Generally Covariant Quantum Theory: Quantum Electrodynamics.

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Abstract

We continue our investigation of the new project launched in [1, 2, 3, 4, 7, 8] by generalizing Quantum Electrodynamics, the theory of electrons and photons, to our setting. At first, we deal with the respective two point functions, define the correct interaction theory as a series of connected Feynman diagrams and finally, we show that for a certain class of spacetime metrics, each diagram is finite and a modified perturbation series is analytic.

1 Introduction.

In this paper, we shall briefly introduce all necessary tools and ideas in order for the reader to understand the construction. Alternatively, he or she may first consult the aforementioned references in the following order: [1] contains the first ideas behind the construction followed up by [4, 8] for information regarding the interacting theory and a generalized Heisenberg picture respectively. Finally, [3] contains some novel physical ideas to make the construction work out and provide one with an analytic theory, some comments regarding quantum gravity in that respect have been dealt with in [7]. The intention of this paper is to show the strength of our framework and apply it to Quantum Electrodynamics, which remains the best tested "quantum field theory". As is well known, QED is not perturbatively renormalizable due to the presence of the Landau pole in the beta function of the renormalization group; our framework will turn out to be perturbatively *finite* as it was for ϕ^4 theory. The key idea behind such wonderful result is a more physical interpretation of the meaning of Lorentz invariance: the latter says that the absolute value of the *propagation* amplitude for a particle to travel from x to y cannot depend upon the four momentum of the free particle. This leaves plenty of *friction* terms left, such as a momentum dependent friction associated to the creation-annihilation process as well as a momentum independent friction term coming from propagation. Traditional quantum theory is all about frictionless propagation and cost-free creation and annihilation processes: it are these idealizations which cause the main trouble as has been explained in [3]. This means we leave all conserved quantities, such as an energy momentum tensor, behind us: the coupling of gravity to quantum

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particles will have to work in a different way. The latter means we dispose also of the cosmological constant problem as has been explained in [3].

As the reader will once again notice, our results do not really depend upon the details of the structure of the interaction vertex, nor the number of loops in a diagram. This shows that the distinction between renormalizable and non-renormalizable theories is completely void and that a perturbative graviton theory is most likely well defined, as has been explained in [7]. The reader who wishes to understand more about the rationale for this construction, and in particular the failure of a covariant operational approach, should consult [2, 6]. In the second reference, I try to build a unique relativistic operational quantum theory, almost from the same principles - with the only exception of operational versus realist- as the one advocated in this series of papers and this theory turned out to be a dead end. This came in 2012 as a big disapointment to me since I did not know what I had done wrong, the physical ideas all seemed to be fine! I could not do anything else than sacrifying unitarity and operationalism and more comments upon *why* this should be done can be found in [2]. It is rather important to understand or at least to know that this proposal comes from a long process of trial and error where, logically, some principles had to be given up.

This paper is organized as follows: in section two, we perform the basic construction of the two point functions for spin 1/2 and spin 1 particles. This section will use results from [1, 3, 8] but I will try to keep the exposition fairly self contained. In section three, we define QED, this will constitute a small enlargement of the results in [1, 4] towards Fermions instead of Bosons. Section four contains then the proof that QED is perturbatively finite relying upon similar methods as in [3]. Comments will be made regarding the violation of unitarity necessary to make the theory analytic. It is important to understand that the conditions placed upon spacetime will be *slightly* more elaborate than is the case for a spin zero particle [3]. In that reference, we obtained that if the volume of the intersection of large Euclidean balls around a point x with the region within the geodesic horizon defined by x scales *at most* in a Euclidean way, then the theory is perturbatively renormalizable. In this paper, the asymptotic behavior of the parallel transporters of spin one and spin one half along spacetime geodesics, as measured in a cosmic reference frame, will also be of utmost importance given that they show up immediately in the interactions. We will have more to say about this in section four when dealing with bounds on Feynman diagrams. Finally, our conclusions hint towards similar results for non abelian gauge theory as well as the introduction of ghost particles.

