A General class of exactly solvable inverted quadratic Liénard type equations

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Abstract

The inverted quadratic Liénard type equation is very useful in various branches of classical and quantum theories, since it admits a position dependent mass dynamics. The objective of the present work is to show that some interesting inverted nonlinear oscillator equations like the inverted version of Mathews-Lakshmanan oscillator belong to a general class of exactly solvable inverted quadratic Liénard equations. This class of equations is generated from a first integral formulated as an integro-differential equation. The obtained results may be used for the identification and integrability of a family of dynamical systems equations.

Keywords: Inverted quadratic Liénard equation, first integral , exact solutions, Mathews-Lakshmanan oscillator.

1. Introduction

The best understanding of the dynamics of quadratic Liénard type differential equations has become a vital requirement since this type of dissipative nonlinear oscillator equations arises often in mathematical modeling of many physical problems involving nonlinear phenomena. The quadratically dissipative Liénard type equation involves then an attractive field of both mathematical and physical investigations. A famous model of this type of position dependent mass equation is the Mathews-Lakshmanan oscillator equation which exhibits periodic motion with harmonic form [1]

$$\ddot{x} - \frac{\mu x}{1+\mu x^2}\dot{x}^2 + \frac{a^2 x}{1+\mu x^2} = 0 \qquad (1)$$

where the dot over a symbol denotes the differentiation with respect to time. In the same perspective, it has been identified that the equation (1) belongs to the general class of quadratic Liénard type equations of the form [2]

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 + a^2 \frac{f(x)}{g(x)^2}\int f(x)dx = 0 \qquad (2)$$

where a is an arbitrary constant, $f(x)$ and $g(x) \neq 0$, are arbitrary functions of dependent variable x, and prime means the differentiation with respect to x. Recently, a number of works has been performed about the inverted version of the Mathews-Lakshmanan oscillator, due to its important applications in classical and quantum sciences [3-5]. By using a non-standard complex Lagrangian, El-Nabulsi [5] has shown that an inverted version of the Mathews-Lakshmanan oscillator equation (1) may be constructed as

$$\ddot{x} + \frac{\mu x}{1+\mu x^2}\dot{x}^2 - \frac{a^2 x}{1+\mu x^2} = 0 \qquad (3)$$

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In [5] it has been demonstrated that for appropriate values of some model parameters, this inverted nonlinear oscillator may also exhibit periodic motion with harmonic form. In this regard the question of generating inverted quadratic Liénard type equations from classical formal approach becomes an interesting open mathematical problem. It then appears reasonable in this situation to theoretically investigating such a relevant problem. In this perspective, it should be noted in [6] that the proposed class of quadratic Liénard type equations may lead under the over-damped dynamical regime, to a class of inverted quadratic Liénard type equations. The problem of constructing general classes of exactly integrable second order nonlinear differential equations is not a simple task, since exact analytical solutions are always not available. So, several approaches of different complexities have then been developed to overcome this mathematical difficulty. Usually, it is desired to transform the original second order nonlinear differential equation into a differential equation for which analytical properties or solutions are well known. This is often performed by means of linearizing or variable transformation. In this way the approach by first integral has also been used for reducing the order of the initial differential equation to computing easily solutions of first order equation. An inverse problem, however, may consist of finding, given a first integral, the related differential equations. An equivalent or alternative way would consist of generating nonlinear differential equations from first order differential equations through differentiation [7]. Such first order differential equations consist of intermediate integrals that also may ensure the exact solvability of generated differential equations in question [8]. In the present research contribution, the problem of constructing inverted Mathews-Lakshmanan oscillator depicted in [5], from classical formal basis, namely from first integral approach consisting of differentiation of a linear integro-differential equation is examined. This precisely raises the fundamental question: Is that the inverted Mathews-Lakshmanan oscillator (3) belongs to a general class of exactly solvable inverted quadratic Liénard type equations formulated from classical formal basis? The present work assumes the existence of such a general class of exactly solvable inverted quadratic Liénard type equations. This existence should allow for the identification and integrability of a wide variety of inverted quadratic Liénard type differential equations encountered in different branches of physics and engineering sciences. In other words, the hard problem of solving inverted quadratic Liénard type differential equations is reduced to solving differential equations mapped into quadratures. In this context, it is first shown that the inverted Mathews-Lakshmanan oscillator (3) carried out in [5] belongs to a general class of exactly solvable inverted quadratic Liénard type equations comparable to the general class of nonlinear oscillator equations (2) by using a first integral formulated in terms of integro-differential equation under differentiation approach (section 2), and secondly some examples of inverted versions of interesting classical quadratic Liénard type equations are given for illustrating the application of the proposed theory (section 3). Finally, an analysis of obtained results (section 4) and some conclusions (section 5) are presented.

