Exact solutions for sine-Gordon equations and f-expansion method

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A large number of methods have been proposed for solving nonlinear differential equations. The Jacobi elliptic function method and the f-expansion methods are generalizations from a few of them. These methods produce not only single-solitons but also multi-soliton solutions. In this work we applied the f-expansion method and found novel solutions besides those known for three main equations of the kind sine-Gordon: Triple Sine-Gordon (TSG), Double Sine-Gordon (DSG) and Simple Sine-Gordon (SSG).

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I. INTRODUCTION

It is of great interest to find exact solutions of nonlinear differential equations in nonlinear physical problems [1]. A large number of methods have been proposed such as the homogeneous balancement method [2–4], the hyperbolic tangent function expansion method [5–9], the hyperbolic secant function expansion method [10, 11], the test function method [12, 13], the nonlinear transformation method [14, 15] and the sine-cosine method [16]. These methods can only get the solutions of solitary waves and shock waves but cannot get the periodic solutions of nonlinear equations. Although Porubov *et al.* [17–19] got exact periodic solutions for a few nonlinear differential equations, they used Weierstrass elliptic functions and involved complicated deductions. Recently, Jacobi elliptic function expansion methods has been proposed and applied for solving some nonlinear equations. The periodic solutions gotten by these methods include solitary wave and shock wave solutions [20–26].

For the f-expansion method the basic idea is the following. For a given nonlinear partial differential equation with two idenpendent variables, for instance: a time variable t and an other space variable x as

$$E(u, u_t, u_x, ...) = 0, \tag{1}$$

we seek wave solutions in the way

$$u = u(\xi), \qquad \xi = k(x - ct). \tag{2}$$

Replacing Eq. (2) into Eq. (1), we have a ordinary nonlinear differential equation for $u(\xi)$. By expanding $u(\xi)$ on a polinom in $f(\xi)$ like

$$u = u(\xi) = \sum_{j=0}^{n} a_j f^j, \tag{3}$$

where a_j is a constant to be determined and n is fixed by balancing the terms of higher degree of the ordinary equation while f satisfies the elliptic equation of first kind

$$f'' = pf + qf^3 \quad \text{ou} \quad f'^2 = pf^2 + \frac{1}{2}qf^4 + r, \tag{4}$$

where the apostrophe means derivative with respect to ξ. Most detailed explanation for the Jacobi elliptic function may be found in Refs. [27, 28].

The balancement is made by remarking that the higher degree of the expansion in Eq. (3) is

$$O\left(u\left(\xi\right)\right) = n. \tag{5}$$

Note the derivative property of the elliptic functions like $(\mathrm{sn}\xi)' = \mathrm{cn}\xi\mathrm{dn}\xi$, where $\mathrm{sn}\xi$ and $\mathrm{cn}\xi$ are respectively the Jacobi elliptic sine and cosine while $\mathrm{dn}\xi$ is the Jacobi elliptic function of third kind. Then once we derive Eq. (3) the degree of the derivative is increased upon one unity in such a way that

$$O\left(\frac{d^p u}{d\xi^p}\right) = n + p, \quad p = 1, 2, 3, ... \tag{6}$$

And from Eq. (5) and Eq. (6) it follows that

$$O\left(u^q \frac{d^p u}{d\xi^p}\right) = (q+1)n + p, \quad q = 1, 2, 3, ... \quad p = 1, 2, 3, ... \tag{7}$$

Thus we may select n in Eq. (3) by equaling the degrees of at least two terms in the ordinary equation [29].

In following section we used this method for solving the Triple Sine-Gordon equation (TSG) while in the section III and section IV we solve the Double Sine-Gordon equation (DSG) and the Simple Sine-Gordon equation (SSG) respectively.