2 Propagators for spin one half and spin one particles.

In this section, we will gather together insights from [1, 3, 4]; the reader who wishes to see all the details is invited to consult those references. As it turns out, the procedure to regularize the two point function for photons is identical to the one for spin zero particles. Therefore, it is more logical to start with the spin one case. The propagator for spin one half particles is not so "canonically" regulated as the integer spin procedure leads to an *appearant* violation of the spin-statistics theorem, which is a desastrous result regarding the well-definedness of the Dyson series. There are, of course, several ways to cure this appearant deficit and we will take the easiest procedure which keeps us the closest to the standard results in Minkowski spacetime. Let me start again by introducing the construction for a spin 0 particle. Here, we wish to write down the two point function $W(x, y)$ as the integral over an on shell four momentum which is Lorentz invariant in the sense that the modulus squared of the amplitude $\phi(x, k^a, y)$ for propagation from x to y does not depend upon the momentum k^a, at least not for a single path. Here, propagation from x to y happens along a geodesic $\gamma(s)$ and Lorentz invariance implies, if we ignore friction terms, that

$$\frac{d}{ds}\phi(x, k^a, \gamma(s)) = i\dot{\gamma}^\mu(s)k_\mu(s)\phi(x, k^a, \gamma(s))$$

where

$$\frac{D}{ds}k^\mu(s) = 0$$

and $\gamma(0) = x, \gamma(1) = y, k^\mu(0) = k^a e_a^\mu(x)$. The i in front is absolutely crucial to preserve the modulus; the solution of the above equation is given by

$$\phi(x, k^a, y) = e^{ik^a w_a}$$

where w^a is the tangent vector to the geodesic with norm equal to the length of the geodesic. Now, it may be that x and y can be connected by several, possibly infinite number of geodesics, which implies that one should sum over all of them

$$\phi(x, k^a, y) = \sum_{w:\exp_x(w)=y} e^{ik^a w_a}.$$

The latter constitutes the necessary generalization of the Fourier transform and more comments about this as well as the connection to a generalized Heisenberg framework have been dealt with in [8]. The two point function now reads

$$W(x, y) = \int \frac{d^4k}{(2\pi)^3}\delta(k^2 - m^2)\theta(k^0)\phi(x, k^a, y)$$

and as is well known, this integral does not exist in the Lebesgue sense but only as a bi-distribution with regard to Schwartz functions. Taking the sum over all geodesics is the correct recipe as has been shown in [4]: there, we studied the case of a cylindrical, flat, universe where space was a circle of length L and showed that the infinite sum over all geodesics was well defined as a distribution which effectively discretized the integral to the standard sum over all momenta of the form $\frac{2\pi n}{L}$ with $n \in \mathbb{Z}$. So, again, on nontrivial topologies, we got the same answer as standard quantum mechanics. As has been explained in [3], we need to regularize this integral in two ways such that it is well defined in the Lebesgue sense and has suitable falloff conditions towards spacetime infinity. The former requires the introduction of a momentum dependent friction associated to the creation and annihilation process of a particle while the latter requires momentum independent friction associated to the propagation of a particle. Friction requires a medium and in [3], it has been explained that this medium is the nontrivial gravitational field, associated to classical degrees of freedom in the universe, which determines a preferred unit timelike vectorfield V^μ and an associated Riemannian metric

$$h_{\mu\nu} = 2V_\mu V_\nu - g_{\mu\nu}.$$

Before we proceed, we must say something about the very important physical properties of the two point function any regularization scheme needs to preserve. From one side, we have that $\overline{W(x,y)} = W(y,x)$ for all x, y and, moreover, $W(x,y) = W(y,x)$ if x and y are spacelike to one and another which we denote from now on as $x \sim y$. The latter means that x and y are only connected by means of spacelike geodesics while in the case both points are causal with respect to another, they might be connected by means of timelike and spacelike geodesics as well. This famous property for spacelike separated points is well known as the spin statistics implication: it means that propagation from x to y is the same as propagation from y to x which implies that x and y can be swapped without changing the amplitude. This result is known as Bose statistics, so in our framework, spin zero particles exhibit Bose statistics: this property should be preserved under any deformation.

All this means that the friction terms for spacelike geodesics are going to be different than those for causal ones. Denote by $\Lambda(x, w^a)^{\alpha'}_{\beta}$, where the primed index refers to $\exp_x(w) = y$, the parallel propagator along the geodesic defined by w and connecting x with y. Also, introduce the shorthand $k^{\mu'}_{\star w} = \Lambda(x, w^a)^{\mu'}_{\nu} k^\nu$, then the friction associated to a creation-annihilation process for events related by means of a causal geodesic is proposed to be [3]

$$e^{-\mu(V^\alpha k_\alpha)^2 - \mu(V_{\beta'} k^{\beta'}_{\star w})^2}.$$

As shown in the same reference, in order to preserve the statistics property, for spacelike related events $x \sim y$, of the Wightman function, we need to symmetrize the exponent of this expression by adding the reflection $R^\alpha_\beta(w)k^\beta$ of k^β as well as in x as in y. Under these assumptions, one still has that $W_\mu(x,y) = W_\mu(y,x)$ for $x \sim y$ and $\overline{W_\mu(x,y)} = W_\mu(y,x)$ where

$$W_\mu(x,y) = \int \frac{d^4k}{(2\pi)^3}\delta(k^2 - m^2)\theta(k^0) \sum_{w:\exp_x(w)=y} e^{ik^a w_a} e^{-\mu(V^\alpha k_\alpha)^2 - \mu(V_{\beta'} k^{\beta'}_{\star w})^2 + R(w)\,\text{symmetric if } w \text{ is spacelike.}}.$$

