Quaternion Dynamics, Part 2 – Identities, Octonions, and Pentuples

Gary D. Simpson

gsim100887@aol.com

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Summary

This text develops various identities for Hamilton's quaternions. The results are presented in order of difficulty. Results are organized as **Axioms**, **Vectors**, **Quaternions**, and **Matrices**. There are also sections for **Octonions** and **Pentuples**. **Axioms** are presented first and are largely without rigorous proof. Subsequent identities are constructed from prior identities. When complex conjugates are discussed, the author's thinking is biased towards the original quaternion having a positive vector portion and the conjugate having a negative vector portion. To genuinely understand what is presented, it is recommended that the reader should visualize the concepts in addition to manipulating them algebraically. The algebra is certainly true, but the visual understanding is more elegant and intuitive. This text will likely be updated occasionally.

0 - Axioms

This section includes a few basic concepts from other areas of mathematics and it includes the concepts that Hamilton added as a basis for quaternions. Scalars are denoted by lower-case letters in regular font. Vectors are denoted by lower-case letters in **bold** font. Quaternions are denoted by UPPER-CASE letters in **bold** font. The symbols **i**, **j**, and **k** denote unit vectors in the principle directions x, y, and z respectively.

A scalar is a real number with no direction. A scalar may have units of measurement such as length, mass, or time.

0.0:

$$a_0 \in R$$

It follows that addition of scalars is associative.

0.0.1:

$$(a_0 + b_0) + c_0 = a_0 + (b_0 + c_0) = a_0 + b_0 + c_0$$

It also follows that addition of scalars is commutative.

0.0.2:

$$a_0 + b_0 = b_0 + a_0$$

A vector in three dimensions is the sum of the vectors in the three principle directions.

0.1:

$$\mathbf{a} = a_i\mathbf{i} + a_j\mathbf{j} + a_k\mathbf{k}\,;\ a_i,\ a_j,\ a_k \in R$$

The author thinks that units of measure such as length, mass, and time can be associated with the coefficients of a vector but that the unit vectors **i**, **j**, and **k** themselves do not have units of measure. Instead, they represent direction only. This is best understood by thinking of an arbitrary vector as being equal to the length of the vector multiplied by a unit vector in the direction of the arbitrary vector. The length would then contain the units of measurement and the unit vector represents direction only. This is consistent with **0.0**.

Vector addition is "head-to-tail". It follows that addition of vectors is associative.

0.1.1:

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) = (a_i + b_i + c_i)\mathbf{i} + \left(a_j + b_j + c_j\right)\mathbf{j} + (a_k + b_k + c_k)\mathbf{k}$$

It also follows that addition of vectors is commutative.

0.1.2:

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} = (a_i + b_i)\mathbf{i} + \left(a_j + b_j\right)\mathbf{j} + (a_k + b_k)\mathbf{k}$$

The length of a vector is the square root of the sum of the squares.

0.1.3:

$$\|\mathbf{a}\| = \sqrt{a_i^2 + a_j^2 + a_k^2}$$

This is essentially the Theorem of Pythagoras.

A unit vector in the direction of any arbitrary non-zero vector can be produced by dividing the vector by its length.

0.1.3.1:

$$\mathbf{u}_a = \frac{1}{\|\mathbf{a}\|}\mathbf{a} = \frac{1}{\|\mathbf{a}\|}\left(a_i\mathbf{i} + a_j\mathbf{j} + a_k\mathbf{k}\right)$$

It follows that:

0.1.3.1.1:

$$\|\mathbf{a}\|\mathbf{u}_a = \left(a_i\mathbf{i} + a_j\mathbf{j} + a_k\mathbf{k}\right)$$

Multiplication by a unit vector twice or by each unit vector in order once will reverse the direction.

0.2:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

It follows that:

0.2.1:

$$\frac{1}{\mathbf{i}} = -\mathbf{i}\,;\ \frac{1}{\mathbf{j}} = -\mathbf{j}\,;\ \frac{1}{\mathbf{k}} = -\mathbf{k}$$

Multiplication of the principle unit vectors is anti-commutative. This is one of the most important features of Hamilton's work.

0.3:

$$\mathbf{ij} = -\mathbf{ji}\,;\mathbf{ik} = -\mathbf{ki}\,;\mathbf{jk} = -\mathbf{kj}$$

Since $\mathbf{k}^2$ = -1 = (**ij**)**k**, it follows that:

0.3.1:

$$\mathbf{k} = \mathbf{ij}$$

Since $\mathbf{i}^2$ = -1 = **i**(**jk**), it follows that:

0.3.2:

$$\mathbf{i} = \mathbf{jk}$$

Since $\mathbf{j}^2$ = -1 = **-**(**ik**)**j**, it follows that:

0.3.3:

$$\mathbf{j} = -\mathbf{ik} = \mathbf{ki}$$

A quaternion is defined as the ratio between two arbitrary vectors. It is the sum of a scalar and a "vector". The "vector" portion of a quaternion is typically not a vector in the normal meaning of the word.