2. General theory

The problem in this section is to generate, given an ansatz for the first integral the related general second order nonlinear differential equation, from which the desired general class of exactly solvable inverted quadratic Liénard type equations should be deduced, and to show that the inverted equation (3) belongs to this class of equations.

2.1 General class of forced quadratic Liénard type equations

Consider, to generate the solvable general class of quadratic Liénard type equations, that the first integral of interest is of the general form

$$a_1(x,\dot{x}) = g(x)\dot{x} + a\int f(x)dx \tag{4}$$

so that

$$\frac{d}{dt}a_1(x,\dot{x}) = 0 \tag{5}$$

Thus, the following theorem may be formulated.

Theorem 2.1

Let us consider the equation (5)*. If the function* $a_1(x,\dot{x})$ *satisfies* (5)*, then* x *satisfies the equation*

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 - a^2\frac{f(x)}{g(x)^2}\int f(x)dx = -aa_1\frac{f(x)}{g(x)^2} \tag{6}$$

Proof

Substitution of the function (4) into the equation (5), gives after a few mathematical manipulation, the equation (6). So, the theorem is proved.

The equation (6) designates the generalized forced quadratic Liénard type oscillator equation with the forcing function

$$F(x(t)) = -aa_1\frac{f(x)}{g(x)^2} \tag{7}$$

One may deduce, now, the desired general class of inverted quadratic Liénard type equations.

2.2 General class of inverted quadratic Liénard type equations

Since the equation (6) represents a generalized forced oscillator equation, the unforced version must then exist, that is to say the function (7) may be set equal zero. Thus, the following corollary, which may be immediately, deduced from the theorem 2.1, gives the desired general class of exactly solvable inverted quadratic Liénard type differential equations**.**

Corollary 2.1

Let $a_1 = 0$*.Then the equation* (6) *is reduced to the equation*

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 - a^2\frac{f(x)}{g(x)^2}\int f(x)dx = 0 \tag{8}$$

Proof

A restriction on a is needed to prove the corollary, that is $a \neq 0$. On the other hand, it is possible to assign specific values to a_1 by defining appropriate initial conditions, since the integral in (4) is an indefinite integral. Let us now consider $a_1 = 0$. Thus, substituting $a_1 = 0$, into the equation (6), yields immediately the equation (8). This proves the corollary.

That being so, the equation (8) may formally, be regarded as the inverted version of the general class of quadratic Liénard type equations (2) recorded in [2]. It is, now, possible to demonstrate that the inverted Mathews-Lakshmanan equation (3) belongs to the general class of inverted quadratic Liénard type equations defined by (8). In this perspective the functions $f(x)$ and $g(x)$ must be appropriately chosen knowing that $a_1 = 0$. Therefore, the following theorem may be stated.

Theorem 2.2

Let us consider the equation (8). *If* $f(x) = \frac{x}{\sqrt{x^2 \pm \beta^2}}$, *and* $g(x) = \sqrt{1 + \mu x^2}$, *where* β *is an arbitrary constant, then the equation* (8) *is mapped into the inverted Mathews-Lakshmanan oscillator equation* (3)

$$\ddot{x} + \frac{\mu x}{1 + \mu x^2} \dot{x}^2 - \frac{a^2 x}{1 + \mu x^2} = 0 \qquad (9)$$

Proof

Substitution of $f(x) = \frac{x}{\sqrt{x^2 \pm \beta^2}}$, and $g(x) = \sqrt{1 + \mu x^2}$, into the equation (8), leads immediately to the equation (9). Thus, the theorem is demonstrated. The following corollary allows for the reduction of (9) to quadratures.

Corollary 2.2

Let us consider the equation (4). *If* $f(x) = \frac{x}{\sqrt{x^2 \pm \beta^2}}$, and $g(x) = \sqrt{1 + \mu x^2}$, then the equation (9) is mapped into

$$\int \frac{\sqrt{1 + \mu x^2}}{\sqrt{x^2 \pm \beta^2}} dx = -a(t - t_0) \qquad (10)$$

where t_0 means a constant of integration.