As it turns out, this makes the integral well defined but the resulting function does not go sufficiently rapid to zero for y to infinity in order for loop integrals to be well defined. In case one has infinite winding of geodesics, the sum within the integral is still defined as a distribution and the integral exists in this generalized sense. To cure for the falloff problem and to make the integral well defined in the Lebesgue sense in case one disposes of an infinite number of geodesics connecting x and y, we replace the original differential equation for the exponential function by one with momentum independent friction. That is,

$$\frac{d}{ds}\phi_\kappa(x, k^a, \gamma(s)) = \left(i\dot\gamma^\mu(s)k_\mu(s) - \kappa\sqrt{h(\dot\gamma(s), \dot\gamma(s))}\right)\phi_\kappa(x, k^a, \gamma(s))$$

where all symbols have been defined previously. The solution is obviously given by

$$\phi_\kappa(x, k^a, w^b) = e^{ik^a w_a} e^{-\kappa\int_0^1 \sqrt{h(w(s), w(s))}ds}$$

where the last term has been called the exponentiated energy of the geodesic connecting x with y. This gives rise to

$$W_{\mu\kappa}(x,y) = \int \frac{d^4k}{(2\pi)^3}\delta(k^2-m^2)\theta(k^0) \sum_{w:\exp_x(w)=y} e^{ik^a w_a} e^{-\kappa\int_0^1 \sqrt{h(w(s),w(s))}ds}$$

$$e^{-\mu(V^\alpha k_\alpha)^2-\mu(V_{\beta'}k^{\beta'}_{\star w})^2+R(w)\,\text{symmetric if}\,w\,\text{is spacelike.}}$$

which has all the desired properties. The only important proproperty we need is that

$$|W_{\mu\kappa}(x,y)| < Ce^{-(\kappa-\epsilon)d(x,y)}$$

where C is a constant depending upon the geometry and μ, $0<\epsilon\ll\kappa$ and d denotes the Riemannian distance defined by h. In [3], we worked with a universal C which was possible because we did not consider geometries with multiple geodesics joining two points. This finishes our discussion for the spin-0 two point function; we now return to the photon two point function, the latter has been shown [1] to be equal to

$$W^{\mu\kappa}_{\alpha\beta'}(x,y) = \int \frac{d^4k}{(2\pi)^3}\delta(k^2-m^2)\theta(k^0) \sum_{w:\exp_x(w)=y} g_{\alpha\beta'}(x,w)e^{ik^a w_a} e^{-\kappa\int_0^1 \sqrt{h(w(s),w(s))}ds}$$

$$e^{-\mu(V^\alpha k_\alpha)^2-\mu(V_{\beta'}k^{\beta'}_{\star w})^2+R(w)\,\text{symmetric if}\,w\,\text{is spacelike.}}$$

where

$$g_{\alpha\beta'}(x,w) = (\Lambda^{-1}(x,w))^{\gamma}_{\beta'}g_{\alpha\gamma}(x)$$

is the parallel transport of the bi-tensor along the geodesic determined by w. To get a bound on this expression is not really desirable since it is coordinate dependent; later on we will consider invariants which can be properly bounded; for now, it is sufficient to know that

$$W^{\mu\kappa}_{\alpha\beta'}(x,y) = e^{-\kappa d(x,y)} \sum_{w:\exp_x(w)=y} g_{\alpha\beta'}(x,w)C_{\mu\kappa}(x,w)$$

and there exists a labelling of w, denoted by w_j, such that, in case of an infinite number of geodesics connecting x with y, there exists an $L>0$, independent of x and y such that

$$\sum_{j=0}^{\infty} e^{jk\frac{L}{d(x,y)+1}}\,|C_{\mu\kappa}(x,w_j)|^k < C(\mu,\kappa,L,k,g,V)(1+d(x,y))$$

where $k\geq 1$ and C depends, amongst others, upon the geometry and the friction parameters. At least, we will assume this to be the case and I think it is certainly true for manifolds with a finite (over $\mathbb{Z}$) first homotopy group. Indeed, it is clear that an infinite number of geodesics between two points requires a nontrivial first homotopy group and arbitrary winding numbers. Under rather generic conditions, we may associate to each homotopy generator a minimal length squared $M(h)>0$ (Gromov) such that the energy of a curve with winding number $n>n_0>0$ between[1] x and y is greater than $d(x,y)+n\frac{M(h)}{d(x,y)+1}$ which

[1]The n_0 serves here to avoid the pathological cases where, for example, the length of two h-geodesics equals the minimal distance d and, moreover, they have a relative winding number of one.

leads to the desired result[2]. These conditions are not always true on non-compact spacetimes in case singularities are present (giving rise to topology change and $M(h) = 0$). The reader should notice that spin-one particles are bosons from the symmetry properties of $W_{\mu\kappa}$.