0.4:

$$\mathbf{Q} = \frac{\mathbf{y}}{\mathbf{x}} = q_0 + q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k} = q_0 + \mathbf{q}\,;\, q_0,\, q_i,\, q_j,\, q_k \in R$$

As presented here, since **y** and **x** are both spatial vectors which have dimensional units of length, it follows that the coefficients of **Q** have no dimensional units. The **Q** coefficients are dimensionless because the units of vector length cancel each other in the division. This implies that quaternion **Q** is an operator rather than an object. For quaternion **Q** to be an object, **0.4** must be applicable to dissimilar vectors. For example, vector **y** might represent force and vector **x** might represent acceleration. Quaternion **Q** would then be an object and it would represent mass.

It follows from **0.0.1** and **0.1.1** that addition of quaternions is associative.

0.4.1:

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) = \mathbf{A} + \mathbf{B} + \mathbf{C}$$

It follows from **0.0.2** and **0.1.2** that addition of quaternions is commutative.

0.4.2:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

The magnitude of a quaternion is the square root of the sum of the squares.

0.4.3:

$$\|\mathbf{Q}\| = \sqrt{q_0^2 + q_i^2 + q_j^2 + q_k^2} = \sqrt{q_0^2 + \|\mathbf{q}\|^2}$$

A unit quaternion in the direction of any arbitrary non-zero quaternion can be produced by dividing the quaternion by its magnitude.

0.4.3.1:

$$\mathbf{U}_Q = \frac{1}{\|\mathbf{Q}\|}\mathbf{Q} = \frac{1}{\|\mathbf{Q}\|}\left(q_0 + q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k}\right) = \frac{1}{\|\mathbf{Q}\|}(q_0 + \mathbf{q})$$

The complex conjugate of a quaternion has the same scalar value but the opposite vector value.

0.4.4:

$$\mathbf{Q}^* = q_0 - \left(q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k}\right) = q_0 - \mathbf{q} = \mathbf{Q} - 2\mathbf{q}\,;\; q_0,\, q_i,\, q_j,\, q_k \in R$$

The author's thinking is biased towards the original quaternion having a positive vector portion and the conjugate having a negative vector portion. A quaternion and its complex conjugate can be combined as a sum or a difference to produce a scalar or a vector.

0.4.4.1:

$$\mathbf{Q} + \mathbf{Q}^* = (q_0 + \mathbf{q}) + (q_0 - \mathbf{q}) = 2q_0$$

0.4.4.2:

$$\mathbf{Q} - \mathbf{Q}^* = (q_0 + \mathbf{q}) - (q_0 - \mathbf{q}) = 2\mathbf{q}$$

Now let us multiply two arbitrary vectors together.

$$\mathbf{ab} = \left(a_i\mathbf{i} + a_j\mathbf{j} + a_k\mathbf{k}\right)\left(b_i\mathbf{i} + b_j\mathbf{j} + b_k\mathbf{k}\right)$$

1.1:

$$\mathbf{ab} = -\left(a_ib_i + a_jb_j + a_kb_k\right) + \left(a_jb_k - a_kb_j\right)\mathbf{i} + (a_kb_i - a_ib_k)\mathbf{j} + \left(a_ib_j - a_jb_i\right)\mathbf{k}$$

The dot product of two vectors is defined as:

1.1.1:

$$\mathbf{a} \cdot \mathbf{b} = a_ib_i + a_jb_j + a_kb_k$$

The dot product is a scalar. It follows that:

1.1.1.1:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

The dot product of a vector with itself is the square of the length of the vector.

1.1.1.2:

$$\mathbf{a} \cdot \mathbf{a} = a_i^2 + a_j^2 + a_k^2 = \|\mathbf{a}\|^2$$

The cross product of two vectors is defined as:

1.1.2:

$$\mathbf{a} \times \mathbf{b} = \left(a_jb_k - a_kb_j\right)\mathbf{i} + (a_kb_i - a_ib_k)\mathbf{j} + \left(a_ib_j - a_jb_i\right)\mathbf{k}$$

The cross product is a vector. It follows that:

1.1.2.1:

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

It also follows from **1.1.2** that the cross product of a vector with itself is the zero vector.

$$\mathbf{a} \times \mathbf{a} = \left(a_ja_k - a_ka_j\right)\mathbf{i} + (a_ka_i - a_ia_k)\mathbf{j} + \left(a_ia_j - a_ja_i\right)\mathbf{k}$$

1.1.2.2:

$$\mathbf{a} \times \mathbf{a} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$

Substitution of **1.1.1** and **1.1.2** into **1.1** produces:

1.1.3:

$$\mathbf{ab} \; = \; -(\mathbf{a}\cdot\mathbf{b}) \; + \; (\mathbf{a}\times\mathbf{b})$$

This relation is easily applied to the unit vectors **i**,, **j**, and **k**. It is completely consistent with Hamilton's definitions from **Axioms**.

