle|| < C$$

where the norm is the standard supremum norm for operators on $\mathbb{R}^4 \otimes \mathbb{R}^4$ defined with respect to an orthonormal basis associated to a unit timelike vectorfield (see later for details). Actually, the bounds on the derivative terms are not important since there are only two of them at most and they, therefore, do not influence convergence of the series. The reader notices that $(2n)!C^n$ blows up to infinity in the limit for n to infinity for any C and therefore the expansion series does not converge unless we *impose* that in the quantum theory vertices with a large number of terms come with surpression factors which are of similar magnitude as n^{-2n} or $e^{-2n \ln n}$. In an operational language, this would mean that the definition of the product of fields depends upon the number of operandi and that, therefore, associativity has to be given up. In our language, it simply means that scattering processes associated to many gravitons are super-exponentially surpressed in terms of the number of participating gravitons. This is *new* physics and a feature which distinguishes gravitons from spin one non-abelian gauge particles where, given the fact that only two self-interaction vertices of valence 3 and 4 exist, no such detailed analysis matters. In [10], we immediately constructed the right interaction vertices for non-abelian gauge theory, a remarkable thing indeed; in this paper, we can only partially generalize this virtue to graviton theories.

The same comments regarding the perturbative finiteness of the theory apply ad-verbatim [3, 9, 10]: the proof has been delivered before and only minor adjustments need to be made. As in [10], I show that any graviton theory has to be generally covariant by means of analysis of spin two. Then, I look for a novel point of view on what it means and subsequently, we derive the correct interaction vertices and propagator. Ghost particles will show up naturally in the analysis as it was the case in [10]. I have omitted many details in the construction as not to duplicate results which exist already in the aforementioned references. I shall explain the physics behind every introduced concept but leave the reader the duty to flesh out the mathematical details by consulting the mentioned citations. This should give a feeling as to *why* things work out as I say they do, the rigorous mathematical treatment being obvious for anyone having gone through the entire series.

2 Massless spin two particles and general covariance.

Given that a (massless) spin-one particle was described by means of a Lorentz vector, it is natural to look for a tensor product representation of the Lorentz group

$$\Lambda^a_b \Lambda^c_d h^{bd}.$$

There exist two irreducible components, the symmetric and anti-symmetric tensors and the massless spin two particle resides in the symmetric part. Indeed, as is well known, we should look for symmetric states carrying helicity ± 2, There are exactly two of them $(++)$ and $(--)$ where $\pm$ denotes the state of helicity ± 1 in the defining representation of the Lorentz group; moreover, we have 2 particles of helicity ± 1 each $(0_i \pm) + (\pm 0_i)$, where $0_i; i = 1, 2$ is a state of helicity zero in the Lorentz representation, and 4 particles of helicity zero making up a total of 10 as it should. Therefore, we need a four dimensional symmetry group to eliminate 8 local degrees of freedom; the only such group being the diffeomorphism group. Hence, it is said that the graviton theory needs to be generally covariant.

The big distinction with gauge theory is that the generators of the diffeomorphism Lie-algebra act quasi-locally, instead of ultra-locally, on the "gauge potential" $h_{\mu\nu}$, where we have gotten from h_{ab} to $h_{\mu\nu}$ by means of the vierbein e^a_μ, associated to the Lorentzian spacetime metric $g_{\mu\nu}$. Indeed, the Lie algebra of the diffeomorphism group is given by the vectorfields $\mathbf{V}$ which are realized by means of the Lie-derivative

$$\delta_{\mathbf{V}} = \mathcal{L}_{\mathbf{V}}.$$

The Lie algebra is preserved given that

$$[\mathcal{L}_{\mathbf{V}}, \mathcal{L}_{\mathbf{W}}] = \mathcal{L}_{[\mathbf{V},\mathbf{W}]}.$$

The Lie derivative on a general tensor field $T^{\mu_1 \cdots \mu_r}_{\nu_1 \ldots \nu_s}$ is given by

$$\mathcal{L}_{\mathbf{V}} T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s} = T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s ; \alpha} V^\alpha - T^{\beta \ldots \mu_r}_{\nu_1 \ldots \nu_s} V^{\mu_1}_{;\beta} - \ldots + T^{\mu_1 \ldots \mu_r}_{\beta \ldots \nu_s} V^{\beta}_{;\nu_1} + \ldots$$

where we have used the Levi-Civita connection associated any spacetime metric. We now come to the definition of what we mean with a generally covariant theory: under the usual action of spacetime diffeomorphisms, the spacetime metric $g_{\mu\nu}$ as well as the graviton polarization $h_{\mu\nu}$ transform as

$$g \to g + \mathcal{L}_{\epsilon \mathbf{V}} g, \;\; h \to h + \mathcal{L}_{\epsilon \mathbf{V}} h.$$