Proof

Substitution of $f(x)=\frac{x}{\sqrt{x^2\pm\beta^2}}$, and $g(x)=\sqrt{1+\mu x^2}$, into the equation (4), gives immediately after integrating the obtained result, the equation (10). So, the corollary is proved. In this perspective the equation (9) admits the first integral $I=a^2\beta^2$, so that

$$\pm I=(1+\mu x^2)\dot{x}^2-a^2x^2 \tag{11}$$

It is worth noting that the value of β may be known from initial conditions by using equation (11) once specific values are assigned to the parameters a and μ. Also, the equation (10) or (11) shows that for a same value of x, that is for a same point there may exit two values for the velocity $\dot{x}$. A physical interpretation of this fact may be found in [9]. The above shows that the inverted Mathews-Lakshmanan oscillator equation (3) investigated in [5] from a complex Lagrangian approach may be generated using classical formal basis consisting of first integral expressed as an integro-differential equation under differentiation. In doing so, the problem is reduced to solving a differential equation mapped into the form of a quadrature. Let us consider, at present, two additional examples of quadratic Liénard type equations in the following to illustrate the proposed theory (section 3).

3. Examples

Example 1. $f(x)=\frac{x}{\sqrt{x^2\pm\beta^2}}$, and $g(x)=\sqrt{1-\mu x^2}$

The corresponding inverted quadratic Liénard type equation may be written as

$$\ddot{x}-\frac{\mu x}{1-\mu x^2}\dot{x}^2-\frac{a^2x}{1-\mu x^2}=0 \tag{12}$$

which admits the first integral I given by the following equation

$$\pm I=(1-\mu x^2)\dot{x}^2-a^2x^2 \tag{13}$$

Example 2. $f(x)=\frac{x}{\sqrt{\beta^2-x^2}}$, and $g(x)=\sqrt{1+\mu x^2}$

The corresponding quadratic Liénard type equation according to equation (8) may be obtained in the form

$$\ddot{x}+\frac{\mu x}{1+\mu x^2}\dot{x}^2+\frac{a^2x}{1+\mu x^2}=0 \tag{14}$$

with the first integral I given by

$$I=(1+\mu x^2)\dot{x}^2+a^2x^2 \tag{15}$$

which corresponds to the first integral derived in [10], since the equation (14) represents a particle constrained to move on a rotating parabola. The equation (14) is also examined in [5] as an alternative to the Mathews-Lakshmanan oscillator equation. These illustrative examples enable then to discuss the obtained results (section 4).

4. Discussion

The problem of investigating both classical and quantum aspects of the inverted version of the famous Mathews-Lakshmanan oscillator has recently become an active research field of mathematical physics. In [5] the classical aspect of the Mathews-Lakshmanan oscillator is explored in its inverted version from complex Lagrangian approach. However, this problem may also be handled from other simple but relevant mathematical methods. In this work, a classical approach consisting of generating quadratic Liénard type differential equations from first integral is developed. Precisely, it is shown here that quadratic Liénard type differential equations may be constructed from differentiation of a first integral formulated as a linear integro-differential equation. As a result, a general class of exactly solvable inverted quadratic Liénard type equations has been carried out. In doing so, it has been shown that the inverted Mathews-Lakshmanan oscillator under question belongs to the developed general class of inverted quadratic Liénard type equations. Other inverted and classical standard oscillator equations are also investigated. An advantage of the proposed theory results from the fact that the inverted Liénard type differential equations under question may be mapped into quadratures. In this perspective first integrals are obtainable, so that not only the solvability of equations is ensured, but Lagrangian and Hamiltonian descriptions may also be performed. Besides, these inverted nonlinear oscillator equations belong to the general class of inverted quadratic Liénard type equations found to be comparable to a general class of nonlinear oscillator equations represented by the relationship (2) for which a wide and rich variety of Lagrangian and Hamiltonian analysis is devoted. It is worth mentioning that the present study suggests a generalization of the non-standard Lagrangian given in [5] in the form

$$L(t,\dot{x},x) = b\dot{x}u(x) + cv(x)\int v(x)dx \tag{16}$$

where $u(x)$ and $v(x)$ are arbitrary functions of x, and b and c are arbitrary constants.

So, considering the above, the following conclusions may be given (section 5).

5. Conclusions

In this contribution the problem of generating inverted quadratic Liénard type oscillator equations is classically investigated instead of using a non-standard Lagrangian formulation. A mathematical theory consisting of generating nonlinear differential equations from classical formal bases has been performed. To be more precise, it is shown that the inverted Mathews-Lakshmanan oscillator equation belongs to a general class of exactly solvable inverted quadratic Liénard type differential equations designed from a first integral represented as a linear integro-differential equation by means of differentiation. As a result, the inverted quadratic Liénard type equations are mapped into quadratures, so that first integrals are obtainable. In this regard a subsequent Lagrangian and Hamiltonian description can be performed for classical and quantum applications.

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