Let us now introduce the two point function for spin-$\frac{1}{2}$ particles: in [1], it was shown that the correct frictionless two point function for particles and anti-particles of spin-$\frac{1}{2}$ was given by

$$W_p(x,y)_i^{j'} = \int_{T^\star\mathcal{M}_x} \frac{d^4k}{(2\pi)^3}\delta(k^2-m^2)\theta(k^0) \sum_{w:\exp_x(w)=y} (\Lambda^{\frac{1}{2}}(x,w))_r^{j'}(-ik_a(\gamma^a)_i^r+m\delta_i^r)e^{ik^a w_a}$$

and

$$W_a(x,y)_{j'}^i = \int_{T^\star\mathcal{M}_x} \frac{d^4k}{(2\pi)^3}\delta(k^2-m^2)\theta(k^0) \sum_{w:\exp_x(w)=y} (-ik_a(\gamma^a)_r^i-m\delta_r^i)((\Lambda^{\frac{1}{2}}(x,w))^{-1})_{j'}^r e^{ik^a w_a}$$

where $\Lambda^{\frac{1}{2}}(x,w)$ is the spin transformation associated to parallel transport of a spinor along a geodesic between x and y determined by w. It has been shown that

$$W_p(x,y)_i^{j'} + W_a(y,x)_i^{j'} = 0$$

for $x \sim y$ so that spin-$\frac{1}{2}$ particles exhibit Fermi statistics. To obtain the correct propagator, it is sufficient to dress the usual Fourier waves with the aforementioned friction terms and to take

$$\widetilde{W}^{\mu\kappa}_{p,a}(x,y) = \frac{1}{2}\left(W^{\mu\kappa}_{p,a}(x,y) - W^{\mu\kappa}_{a,p}(y,x)\right)$$

if $x \sim y$ and

$$\widetilde{W}^{\mu\kappa}_{p,a}(x,y) = W^{\mu\kappa}_{p,a}(x,y)$$

otherwise. Note that it is necessary to take the antisymmetric difference in order to preserve the spin-statistics result as $W^{\mu\kappa}_{p,a}(x,y)$ does not obey it in general as the reader may verify. Again, the reader may infer that on a general class of backgrounds

$$\widetilde{W}^{\mu\kappa}_{p}(x,y) = e^{-\kappa d(x,y)} \sum_{w:\exp_x(w)=y} \Lambda^{\frac{1}{2}}(x,w)\left(C^{\mu\kappa}_{p;b}(x,w)\gamma^b + C^{\mu\kappa}_{p}(x,w)1\right)$$

and

$$\widetilde{W}^{\mu\kappa}_{a}(x,y) = e^{-\kappa d(x,y)} \sum_{w:\exp_x(w)=y} \left(C^{\mu\kappa}_{a;b}(x,w)\gamma^b + C^{\mu\kappa}_{a}(x,w)1\right)(\Lambda^{\frac{1}{2}}(x,w))^{-1}$$

[2] Indeed, one can bound the sum over all w as the sum over winding numbers n of

$$C_{\mu\kappa}e^{-\kappa\frac{nM(h)}{d(x,y)+1}}$$

which has the above mentioned properties since

$$C^k_{\mu\kappa}\sum_n e^{-\kappa k\frac{nM(h)}{d(x,y)+1}} = C^k_{\mu\kappa}\frac{1}{1-e^{-\kappa k\frac{M(h)}{d(x,y)+1}}} \leq C(\mu,\kappa,h,k)(d(x,y)+1).$$

The division through $d(x,y)+1$ stems from infinitely large homotopy classes and can be ignored when all nontrivial topology resides in a compact region of spacetime.

where every coefficient function has the dimension of mass and satisfies, moreover, an identical property as the coefficient function for the spin-one two point function with bounds holding in any reference frame[3] with $e_0 = V$. Indeed, the latter class of reference frames will become important in the sequel since they constitute also a vierbein for the Riemannian metric h; the transporters $\Lambda^{\frac{1}{2}}(x, w)$ are all expected to be expressed in such basis with a rotation invariant operator norm

$$\sqrt{\mathrm{Tr}(A^{\dagger}A)}.$$

It are these estimates, and in particular the linear dependency of the bound upon $d(x, y)$, which are of crucial importance to adress finiteness of Feynman diagrams, see [3] to obtain an understanding for this claim.