It follows that a vector multiplied by itself produces the opposite of the square of the length.

$$\mathbf{a}^2 \; = \; \mathbf{aa} \; = \; -(\mathbf{a}\cdot\mathbf{a}) \; + \; (\mathbf{a}\times\mathbf{a}) \; = \; -\|\mathbf{a}\|^2 \; + \; \mathbf{0}$$

1.1.3.1:

$$\mathbf{a}^2 \; = \; -\|\mathbf{a}\|^2$$

and also

1.1.3.2:

$$\mathbf{aa}^* = \; -\mathbf{a}^2 = \; \|\mathbf{a}\|^2$$

Reversing the order of the multiplication produces the conjugate.

$$\mathbf{ba} = \; \left(b_i\mathbf{i} \; + \; b_j\mathbf{j} \; + \; b_k\mathbf{k}\,\right)\left(a_i\mathbf{i} \; + \; a_j\mathbf{j} \; + \; a_k\mathbf{k}\,\right) = \; -(\mathbf{b}\cdot\mathbf{a}) \; + \; (\mathbf{b}\times\mathbf{a})$$

1.1.4:

$$\mathbf{ba} \; = \; -(\mathbf{a}\cdot\mathbf{b}) \; - \; (\mathbf{a}\times\mathbf{b}) \; = \; (\mathbf{ab})^*$$

It follows that **ab** and **ba** can be combined as a sum and difference as follows:

1.1.5:

$$\mathbf{ab} \; + \; \mathbf{ba} \; = \; -2(\mathbf{a}\cdot\mathbf{b})$$

1.1.6:

$$\mathbf{ab} \; - \; \mathbf{ba} \; = \; 2(\mathbf{a}\times\mathbf{b})$$

Next let us consider the question of association as applied to the multiplication of three vectors. Is the following expression true?

1.2:

$$(\mathbf{ab})\mathbf{c} \; = \; \mathbf{a}(\mathbf{bc})\,; \; or \; (\mathbf{ab})\mathbf{c} \; - \; \mathbf{a}(\mathbf{bc}) = 0 \; + \; \mathbf{0}\,; ?????$$

The details of this multiplication are very tedious. Therefore, the author will break it into portions to describe the left-hand side and right-hand side respectively.

$$(\mathbf{ab})\mathbf{c} = [-(\mathbf{a}\cdot\mathbf{b}) \ + \ (\mathbf{a}\times\mathbf{b})]\mathbf{c}$$

$$(\mathbf{ab})\mathbf{c} = [-(\mathbf{a}\cdot\mathbf{b})\mathbf{c} \ + \ (\mathbf{a}\times\mathbf{b})\mathbf{c}]$$

The part of this that is difficult is (**a**x**b**)**c**. Therefore, the author will expand upon that.

$$(\mathbf{a}\times\mathbf{b})\mathbf{c} = [-(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} \ + \ (\mathbf{a}\times\mathbf{b})\times\mathbf{c}]$$

Therefore:

1.2.1:

$$(\mathbf{ab})\mathbf{c} = [-(\mathbf{a}\cdot\mathbf{b})\mathbf{c} \ - \ (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} \ + \ (\mathbf{a}\times\mathbf{b})\times\mathbf{c}]$$

A similar treatment for the right-hand side of **1.2** gives:

$$\mathbf{a}(\mathbf{bc}) \ = \ \mathbf{a}[-(\mathbf{b}\cdot\mathbf{c}) \ + \ (\mathbf{b}\times\mathbf{c})]$$

$$\mathbf{a}(\mathbf{bc}) \ = \ [-\mathbf{a}(\mathbf{b}\cdot\mathbf{c}) \ + \ \mathbf{a}(\mathbf{b}\times\mathbf{c})]$$

1.2.2:

$$\mathbf{a}(\mathbf{bc}) \ = \ [-\mathbf{a}(\mathbf{b}\cdot\mathbf{c}) \ - \ \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \ + \ \mathbf{a}\times(\mathbf{b}\times\mathbf{c})\,]$$

It is not obvious to the author that **1.2.1** and **1.2.2** are equivalent. There is still some question concerning the cross product terms.