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II. TRIPLE SINE-GORDON SOLUTIONS

A. Tangent transformation

In order for solving the TSG

$$\Theta_{xt} = \alpha \sin\Theta + \beta \sin 2\Theta + \gamma \sin 3\Theta, \tag{8}$$

we should make two trasnformations

$$u = \tan\tfrac{\Theta}{2} \qquad \text{and} \quad \xi = k\,(x - ct)\,, \tag{9}$$

where k is the wave vector and c is the wave velocity in the travelling wave system. So Eq. (8) changes for

$$\begin{aligned} &k^2c(1+u^2)^2u'' \\ &\quad -2k^2cu(1+u^2)(u')^2 + au + 2bu^3 + du^5 = 0, \end{aligned} \tag{10}$$

where $a = \alpha + 2\beta + 3\gamma, \quad b = \alpha - 5\gamma \quad \text{e} \quad d = \alpha - 2\beta + 3\gamma.$

By balancing the Eq. (10) we note that n is arbitrary and so can make $n = 1$ into the expansion of Eq. (3). By making use of the elliptic equation of Eq. (4) we get the solutions:

1. Trivial case, $a_1 = 0$:

$$\Theta^{\pm}_{1T\pm}(x,t) = 2\arctan\left[\pm\sqrt{\frac{-\alpha + 5\gamma \pm \sqrt{(\alpha-5\gamma)^2 - (\alpha+2\beta+3\gamma)(\alpha-2\beta+3\gamma)}}{\alpha - 2\beta + 3\gamma}}\right]. \tag{11}$$

2. Non trivial case, $a_0 = 0$:

$$\Theta^{\pm}_{2T}(x,t) = \\ 2\arctan\left\{\pm\sqrt{\frac{k^2cp + \alpha + 2\beta + 3\gamma}{2k^2cr}}\,f\left[k(x-ct)\right]\right\}, \tag{12a}$$

with the dispersion relations

$$(2qr - p^2)(k^2c)^2 - 2(a-b)pk^2c + (2ab - a^2) = 0 \tag{12b}$$

e

$$(2qr - p^2)(k^2c)^2 - (a-d)pk^2c + ad = 0. \tag{12c}$$

The sine and cosine transformations cannot be made in TSG to get solutions. It is necessary seeks other transformations.

III. DOUBLE SINE-GORDON SOLUTIONS

A. Tangent transformation

The DSG has the form

$$\Theta_{xt} = \alpha \sin\Theta + \beta \sin 2\Theta. \tag{13}$$

Hence it is the limit of the TSG whenever $\gamma = 0$. Its solutions should also fulfill this requirement.

1. Trivial case, $a_1 = 0$:
   By making $\gamma = 0$ in solution (11) we get

$$\Theta_{1D\pm}(x,t) = 2\arctan\left[\pm\sqrt{\frac{2\beta+\alpha}{2\beta-\alpha}}\right]. \tag{14}$$

   By solving (13) we may get non trivial solutions by two way:

2. Case 1, $a_0 = 0$ and $r \neq 0$:

$$\Theta_{2D\pm}(x,t) = \\ 2\arctan\left\{\pm\sqrt{\frac{pk^2c + (\alpha+2\beta)}{2rk^2c}}\,f[k(x-ct)]\right\}, \tag{15a}$$

   with the following dispersion relation

$$(2qr - p^2)(k^2c)^4 + 4\beta p(k^2c)^2 + (\alpha^2 - 4\beta^2) = 0. \tag{15b}$$

3. Case 2, $a_0 = 0$ and $r = 0$:

$$\Theta_{3D\pm}(x,t) = 2\arctan\left\{\pm\sqrt{\frac{q(\alpha+2\beta)}{2p\alpha}}\,f[k(x-ct)]\right\}, \tag{16a}$$

   with the next dispersion relation

$$k^2c = \frac{\alpha + 2\beta}{p}. \tag{16b}$$

We see that the solution (15a) is a particular case of the solution (12a). But we also found other solutions (16a).