In [3], we introduced important concepts such as the one of spacelike and timelike geodesic horizon or geodesic horizon of a point x. The latter is defined as the boundary of the space of all events y which can be reached by a geodesic emanating from x. As is well known, some events require acceleration to be reached and we have previously studied such cosmology.

3 Defining Quantum Electrodynamics.

From now on, we will drop the μ, κ and tilde references in the notation of the two point function and we will just refer to $W_{\alpha\beta'}(x, y)$ as the physical photon two point function with the necessary friction terms such that all bounds above hold. I have never explicitely stated this, but the definition of the Feynman propagator can also be extended to a spacetime with closed timelike curves; for photons for example, one could state that

$$\Delta_{F\alpha\beta'}(x, y) = \frac{1}{2}\left(W_{\alpha\beta'}(x, y) + W_{\beta'\alpha}(y, x)\right)$$

if $y \in J^{+}(x) \cap J^{-}(x)$ and frame the definition as usual otherwise. It may, however, be that on such spacetime no spacelike region to an event exists and therefore one does not have a global result regarding particle statistics. This would suggest an ambiguity in the theory which can only be resolved by speaking about the same theory on different spacetimes, Fewster and Verch [9] have interesting things to say about that. Taking, however, the statistics result for granted, we obtain for Fermions that the particle Feynman propagator looks like

$$\Delta_{F;p}(x, y)_{i}^{j'} = W_p(x, y)_{i}^{j'}$$

if $y \notin J^{-}(x)$ and

$$-W_a(y, x)_{i}^{j'}$$

if $y \in J^{-}(x) \setminus J^{+}(x)$ and finally, in case of closed timelike curves

$$\frac{1}{2}\left(W_p(x, y)_{i}^{j'} - W_a(y, x)_{i}^{j'}\right)$$

[3] If one allows for arbitrary large boosts of such special reference frame, then no bound needs to hold.

while the anti-particle propagator satisfies

$$\Delta_{F;p}(x,y)_i^{j'} = -\Delta_{F;a}(y,x)_i^{j'}$$

as usual. One should notice that we did not define the coincidence limit $\Delta_{F,p}(x,x)^i_j$ which, unlike for integer spin particles, has no natural value given that

$$W_p(x,x)^i_j \neq -W_a(x,x)^i_j$$

as the reader may wish to verify, see [1]. Therefore, the natural thing to do is to forbid diagrams requiring such expression, or to pick an arbitary value which gives effectively one extra parameter in the theory. As in [3, 4] we define the interacting theory as a sum over connected Feynman diagrams between in IN and OUT states $|\mathrm{IN}\,(x_1,a_1),\ldots,(x_n,a_n)\rangle$ and $|\mathrm{OUT}\,(y_1,b_1),\ldots,(y_m,b_m)\rangle$ respectively where a_i, b_j is either a covariant spacetime index or a covariant, corresponding to a particle in the IN state and an anti-particle in the OUT state, or contravariant spinor index with the opposite interpretational conventions. Here, it is understood that all x_i (y_j) belong to non-intersecting spacelike, but not necessarily achronal, hypersurfaces S_I (S_F) such that S_F is in the future of S_I in a well defined sense, see [3] for the exact definition. The diagrams we consider are such that any internal vertex is connected to an IN or OUT vertex, no IN (OUT) vertices are connected by a single propagator to an IN (OUT) vertex since otherwise there would exist an IN (OUT) vertex where a particle would arrive (leave) in contrast to the meaning of IN and OUT. One should be aware that the definition of the Feynman propagator does *not* imply that all processes are travelling forwards in time: all the definition of the Feynman propagator says is that the amplitude for a process "going backwards in time" equals plusminus the amplitude for the opposite process "going forwards in time". Therefore, we can state that an IN electron is leaving towards its past instead of a positron arriving from the past at the specified IN position: indeed, the relationship between the particle and anti-particle Feynman propagator does not immediately reveal the correct interpretation. What we state is that the correct interpretation is given by putting the IN vertices as first argument in the Feynman propagator and the OUT vertices as last argument; we don't care about a unique interpretation for the internal vertices.