Let **u** = **a**x**b**.

$$\mathbf{u} \ = \ \mathbf{a}\times\mathbf{b} \ = \ \left(a_j b_k \ - \ a_k b_j\right)\mathbf{i} \ + \ (a_k b_i \ - \ a_i b_k)\mathbf{j} \ + \ \left(a_i b_j \ - \ a_j b_i\right)\mathbf{k}$$

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{u}\times\mathbf{c} = \ \left(u_j c_k \ - \ u_k c_j\right)\mathbf{i} \ + \ (u_k c_i \ - \ u_i c_k)\mathbf{j} \ + \ \left(u_i c_j \ - \ u_j c_i\right)\mathbf{k}$$

The next step substitutes the coefficients of **u** from two lines above into the equation immediately above.

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times\mathbf{c} \ = \ &\Big((a_k b_i \ - \ a_i b_k)c_k \ - \ \left(a_i b_j \ - \ a_j b_i\right)c_j\Big)\,\mathbf{i} \\ &+ \ \Big(\left(a_i b_j \ - \ a_j b_i\right)c_i \ - \ \left(a_j b_k \ - \ a_k b_j\right)c_k\Big)\,\mathbf{j} \\ &+ \ \Big(\left(a_j b_k \ - \ a_k b_j\right)c_j \ - \ (a_k b_i \ - \ a_i b_k)c_i\Big)\,\mathbf{k}\end{aligned}$$

Let **u** = **b**x**c**.

$$\mathbf{u} \ = \ \mathbf{b}\times\mathbf{c} \ = \ \left(b_j c_k \ - \ b_k c_j\right)\mathbf{i} \ + \ (b_k c_i \ - \ b_i c_k)\mathbf{j} \ + \ \left(b_i c_j \ - \ b_j c_i\right)\mathbf{k}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\times\mathbf{u} = \ \left(a_j u_k \ - \ a_k u_j\right)\mathbf{i} \ + \ (a_k u_i \ - \ a_i u_k)\mathbf{j} \ + \ \left(a_i u_j \ - \ a_j u_i\right)\mathbf{k}$$

The next step substitutes the coefficients of **u** from two lines above into the equation immediately above.

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \;=\; \Big(a_j\big(b_i c_j \;-\; b_j c_i\big) \;-\; a_k(b_k c_i \;-\; b_i c_k)\Big)\,\mathbf{i} \;+\; \Big(a_k\big(b_j c_k \;-\; b_k c_j\big) \;-\; a_i\big(b_i c_j \;-\; b_j c_i\big)\Big)\,\mathbf{j}$$
$$+\; \Big(a_i(b_k c_i \;-\; b_i c_k) \;-\; a_j\big(b_j c_k \;-\; b_k c_j\big)\Big)\,\mathbf{k}$$

These two expressions still do not appear to be equivalent. Therefore, a term by term comparison is required.

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \;=\; &\Big((a_k b_i c_k \;-\; a_i b_k c_k) \;-\; \big(a_i b_j c_j \;-\; a_j b_i c_j\big)\Big)\,\mathbf{i} \\ &+\; \Big(\big(a_i b_j c_i \;-\; a_j b_i c_i\big) \;-\; \big(a_j b_k c_k \;-\; a_k b_j c_k\big)\Big)\,\mathbf{j} \\ &+\; \Big(\big(a_j b_k c_j \;-\; a_k b_j c_j\big) - \;(a_k b_i c_i \;-\; a_i b_k c_i)\Big)\,\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \;=\; &\Big(\big(a_j b_i c_j \;-\; a_j b_j c_i\big) \;-\; (a_k b_k c_i \;-\; a_k b_i c_k)\Big)\,\mathbf{i} \\ &+\; \Big(\big(a_k b_j c_k \;-\; a_k b_k c_j\big) \;-\; \big(a_i b_i c_j \;-\; a_i b_j c_i\big)\Big)\,\mathbf{j} \\ &+\; \Big((a_i b_k c_i \;-\; a_i b_i c_k) \;-\; \big(a_j b_j c_k \;-\; a_j b_k c_j\big)\Big)\,\mathbf{k}\end{aligned}$$

The author can now state that in general, these two cross product expressions are not equal.