B. Tangent transformation up to second order

Once n is arbitrary in Eq. (10) even for $\gamma = 0$ we may set $n = 2$ to seek other solutions for the DSG . So we get more solutions in the way

$$\Theta^{\pm}_{4D\pm} = \tag{17a}$$
$$2\arctan\left\{\pm\sqrt{\frac{\alpha}{2\beta - \alpha}} \pm \frac{2p}{r}\sqrt{\frac{\alpha}{2\beta-\alpha}} f^2\left[k(x-ct)\right]\right\},$$

$$k^2c = \frac{2\beta - \alpha}{4p}. \tag{17b}$$

C. Cosine transformation

Now with the next transformations in (13)

$$u = \cos\Theta \qquad \text{and} \;\; \xi = k\left(x - ct\right), \tag{18}$$

we have

$$k^2c(1-u^2)u'' \tag{19}$$
$$+k^2cu(u')^2 - \alpha(1-u^2)^2 - 2\beta u(1-u^2)^2 = 0.$$

By making balancement we note that the one possibility is $n = 1$.

So we have more solutions for Eq. (13) in the form

1. Trivial case, $a_1 = 0$:

$$\Theta^{\pm l}_{5D}(x,t) \;=\; \pm l\pi, \tag{20}$$
$$\Theta_{6D}(x,t) \;=\; \arccos\left(-\frac{\alpha}{2\beta}\right), \tag{21}$$

where $l = 0, 1, 2, 3, ...$

2. Non trivial case, $a_0 = 0$:

$$\Theta_{7D\pm}(x,t) = \tag{22a}$$
$$\arccos\left\{-\frac{\alpha}{4\beta} \pm \sqrt{-\frac{q}{p}\left(\frac{\alpha^2 + 16\beta^2}{16\beta^2}\right)} f\left[k(x-ct)\right]\right\},$$

with the dispersion relation

$$k^2c = -\frac{\alpha^2 + 16\beta^2}{4\beta p} \tag{22b}$$

and the constraint relation

$$\frac{2qr}{p^2} = \left(\frac{16\beta^2 - \alpha^2}{16\beta^2 + \alpha^2}\right)^2. \tag{22c}$$

The sine transformation cannot be made in DSG to get solutions.

IV. SIMPLE SINE-GORDON SOLUTIONS

A. Sine transformation

The SSG has the form

$$\Theta_{xt} = \alpha\sin\Theta. \tag{23}$$

We can see that SSG is the limit of DSG and TSG whenever $\beta = 0$ and $\gamma = 0$. Then a lot of solutions may be obtained by making these limits in solutions of DSG and TSG. We seek by solutions that cannot be obtained from DSG and TSG through two transformations: the sine transformation and cosine transformation in second order.

With the following transformations in Eq. (23)

$$u = \sin\tfrac{\Theta}{2} \qquad \text{and} \qquad \xi = k(x-ct), \tag{24}$$

we obtain

$$k^2c(1-u^2)u'' + k^2cu(u')^2 + \alpha u(1-u^2)^2 = 0. \tag{25}$$

By balancing we note that $n = 1$ only. So we have

1. Trivial case, $a_1 = 0$:

$$\Theta^{\pm l}_{1S} \;=\; \pm l\pi, \tag{26}$$

where $l = 0, 1, 2, \ldots$

2. Non trivial case, $a_0 = 0$:

$$\Theta_{2S\pm} = \arcsin\left\{\pm\sqrt{\frac{k^2cq}{2\alpha}} f[k(x-ct)]\right\}, \tag{27a}$$

$$qr(k^2c)^2 + 2\alpha p(k^2c) + 2\alpha \;=\; 0, \tag{27b}$$
$$(k^2c)^2pq \;=\; 0. \tag{27c}$$

B. Cosine transformation up to second order

Make the transformation given by Eq. (18). Then we go up to Eq. (19) in the same a way. By making $\beta = 0$ in this equation, we obtain the corresponding equation to SSG in the form

$$kc^2(1-u^2)u'' + kc^2u(u')^2 - \alpha(1-u^2)^2 = 0. \tag{28}$$

We note that it may be made $n = 2$ in the expansion given by Eq. (3).