As explained in [1], the only interaction vertex or intertwiner is given by

$$e^\mu_a(x)(\gamma^a)^i_j$$

which has no internal symmetries, so the symmetry factor of a diagram equals always one. An internal vertex with label k is therefore represented by a triple (μ_k, i_k, j_k) where the index j_k is covariant and the remaining two contravariant. Take then the series $(b_m,\ldots,b_1,(\mu_1,i_1,j_1),\ldots,(\mu_V,i_V,j_V),a_1,\ldots,a_n)$ where V represents the number of internal vertices and define the rule that the transposition of a spacetime index with any other index corresponds to plus one, while the transposition of a spinor index with another spinor index corresponds to minus one. Moreover, only covariant and contravariant spinor indices of different vertices can couple to one and another; then, the reader verifies that the overall sign of a diagram is well defined, taken into account the properties of the Fermi Feynman propagator, and independent of the labelling of the internal vertices. With all this in mind, we write formally

$$\langle \mathrm{OUT}\,(y_1, b_1), \ldots, (y_m, b_m) | \mathrm{IN}\,(x_1, a_1), \ldots, (x_n, a_n)\rangle = \sum_D (-i\lambda)^V \epsilon(D)$$

$$\int dz_1 \sqrt{h(z_1)} \ldots \int dz_V \sqrt{h(z_V)} \prod \Delta_{F;\, a_l a_{p(l)}}(\alpha_l, \alpha_{p(l)}) \prod \Delta_{F,p}(\alpha_k, \alpha_{r(k)})^{j_{r(k)}}_{i_k} \prod (\gamma^{a_m})^{i_m}_{j_m}$$

where $\epsilon(D) = \pm 1$ the sign of the diagram which has been fixed by the consistent choice for the particle Feynman propagator in the Fermi sector and $\alpha \in \{z_k, x_i, y_j\}$. I say formally, since experience [3] has shown that the series does not converge albeit every diagram gives a finite contribution which we will show explicitely in the next section where we shall estimate the magnitude of a diagram. Corrections to unitarity should therefore occur and we will comment upon that later on.

In [3, 4] did we make a distinction between a Type I, II, III quantum theory and we explained why some spacetimes excluded one type but not another. Therefore, by "spacetime" in the sequel, we mean that portion of spacetime to which our analysis is applicable. The reader might want to read upon those fine details prior to proceeding.

4 Perturbative Finiteness of QED.

We repeat the assumptions which went into the argument in [3] and later on specify some additional constraints the geometry has to satisfy in order for our proof to hold. I do not think those assumptions may be significantly changed without affecting the "basic" structure of the theory and we will keep possible generalizations of the structure as well as the analysis for future work. In particular, we assumed that our Riemannian geometry is exponentially finite meaning that

$$\int dy \sqrt{h(y)} P(d(x,y)) e^{-\kappa d(x,y)} < R(P, \kappa)$$

for every polynomial P and $\kappa > 0$. Moreover, the balls of radius r around x have the following volume bound

$$\mathrm{Vol}(B(x,r)) \leq K r^4$$

meaning that the Riemannian geometry is "dominated" by an asymptotically Euclidean space in some sense and that hyperbolic behaviour should not occur on large scales. One can already guess that we will also need norm bounds on the geodesic transporters $\Lambda(x,w)^{a'}_{b}$ and $\Lambda^{\frac{1}{2}}(x,w)^{i'}_{j}$ with respect to any cosmic reference frame $e_0 = V$. Those issues were of course completely irrelevant in the spin-0 theory given that there are no internal degrees of freedom to "feel" those transporters and these matters are not important in Minkowski either but they might be of importance in a more general cosmology.

The contribution of a diagram is given by

$$\int dz_1 \sqrt{h(z_1)} \ldots \int dz_V \sqrt{h(z_V)} \prod e^{-\kappa d(\alpha_l, \alpha_{p(l)})} \sum_{w_l : \exp_{\alpha_l}(w_l) = \alpha_{p(l)}} (\Lambda^{-1}(\alpha_l, w_l))^{b_l}_{a_{p(l)}} \eta_{b_l a_l} C_{\mu\kappa}(\alpha_l, w_l)$$

$$\prod e^{-\kappa d(\alpha_k,\alpha_{r(k)})} \sum_{w_k:\exp_{\alpha_k}(w_k)=\alpha_{r(k)}} (\Lambda^{\frac{1}{2}}(\alpha_k,w_k))^{j_{r(k)}}_{s_k} \left(C^{\mu\kappa}_{p;b}(\alpha_k,w_k)\gamma^b + C^{\mu\kappa}_p 1\right)^{s_k}_{i_k} \prod (\gamma^{a_m})^{i_m}_{j_m}$$

where index notation has been used just as before. The very structure of this formula reveals that loops are not going to be of importance, just as it was the case for ϕ^4 theory [3]. As a general matter, one has that

$$V - I = C - L$$

where V is the number of internal vertices, I the number of internal edges, C the number of components and finally L the number of loops. For QED, one has moreover that