1.2.3:

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \;\neq\; \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

There are some shared terms between these expressions. It is still possible that the difference between these two forms of the cross product will offset the other differences in **1.2.1** and **1.2.2** to cause **1.2** to be true. Essentially, the question is "Does **1.2.1** minus **1.2.2** equal zero?".

$$\begin{aligned}&(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \;-\; \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \\ &\qquad =\; \big(-a_i b_k c_k - \; a_i b_j c_j + a_j b_j c_i + \; a_k b_k c_i\big)\mathbf{i} \\ &\qquad +\; \big(-a_j b_i c_i - a_j b_k c_k + \; a_k b_k c_j + a_i b_i c_j\;\big)\mathbf{j} \\ &\qquad +\; \big(-a_k b_j c_j - \; a_k b_i c_i + \; a_i b_i c_k + \; a_j b_j c_k\big)\mathbf{k}\end{aligned}$$

1.2.4:

$$\begin{aligned}&(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \;-\; \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \\ &\qquad =\; \big(-a_i\big(b_k c_k + b_j c_j\;\big) \;+\; \big(a_j b_j + \; a_k b_k\big)c_i\big)\mathbf{i} \\ &\qquad +\; \big(-a_j(b_i c_i + \; b_k c_k) \;+\; (a_k b_k + \; a_i b_i)c_j\big)\mathbf{j} \\ &\qquad +\; \big(-a_k\big(b_j c_j + \; b_i c_i\big) + \; \big(a_i b_i + \; a_j b_j\big)c_k\big)\mathbf{k}\end{aligned}$$

This looks promising. Notice that the terms inside the inner parentheses are similar to vector dot products. Next, let us compare the scalar terms.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) - (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = ?????$$

Let **u** = **b**x**c**.

$$\mathbf{u} = (b_j c_k - b_k c_j)\mathbf{i} + (b_k c_i - b_i c_k)\mathbf{j} + (b_i c_j - b_j c_i)\mathbf{k}$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \mathbf{u} = a_i(b_j c_k - b_k c_j) + a_j(b_k c_i - b_i c_k) + a_k(b_i c_j - b_j c_i)$$

Let **u** = **a**x**b**.

$$\mathbf{u} = (a_j b_k - a_k b_j)\mathbf{i} + (a_k b_i - a_i b_k)\mathbf{j} + (a_i b_j - a_j b_i)\mathbf{k}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{u} \cdot \mathbf{c} = (a_j b_k - a_k b_j)c_i + (a_k b_i - a_i b_k)c_j + (a_i b_j - a_j b_i)c_k$$

Let us rearrange the right-hand side of this equation.

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = a_i(b_j c_k - b_k c_j) + a_j(b_k c_i - b_i c_k) + a_k(b_i c_j - b_j c_i)$$

Therefore, the scalar terms are equal.

1.2.5:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

The next task is to compare the remaining vector terms.

$$\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} = ??????$$

$$\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) = a_i(b_i c_i + b_j c_j + b_k c_k)\mathbf{i} + a_j(b_i c_i + b_j c_j + b_k c_k)\mathbf{j} + a_k(b_i c_i + b_j c_j + b_k c_k)\mathbf{k}$$

$$(\mathbf{a} \cdot \mathbf{b})\mathbf{c} = (a_i b_i + a_j b_j + a_k b_k)c_i\mathbf{i} + (a_i b_i + a_j b_j + a_k b_k)c_j\mathbf{j} + (a_i b_i + a_j b_j + a_k b_k)c_k\mathbf{k}$$

1.2.6:

$$\begin{aligned}\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \\ &= \big(a_i(b_j c_j + b_k c_k) - (a_j b_j + a_k b_k)c_i\big)\mathbf{i} \\ &+ \big(a_j(b_i c_i + b_k c_k) - (a_i b_i + a_k b_k)c_j\big)\mathbf{j} \\ &+ \big(a_k(b_i c_i + b_j c_j) - (a_i b_i + a_j b_j)c_k\big)\mathbf{k}\end{aligned}$$

Now, compare **1.2.4** with **1.2.6**. It should be noted that:

1.2.7:

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} - \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -[\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}]$$

or

1.2.7.1:

$$-(\mathbf{a} \cdot \mathbf{b})\mathbf{c} \;+\; (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \;\;=\; -\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) \;+\;\; \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

Therefore, **1.2.5** combined with **1.2.7** result in **1.2** being true. Vector multiplication is associative.

1.2.8:

$$(\mathbf{ab})\mathbf{c} \;=\; \mathbf{a}(\mathbf{bc}) \;=\; \mathbf{abc}$$

Next let us consider some identities involving vector dot products and vector cross products. Since scalar multiplication is associative and commutative, the following identities are obviously true:

1.3.1:

$$a_0(\mathbf{b} \cdot \mathbf{c}) \;=\; (\mathbf{b} \cdot \mathbf{c})a_0 \;=\; (a_0\mathbf{b} \cdot \mathbf{c}) \;=\; (\mathbf{b} \cdot a_0\mathbf{c})$$

1.3.2:

$$a_0(\mathbf{b} \times \mathbf{c}) \;=\; (\mathbf{b} \times \mathbf{c})a_0 \;=\; (a_0\mathbf{b} \times \mathbf{c}) \;=\; (\mathbf{b} \times a_0\mathbf{c})$$

2 - Quaternions

Now let us multiply two arbitrary quaternions together.