Subsequent application gives

$$(g + \mathcal{L}_{\epsilon \mathbf{V}} g) + \mathcal{L}_{\mathbf{W}}(g + \mathcal{L}_{\epsilon \mathbf{V}} g) = g + \mathcal{L}_{\epsilon(\mathbf{V}+\mathbf{W})} g + \mathcal{L}_{\epsilon \mathbf{W}} \mathcal{L}_{\epsilon \mathbf{V}} g$$

and the property

$$[\delta_{\epsilon \mathbf{V}}, \delta_{\epsilon \mathbf{W}}] = \delta_{\epsilon^2 [\mathbf{V},\mathbf{W}]}$$

is needed for this to be an action. In order for $g_{\mu\nu}$ to remain *stationary* we therefore form the combination

$$g_{\mu\nu} + l_p h_{\mu\nu}$$

and define

$$\delta'_{\epsilon \mathbf{V}} h = \delta_{\epsilon \mathbf{V}} h + (l_p)^{-1} \delta_{\epsilon \mathbf{V}} g, \;\; \delta'_{\epsilon \mathbf{V}} g = 0$$

where the Plank length has been inserted because the graviton propagator has dimension mass^2. It is readily verified that

$$\delta'_{\epsilon(\mathbf{V}+\mathbf{W})} = \delta'_{\epsilon \mathbf{V}} + \delta'_{\epsilon \mathbf{W}}$$

and

$$[\delta'_{\epsilon \mathbf{V}}, \delta'_{\epsilon \mathbf{W}}] = \delta'_{\epsilon^2 [\mathbf{V},\mathbf{W}]}$$

given that

$$\delta'_{\epsilon \mathbf{V}} \delta'_{\epsilon \mathbf{W}} = \delta_{\epsilon \mathbf{V}} \delta'_{\epsilon \mathbf{W}}.$$

The symmetries of a graviton theory require that internal interaction vertices between gravitons are constructed from scalar densities under the action δ' while interactions with ghost particles are constructed from tensor densities under the action δ. The rationale is the same as the one in non-abelian gauge theory where one adds all *covariant* interaction terms which do not stem from a local gauge symmetric scalar density to the theory and couples them to ghost particles.

As is well known [3, 4, 9, 10], the interaction vertices and two point function are all we need to define a generally covariant quantum theory; we do not have any problems regarding the definition of a covariant measure. In the next section, we introduce the graviton propagator as well as the mandatory spin-0 ghost particles.

3 The regularized graviton propagator and spin-zero ghost particles.

The unregularized graviton propagator can be defined from a generalized Schrodinger equation as we did for spin $0, \frac{1}{2}, 1$ particles in [1]. For simplicity of presentation, we shall assume that any two points in spacetime can be connected by means of a unique geodesic, the extension to the generic situation of an arbitrary number of geodesics (including no geodesic at all) is discussed in [3, 8, 9]. These fine points would just obscure the simplicity of the result and it has been shown before [8, 9] that they can be suitably taken into account. The result is that the regularized Feynman propagator has to be of the form

$$\Delta_F^{\mu\kappa}(x,y)_{\alpha\beta,\alpha'\beta'} = \left(g_{\alpha\alpha'}(x,y)g_{\beta\beta'}(x,y) + g_{\beta\alpha'}(x,y)g_{\alpha\beta'}(x,y) - \frac{1}{2}g_{\alpha\beta}(x)g_{\alpha'\beta'}(y) \right) \Delta_F^{\mu\kappa}(x,y).$$

Here, (un)primed indices refer to y (x) and

$$g_{\alpha\beta'}(x,y) = \Lambda^{-1}(x,y)^{\beta}_{\beta'} g_{\alpha\beta}(x)$$

where

$$\Lambda(x,y)^{\beta'}_{\beta}$$

denotes the transporter along the geodesic from x to y. The factor $\frac{1}{2}$ in the last term stems from the demand that

$$\Delta_F^{\mu\kappa}(x,y)_{\alpha\beta,\alpha'\beta'} g^{\alpha\beta}(x) = \Delta_F^{\mu\kappa}(x,y)_{\alpha\beta,\alpha'\beta'} g^{\alpha'\beta'}(y) = 0$$

because

$$g_{\alpha\alpha'}(x,y)g_{\beta\beta'}(x,y)g^{\alpha\beta}(x) = g_{\alpha'\beta'}(y).$$