$$2I + (n' + m') = 3V$$

where $0 \leq n' \leq n$, $0 \leq m' \leq m$ is the number of IN and OUT vertices connected to an internal vertex. Hence,

$$\frac{3}{2}V - \frac{n'+m'}{2} \geq L = C + \frac{V}{2} - \frac{n'+m'}{2} \geq 0$$

which implies that $V \geq \frac{n'+m'}{3}$. To simplify the analysis and to eliminate the distinction between internal and external photon or Fermion lines we consider that

$$\left| \sum_{w_k:\exp_{\alpha_k}(w_k)=\alpha_{r(k)}} (\Lambda^{\frac{1}{2}}(\alpha_k,w_k))^{j_{r(k)}}_{s_k} \left(C^{\mu\kappa}_{p;b}(\alpha_k,w_k)\gamma^b + C^{\mu\kappa}_p 1\right)^{s_k}_{i_k} \right| \leq$$

$$2 \sum_{w_k:\exp_{\alpha_k}(w_k)=\alpha_{r(k)}} \sqrt{\mathrm{Tr}\left((\Lambda^{\frac{1}{2}}(\alpha_k,w_k))^{\dagger}(\Lambda^{\frac{1}{2}}(\alpha_k,w_k))\right)} \sqrt{\sum_b \left|C^{\mu\kappa}_{p;b}(\alpha_k,w_k)\right|^2 + |C^{\mu\kappa}_p|^2}$$

which can be further bounded to

$$2\left(\sup_{w_k:\exp_{\alpha_k}(w_k)=\alpha_{r(k)}} \sqrt{\mathrm{Tr}\left((\Lambda^{\frac{1}{2}}(\alpha_k,w_k))^{\dagger}(\Lambda^{\frac{1}{2}}(\alpha_k,w_k))\right)}\right)\left(\sum_{w_k:\exp_{\alpha_k}(w_k)=\alpha_{r(k)}} \left(\sum_b \left|C^{\mu\kappa}_{p;b}(\alpha_k,w_k)\right| + \left|C^{\mu\kappa}_p\right|\right)\right)$$

Finally, this is majorated by

$$D(\mu,\kappa,g,V)\left(\sup_{w_k:\exp_{\alpha_k}(w_k)=\alpha_{r(k)}} \sqrt{\mathrm{Tr}\left((\Lambda^{\frac{1}{2}}(\alpha_k,w_k))^{\dagger}(\Lambda^{\frac{1}{2}}(\alpha_k,w_k))\right)}\right)(1+d(\alpha_k,\alpha_{r(k)}))$$

which is the bound we need. Given that $\left|(\gamma^a)^i_j\right|$ equals one 16 times and zero 48 times gives then that the contribution of a diagram is bounded by

$$(16)^V \int dz_1 \sqrt{h(z_1)} \ldots \int dz_V \sqrt{h(z_V)}$$

$$\prod e^{-\kappa d(\alpha_l,\alpha_{p(l)})}\left(\sup_{w_l:\exp_{\alpha_l}(w_l)=\alpha_{p(l)}} \sqrt{\mathrm{Tr}(\Lambda^{-1}(\alpha_l,w_l)^{\dagger}\Lambda^{-1}(\alpha_l,w_l))}\right) C(\mu,\kappa,g,V)(1+d(\alpha_l,\alpha_{p(l)}))$$

$$\prod e^{-\kappa d(\alpha_k,\alpha_{r(k)})} D(\mu,\kappa,g,V)\left(\sup_{w_k:\exp_{\alpha_k}(w_k)=\alpha_{r(k)}} \sqrt{\mathrm{Tr}\left((\Lambda^{\frac{1}{2}}(\alpha_k,w_k))^{\dagger}(\Lambda^{\frac{1}{2}}(\alpha_k,w_k))\right)}\right)(1+d(\alpha_k,\alpha_{r(k)})).$$

Now, it is easy to see that an appropriate requirement for the parallel propagators to satisfy is that

$$\sup_{w_l:\exp_{\alpha_l}(w_l)=\alpha_{p(l)}} \sqrt{\mathrm{Tr}(\Lambda(\alpha_l,w_l)^\dagger \Lambda(\alpha_l,w_l))} \leq F(g,V)e^{\delta d(\alpha_l,\alpha_{p(l)})}$$

with $\delta < \kappa$ and likewise for

$$\sup_{w_k:\exp_{\alpha_k}(w_k)=\alpha_{r(k)}} \sqrt{\mathrm{Tr}\left((\Lambda^{\frac{1}{2}}(\alpha_k,w_k))^\dagger(\Lambda^{\frac{1}{2}}(\alpha_k,w_k))\right)}.$$