Some solutions are:

1. Case 1, $r = 0$:

$$\Theta_{3S\pm}(x,t) = \arccos\left\{\pm\left[1+\frac{q}{p}f^2\left[k(x-ct)\right]\right]\right\}, \tag{29a}$$

$$p^2(k^2c)^2 - \alpha^2 = 0. \tag{29b}$$

2. Case 2, $r \neq 0$:

   There are two solutions in the form

$$\Theta^{\pm}_{4S\pm}(x,t) = \tag{30a}$$
$$\arccos\left\{\pm\left(1+\frac{p}{r}\right) \pm \frac{\sqrt{p^2-2qr}}{r}f^2\left[k(x-ct)\right]\right\},$$

$$(qr)^2(k^2c)^2 - \alpha^2(p^2-2qr) = 0 \tag{30b}$$

e

$$\Theta^{\pm}_{5S\pm}(x,t) = \tag{31a}$$
$$\arccos\left\{\pm\frac{p}{\sqrt{p^2-2qr}} \pm \frac{q}{\sqrt{p^2-2qr}}f^2\left[k(x-ct)\right]\right\},$$

$$(p^2-2qr)(k^2c)^2 - \alpha^2 = 0. \tag{31b}$$

   It is worth noting from the solutions of Eq. (31a) that by making $r = 0$ we obtain the solutions for the Caso 1.

V. DISCUSSION

In this letter we have derived a few solutions for sine-Gordon equations. As an equation may be obtained from other equation vanishing one or two parameters, then by vanishing these parameters in the solutions or into any step to get these solutions, we also obtain further solutions for these equations. It is sufficient to explore the cases which cannot be obtained with this procedure. For example, it is not necessary to make the tangent transformation for DSG and SSG since the solutions with this transformation can be obtained from TSG. It is sufficient to make $\gamma = 0$ and/or $\beta = 0$ into solutions. However we cannot make the cosine transformation in TSG, so we make in DSG to obtain solutions. In the same a way it is necessary to make the sine and cosine transformations up to second order in SSG since these transformations cannot be made into the other equations. Although we had focused our attention in this procedure, which is obtain solutions of equation each other, we made the transformation of tangent for DSG in order to furnish a example, even by making so for TSG previously. It may be noted that the equations sine-Gordon with less terms allow more transformations with facility.

TABLE I: Explicit values for p, q and r of hyperbolic, circular, elliptic functions which satisfy the elliptic equation.

functions	p	q	r
sn	$-(1+m^2)$	$2m^2$	1
sin	-1	0	1
tanh	-2	2	1
cn	$2m^2-1$	$-2m^2$	$1-m^2$
cos	-1	0	1
sech	1	-2	0
dn	$2-m^2$	-2	$-(1-m^2)$
1	2	-2	-1
sech	1	-2	0
ns = 1/sn	$-(1+m^2)$	2	m^2
cossec	-1	2	0
cotanh	-2	2	1
nc = 1/cn	$2m^2-1$	$2(1-m^2)$	$-m^2$
sec	-1	2	0
cosh	0	0	-1
nd = 1/dn	$2-m^2$	$-2(1-m^2)$	-1
1	2	-2	-1
cosh	1	0	-1
sc = sn/cn	$2-m^2$	$2(1-m^2)$	1
tan	2	2	1
sinh	1	0	1
cs = cn/sn	$2-m^2$	2	$1-m^2$
cotan	2	2	1
cossech	1	2	0
sd = sn/dn	$2m^2-1$	$-2m^2(1-m^2)$	1
sin	-1	0	1
sinh	1	0	1
ds = dn/sn	$2m^2-1$	2	$-m^2(1-m^2)$
cossec	-1	2	0
cossech	1	2	0
cd = cn/dn	$-(1+m^2)$	$2m^2$	1
cos	-1	0	1
1	-2	2	1
dc = dn/cn	$-(1+m^2)$	2	m^2
sec	-1	2	0
1	-2	2	1
exp	1	0	0

Finally, some solutions which were derived in this paper were not found in literature. The set of the solutions of these three equations SSG, DSG and TSG, gotten with the f-expansion method contain further solutions than the works from Refs. 23 and 25. Even so we do not explore all possibilities of solutions of the algebric systems.

In Table I the main functions that satisfy the elliptic equation are found. To become explicit a solution we must select a function with its values of p, q and r and replace them into solutions found.

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