Therefore, in the definition of the propagator, we at least eliminated the graviton trace degrees of freedom. Here, as mentioned in the introduction and explained in [3, 9], $\mu, \kappa > 0$ are two friction parameters of dimension length squared and length inverse associated to the creation-annihilation process and the propagation of information along the geodesic respectively. The way both of them are implemented is by means of a preferred, unit norm, timelike vectorfield V^μ associated to the Lorentzian geometry $g_{\mu\nu}$, the existence of which is a generic property for general backgrounds as is well known by relativists. The vectorfield is unique in the sense that there exists a procedure giving exactly one V^μ; however, different procedures may lead to other answers. In any case V^μ uniquely defines a Riemannian tensor $q_{\mu\nu}$ and a class of reference frames connected by an $SO(3)$ transformation which constitute a vierbein for both g and q. With respect to such vierbein, the propagator reduces to

$$\Delta_F^{\mu\kappa}(x,y)_{\alpha\beta,\alpha'\beta'} = e^a_\alpha(x)e^b_\beta(x)e^{a'}_{\alpha'}(y)e^{b'}_{\beta'}(y)\left(\Lambda(x,y)_{a'a}\Lambda(x,y)_{b'b} + \Lambda(x,y)_{a'b}\Lambda(x,y)_{b'a} - \frac{1}{2}\eta_{ab}\eta_{a'b'}\right)$$

$$\Delta_F^{\mu\kappa}(x,y)$$

where lowering of the a', b' indices occured with respect to $\eta_{a'b'}$. We have shown in [3, 9] that $\Delta_F^{\mu\kappa}(x,y)$ can be bounded by $\frac{C(g,V,\kappa,\epsilon)}{\mu}e^{-(\kappa-\epsilon)d(x,y)}$ where $d(x,y)$ is the global Riemannian distance associated to $q_{\mu\nu}$, $0 < \epsilon \ll \kappa$ and $C(g,V,\kappa,\epsilon)$ is a dimensionless constant which only depends upon g, V and κ, ϵ.

Now, as explained in the introduction, any internal interaction vertex with $2n$-legs is going to contribute an amplitude which is bounded by

$$(l_p)^{2n}(2n)!D^n$$

where D constituted a universal bound on the propagator in the sup-norm attached to our class of reference frames. Hence, we look for a norm estimate of the tensor

$$W(x,y)^{ba'}_{ab'} = \Lambda(x,y)^{a'}_{a}\Lambda(x,y)^{b}_{b'} + \Lambda(x,y)^{a'b}\Lambda(x,y)_{b'a} - \frac{1}{2}\delta^b_a\delta^{a'}_{b'}$$

which naturally leads to bounds of the kind

$$\sqrt{\mathrm{Tr}\left(\Lambda(x,y)^\dagger\Lambda(x,y)\right)} < D(g,V)e^{\delta d(x,y)}$$

with regard to this special class of reference frames, where $0 < 2\delta < \kappa - \epsilon$ and the factor of two emerges because we are dealing with a particle of spin two. We hinted in the introduction that it would be desirable to surpress these vertices with a factor $a(n)$ such that $a(n)(2n)! \to 1$ in the limit for n to infinity. This would lead to a bound of the kind

$$1 > l_p^2\frac{C(g,V,\kappa,\epsilon)}{\mu}||W(x,y)^{ba'}_{ab'}|| > 0$$

which implies that

$$\mu > l_p^2\alpha(g,V,\kappa,\epsilon)$$

is the kind of bound on the friction term μ one should anticipate in a graviton theory on a generic background. Finally, it has been explained in full detail in [3, 9] why the specific friction terms added do not lead to a violation of local Lorentz invariance in a well defined sense; therefore, there is no conflict with observation whatsoever.