$$\mathbf{AB} = (a_0 + \mathbf{a})(b_0 + \mathbf{b}) = a_0 b_0 + b_0\mathbf{a} + a_0\mathbf{b} + \mathbf{ab}$$

$$\mathbf{AB} = (a_0 + \mathbf{a})(b_0 + \mathbf{b}) = a_0 b_0 + b_0\mathbf{a} + a_0\mathbf{b} + [-(\mathbf{a}\cdot\mathbf{b}) + (\mathbf{a}\times\mathbf{b})]$$

$$\mathbf{AB} = (a_0 + \mathbf{a})(b_0 + \mathbf{b}) = (a_0 b_0 - \mathbf{a}\cdot\mathbf{b}) + (b_0\mathbf{a} + a_0\mathbf{b} + \mathbf{a}\times\mathbf{b})$$

When two vectors were multiplied together in **1.1.3**, the result was the negative of the vector dot product plus the vector cross product. The author desires to maintain symmetry between vector multiplication and quaternion multiplication. Therefore, the author will revise the above relation for **AB** to introduce the quaternion dot product and the quaternion cross product. To do so, subtract $2a_0b_0$ from the scalar group and add $2a_0b_0$ to the vector group.

$$\mathbf{AB} = (a_0 + \mathbf{a})(b_0 + \mathbf{b}) = (-2a_0 b_0 + a_0 b_0 - \mathbf{a}\cdot\mathbf{b}) + (2a_0 b_0 + b_0\mathbf{a} + a_0\mathbf{b} + \mathbf{a}\times\mathbf{b})$$

$$\mathbf{AB} = (a_0 + \mathbf{a})(b_0 + \mathbf{b}) = (-a_0 b_0 - \mathbf{a}\cdot\mathbf{b}) + [b_0(a_0 + \mathbf{a}) + a_0(b_0 + \mathbf{b}) + \mathbf{a}\times\mathbf{b}]$$

2.1:

$$\mathbf{AB} = (a_0 + \mathbf{a})(b_0 + \mathbf{b}) = -(a_0 b_0 + \mathbf{a}\cdot\mathbf{b}) + (b_0\mathbf{A} + a_0\mathbf{B} + \mathbf{a}\times\mathbf{b}) = -\mathbf{A}\cdot\mathbf{B} + \mathbf{A}\times\mathbf{B}$$

It is worth mentioning that if vector **a** and vector **b** are not collinear, then **2.1** creates a space since **a** x **b** is orthogonal to the plane created by (b_0**a** + a_0**b**).

The dot product between two quaternions can now be defined as:

2.1.1:

$$\mathbf{A}\cdot\mathbf{B} = a_0 b_0 + a_i b_i + a_j b_j + a_k b_k = a_0 b_0 + \mathbf{a}\cdot\mathbf{b}$$

The dot product is a scalar. Therefore, it follows that:

2.1.1.1:

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$$

The dot product of a quaternion with itself is the square of the magnitude of the quaternion.

2.1.1.2:

$$\mathbf{A}\cdot\mathbf{A} = a_0^2 + a_i^2 + a_j^2 + a_k^2 = \|\mathbf{A}\|^2 = a_0^2 + \|\mathbf{a}\|^2$$

The cross product between two quaternions can now be defined as:

2.1.2:

$$\mathbf{A} \times \mathbf{B} = b_0\mathbf{A} + a_0\mathbf{B} + \mathbf{a} \times \mathbf{b}$$

It follows that:

$$\mathbf{B} \times \mathbf{A} = a_0\mathbf{B} + b_0\mathbf{A} + \mathbf{b} \times \mathbf{a}$$

$$\mathbf{B} \times \mathbf{A} = b_0\mathbf{A} + a_0\mathbf{B} - \mathbf{a} \times \mathbf{b}$$

2.1.2.1:

$$\mathbf{B} \times \mathbf{A} = \mathbf{A} \times \mathbf{B} - 2(\mathbf{a} \times \mathbf{b})$$

It also follows from **2.1.2** that the cross product of a quaternion with itself is:

2.1.2.2:

$$\mathbf{A} \times \mathbf{A} = a_0\mathbf{A} + a_0\mathbf{A} + \mathbf{a} \times \mathbf{a} = 2a_0\mathbf{A}$$

Substitution of **2.1.1** and **2.1.2** into **2.1** produces:

2.1.3:

$$\mathbf{AB} = (a_0 + \mathbf{a})(b_0 + \mathbf{b}) = -(\mathbf{A} \cdot \mathbf{B}) + (\mathbf{A} \times \mathbf{B})$$

It follows that a quaternion multiplied by itself produces:

$$\mathbf{A}^2 = \mathbf{AA} = -(\mathbf{A} \cdot \mathbf{A}) + (\mathbf{A} \times \mathbf{A})$$