Denoting with $E(\mu,\kappa,g,V) = \max\{C(\mu,\kappa,g,V), D(\mu,\kappa,g,V)\}$, we have that the bound on the diagram simplifies to

$$(16)^V(F(g,V)E(\mu,\kappa,g,V))^{\frac{3}{2}V+\frac{n+m}{2}} \int dz_1\sqrt{h(z_1)}\ldots\int dz_V\sqrt{h(z_V)} \prod_{\text{all lines}} e^{-(\kappa-\delta)d(\alpha,\beta)}(1+d(\alpha,\beta)).$$

For sake of simplicity, we can ignore the linear term $(1+d(\alpha,\beta))$ by adding a small $0<\epsilon<\kappa-\delta$ in the exponential and muliplying with another constant. Hence, we are left with

$$(G(\mu,\kappa,g,V))^{\frac{3}{2}V+\frac{n+m}{2}} \int dz_1\sqrt{h(z_1)}\ldots\int dz_V\sqrt{h(z_V)} \prod_{\text{all lines}} e^{-(\kappa-\delta-\epsilon)d(\alpha,\beta)}$$

and these are precisely the same integrals as for ϕ^4 theory [3] except that we have now trivalent vertices instead of vertices with four legs, the distinction between Fermi and photon lines being gone. To evaluate such integrals, we use as before [3] that

$$d(x,y)+d(y,z) \geq d(x,z)+\frac{1}{2}d(y,\frac{x+z}{2})$$

for $d(y,\frac{x+z}{2}) \geq \frac{3}{2}d(x,z)$ where $\frac{x+z}{2}$ is a formal notation for some midpoint of x,z. Note that this general bound is somewhat weaker than it is in the Euclidean case where we could drop the factor $\frac{1}{2}$; also it is now possible for a component of the diagram to be connected to an odd number of external vertices. In particular, it might be connected to a single external vertex so that the dependency of the bound on the associated spacetime point is integrated out. This is *the* major qualitative difference with ϕ^4 theory; we shall now estimate the integral: it is bounded by

$$R(1,\kappa-\delta-\epsilon)^V$$

which we derived before [3]. We proved this by induction on the number of internal vertices: if $V=0$, then we simply have a product of numbers $e^{-(\kappa-\delta-\epsilon)d(\alpha,\beta)}$ which is smaller or equal to one. Suppose it is true for $V \geq 0$, then we prove it for $V+1$: take any internal vertex connected by an edge to an exterior vertex α and remove it. The effect is that we obtain a new diagram with three new external vertices from which we remove the edge with α; the remainer is bounded by

$$R(1,\kappa-\delta-\epsilon)^{V-1}.$$

Now, we identify the three vertices again and perform the remaining integration over this vertex which gives a factor of $R(1,\kappa-\delta-\epsilon)$ (due to the leg with α)

which proves the result.

In order to obtain nonperturbative results, we need an estimate on the number of diagrams with n IN and m OUT vertices; we leave such investigations for the future. It might be that unitarity has to be sacrified at the level of the interaction series to make the latter analytic.

5 Conclusions.

The kind of gauge invariance introduced in [1] does not coincide with its standard form in quantum field theory but agrees effectively with this notion in the unregularized Minkowski theory. This is logical given that our regularized scheme does not contain any conserved quantities and therefore gauge invariance must have a different meaning since the standard notion crucially depends upon the conservation law for the Fermi current. Hence, we might expect small, but nonvanishing amplitudes associated to processes involving photons with a "longitudonal" polarization in a background sufficiently close to Minkowski spacetime. As a general comment, our theory will not satisfy any Ward identity due to the absence of (local) symmetries. Photons with a longitudonal polarization seem not to have been observed with sufficient degree of certainty but it is not excluded that they might exist as several theorists still contemplate theories with a small photon mass.

As explained in [3, 7], the proof of our result did not depend upon the details of the interaction series as our bound came from a very general argument. This shows that also the perturbative graviton theory is going to be well defined and we postpone such adventures for future investigations. The reader may notice that our argument for the

$$R(1, \kappa - \delta - \epsilon)^V$$

behavior of the amplitudes did not really depend upon the Euclidean volume bound on large balls. Indeed, one might weaken the definition of an exponentially finite geometry to the extend that integrals of the kind

$$\int dy \sqrt{h(y)} P(d(x,y)) e^{-\gamma d(x,y)} < R(P, \gamma)$$

are finite for $\gamma > \zeta > 0$. In that way, we could make a Type III theory for hyperbolic universes, see [3] but the bounds taking into account the relative distances between exterior vertices cannot be reproduced anymore for diagrams with many internal vertices. This was my main reason to ignore a Type III theory for the $\Lambda > 0$ cosmology in [3], but it does not need to be so.

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