Regarding ghosts, there is not much to say and all relevant details have been fleshed out in [1, 10]. The only difference with non-abelian gauge theory is that the ghosts must effectively transform in the vector representation of the diffeomorphism group, which we denote by v^α; the two point functions being

$$W_p(x,y)^{\alpha\beta'} = g^{\alpha\beta'}(x,y)\theta(x)\overline{\theta}(y)W(x,y)$$

and

$$W_a(x,y)^{\alpha\beta'} = g^{\alpha\beta'}(x,y)\overline{\theta}(x)\theta(y)W(x,y)$$

leading to a well defined Feynman propagator. Here, we have dropped all reference regarding the friction parameters. As mentioned in section two, all covariant interaction terms with regard to the action δ, which do not stem from the Einstein-Hilbert action, must be coupled to ghosts.

4 The proof that the theory is perturbatively finite.

This section is obvious for anyone having gone through [3, 9]; effectively, it boils down to the fact that we have shown for a generic class of geometries that

$$||P(\nabla^s_{e_a(x)}, \nabla^s_{e_b(y)})\Delta^{\mu\kappa}_F(x,y)|| < \frac{C(g,V,\kappa,\mu,\epsilon)}{\mu} e^{-(\kappa-\epsilon-2\delta)d(x,y)}$$

where the sup-norm has been taken with respect to any vierbein with $e_0 = V$, $P(x_a, y_b) = x_a x_c, y_b y_d, x_a y_b, x_a, y_b, 1$ and ∇^s is the spin covariant derivative. In the case of gravitation, we will need supplementary bounds on the Riemann tensor of g such as

$$R_{abcd}(x)\delta^{aa'}\delta^{bb'}\delta^{cc'}\delta^{dd'}R_{a'b'c'd'}(x) < C$$

where the Lorentz indices are taken with respect to the g, q tetrads. This implies that all interaction intertwiners $Z(x)$ are uniformly bounded in these Lorentz frames. Therefore, the contribution of any Feynman diagram is estimated by

$$\prod_{i=1}^{V} C(Z_i)\left(\frac{C(g,V,\kappa,\mu,\epsilon)}{\mu}\right)^E \int dz_1\sqrt{h(z_1)}\ldots\int dz_V\sqrt{h(z_V)} \prod_{\text{edges }(\alpha,\beta)} e^{-(\kappa-\epsilon-2\delta)d(\alpha,\beta)}$$

where $C(Z_i)$ is a constant depending upon the intertwiners Z_i, V is the number of internal vertices, E the number of edges (internal and external) and α, β are the coordinates of an internal or external vertex respectively. It has been shown in [3, 9] what kind of bounds one can impose on such integrals proving the assertion that the theory is perturbatively finite.

As commented in [3, 9] it may be necessary that additional surpression factors on diagrams with a large number of internal vertices are necessary are necessary to make the perturbation series analytic; we leave the investigation of such issues open to a forthcoming book publication about this topic.

5 Conclusions.

Although we have not filled in all details of the construction, any reader interested in those fine points should consult [3, 9] where they have been explained in the utmost detail. Admitted, we have skipped a small issue in [10] and in this publication regarding the bounds placed upon the covariant derivatives of Feynman propagator, something which was not required for ϕ^4 theory and QED, but any reader with a minimal amount of skill can fill in those small gaps whose detailed treatment would only ask for more space while adding little physical insight.

I leave those details for a forthcoming book publication on the matter where it is the luxurious space granted by a book which allows me to dwell on such fine points. In this publication, we have shown that under reasonable conditions regarding the background geometry, the graviton theory is perturbatively finite. Moreover, it can be made analytic in a suitable range of the physical constants if large vertices as well as diagrams with a large number of vertices are (super-exponentially) surpressed. These deviations from standard quantum mechanics appear mandatory for obtaining a consistent theory of gravitons; it appears logical to me, as I have repeatedly stated, that such modifications are also necessary regarding the so-called non-perturbative approaches towards quantum gravity.

All this is very exciting since it would mean that friction as well as some finite number of colliding gravitons dominate interactions at the Planck scale and that these were the missing ingredients in the standard, unitary, theory. It was, moreover, the principle of general covariance [2, 1] which pointed into the direction of these novel physical ingredients and this constitutes a powerful motivation indeed. It is important to realize that no strings or extended objects have been necessary to obtain these wonderful features and that the point-particle theory is alive and well.

I close here the series of papers on generally covariant quantum theory and postpone the treatment of the remaining little, but rather obvious, details to a forthcoming book publication on the matter given that the generality and depth of the obtained results are truly wonderful.

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