2.1.3.1:

$$\mathbf{A}^2 = -\|\mathbf{A}\|^2 + 2a_0\mathbf{A}$$

A quaternion multiplied by its conjugate produces a scalar equal to the square of the magnitude of the quaternion. This is also equal to the dot product of **A** with itself. See **2.1.1.2** for **A** dot **A**.

$$\mathbf{AA}^* = (a_0 + \mathbf{a})(a_0 - \mathbf{a}) = a_0^2 - \mathbf{a}^2 = a_0^2 - \big(-(\mathbf{a} \cdot \mathbf{a}) + (\mathbf{a} \times \mathbf{a})\big)$$

2.1.3.2:

$$\mathbf{AA}^* = a_0^2 + \mathbf{a} \cdot \mathbf{a} = a_0^2 + \left(a_i^2 + a_j^2 + a_k^2\right) = \|\mathbf{A}\|^2 = \|\mathbf{A}^*\|^2 = \mathbf{A}^*\mathbf{A} = \mathbf{A} \cdot \mathbf{A} = \mathbf{A}^* \cdot \mathbf{A}^*$$

Reversing the order of multiplication produces the following:

$$\mathbf{BA} = (b_0 + \mathbf{b})(a_0 + \mathbf{a}) = -(\mathbf{B} \cdot \mathbf{A}) + (\mathbf{B} \times \mathbf{A})$$

$$\mathbf{BA} = -(\mathbf{A} \cdot \mathbf{B}) + [(\mathbf{A} \times \mathbf{B}) - 2(\mathbf{a} \times \mathbf{b})]$$

2.1.4:

$$\mathbf{BA} = \mathbf{AB} - 2(\mathbf{a} \times \mathbf{b})$$

Reversing the order of multiplication does NOT produce the negative of the prior multiplication (i.e., **BA** ≠ -**AB**) NOR does it produce the conjugate (i.e., **BA** ≠ (**AB**)*).

The products **AB** and **BA** can also be combined as a sum and a difference.

2.1.5:

$$\mathbf{AB} + \mathbf{BA} = 2(\mathbf{AB} - \mathbf{a} \times \mathbf{b}) = 2(-\mathbf{A} \cdot \mathbf{B} + b_0\mathbf{A} + a_0\mathbf{B})$$

2.1.6:

$$\mathbf{AB} - \mathbf{BA} = 2(\mathbf{a} \times \mathbf{b})$$

A product such as **AB** can be multiplied by **A*** or **B*** to produce a scalar multiplied by either **B** or **A**.

2.1.7.1:

$$(\mathbf{A}^*)\mathbf{AB} = (\mathbf{A}^*\mathbf{A})\mathbf{B} = \|\mathbf{A}\|^2\mathbf{B}$$

2.1.7.2:

$$\mathbf{AB}(\mathbf{B}^*) = \mathbf{A}(\mathbf{BB}^*) = \mathbf{A}\|\mathbf{B}\|^2$$

The conjugate of **AB** is determined as follows:

$$\mathbf{AB} = (a_0 + \mathbf{a})(b_0 + \mathbf{b}) = -(\mathbf{A} \cdot \mathbf{B}) + (\mathbf{A} \times \mathbf{B})$$

$$\mathbf{AB} = (2a_0b_0 - \mathbf{A} \cdot \mathbf{B}) + (b_0\mathbf{a} + a_0\mathbf{b} + \mathbf{a} \times \mathbf{b})$$

$$(\mathbf{AB})^* = (2a_0b_0 - \mathbf{A} \cdot \mathbf{B}) - (b_0\mathbf{a} + a_0\mathbf{b} + \mathbf{a} \times \mathbf{b})$$

$$(\mathbf{AB})^* = (4a_0b_0 - \mathbf{A} \cdot \mathbf{B}) - (2a_0b_0 + b_0\mathbf{a} + a_0\mathbf{b} + \mathbf{a} \times \mathbf{b})$$

$$(\mathbf{AB})^* = (4a_0b_0 - \mathbf{A} \cdot \mathbf{B}) - (b_0\mathbf{A} + a_0\mathbf{B} + \mathbf{a} \times \mathbf{b})$$

2.1.8:

$$(\mathbf{AB})^* = (4a_0b_0 - \mathbf{A} \cdot \mathbf{B}) - (\mathbf{A} \times \mathbf{B})$$

The terms **AB** and (**AB**)* can now be combined as a sum and a difference.

2.1.8.1:

$$\mathbf{AB} + (\mathbf{AB})^* = 2(2a_0b_0 - \mathbf{A} \cdot \mathbf{B})$$

2.1.8.2:

$$\mathbf{AB} - (\mathbf{AB})^* = -2(2a_0b_0 - \mathbf{A} \times \mathbf{B})$$

Next let us consider the question of association as applied to the multiplication of three quaternions. Is the following expression true?

2.2:

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})\,;\ (\mathbf{AB})\mathbf{C} - \mathbf{A}(\mathbf{BC}) = 0 + \mathbf{0}\,; ?????$$

Fortunately, this question is essentially answered by the equivalent question for vectors in **1.2.** The author will break the problem into two parts representing the left-hand side and the right-hand side respectively.

The left-hand side is:

$$(\mathbf{AB})\mathbf{C} = [(a_0 + \mathbf{a})(b_0 + \mathbf{b})](c_0 + \mathbf{c}) = [a_0b_0 + b_0\mathbf{a} + a_0\mathbf{b} + \mathbf{ab}](c_0 + \mathbf{c})$$

$$(\mathbf{AB})\mathbf{C} = [a_0b_0c_0 + b_0c_0\mathbf{a} + a_0c_0\mathbf{b} + c_0\mathbf{ab}] + [a_0b_0\mathbf{c} + b_0\mathbf{ac} + a_0\mathbf{bc} + (\mathbf{ab})\mathbf{c}]$$

2.2.1:

$$(\mathbf{AB})\mathbf{C} = a_0b_0c_0 + (b_0c_0\mathbf{a} + a_0c_0\mathbf{b} + a_0b_0\mathbf{c}) + (c_0\mathbf{ab} + b_0\mathbf{ac} + a_0\mathbf{bc}) + (\mathbf{ab})\mathbf{c}$$

The right-hand side is:

$$\mathbf{A}(\mathbf{BC}) = (a_0 + \mathbf{a})[(b_0 + \mathbf{b})(c_0 + \mathbf{c})] = (a_0 + \mathbf{a})[b_0c_0 + c_0\mathbf{b} + b_0\mathbf{c} + \mathbf{bc}]$$

$$\mathbf{A}(\mathbf{BC}) = [a_0b_0c_0 + a_0\,c_0\mathbf{b} + a_0b_0\mathbf{c} + a_0\mathbf{bc}] + [b_0c_0\mathbf{a} + c_0\mathbf{ab} + b_0\mathbf{ac} + \mathbf{a}(\mathbf{bc})]$$

2.2.2:

$$\mathbf{A}(\mathbf{BC}) = a_0b_0c_0 + (b_0c_0\mathbf{a} + a_0\,c_0\mathbf{b} + a_0b_0\mathbf{c}\,) + (c_0\mathbf{ab} + b_0\mathbf{ac} + a_0\mathbf{bc}) + \mathbf{a}(\mathbf{bc})$$

All of the terms in **2.2.1** and **2.2.2** are equal with the possible exception of the final term of each. Since **1.2** showed that (**ab**)**c** = **a**(**bc**), it follows that **2.2.1** and **2.2.2** are equal. Therefore, the multiplication of quaternions is associative.

Hamilton's original use for quaternions was as the ratio between two non-collinear vectors.

2.3:

$$\mathbf{Q} = \frac{\mathbf{y}}{\mathbf{x}}$$

This can be written as:

2.3.1:

$$\mathbf{Qx} = \mathbf{y}\,;\mathbf{xQ}^* = \mathbf{y}$$

It was shown above that the square of a vector is equal to the opposite of the square of the length of the vector (see **1.1.3.2**). This allows **2.3.1** to be solved fairly easily.

$$\mathbf{Qx}(\mathbf{x}^*) = \mathbf{y}(\mathbf{x}^*)\,;\ (\mathbf{x}^*)\mathbf{xQ}^* = (\mathbf{x}^*)\mathbf{y}$$

$$\mathbf{Q}\|\mathbf{x}\|^2 = \mathbf{yx}^*\,;\ \|\mathbf{x}\|^2\mathbf{Q}^* = (\mathbf{x}^*)\mathbf{y}$$

$$\mathbf{Q} = \frac{1}{\|\mathbf{x}\|^2}(\mathbf{yx}^*)\,;\mathbf{Q}^* = \frac{1}{\|\mathbf{x}\|^2}(\mathbf{x}^*\mathbf{y})$$

2.3.1.1:

$$\mathbf{Q} = \frac{1}{\|\mathbf{x}\|^2}(\mathbf{x}\cdot\mathbf{y} + \mathbf{x}\times\mathbf{y})\,;\mathbf{Q}^* = \frac{1}{\|\mathbf{x}\|^2}(\mathbf{x}\cdot\mathbf{y} - \mathbf{x}\times\mathbf{y})$$

These are conjugates.

If the two vectors are collinear (i.e., **x** x **y** = 0) then **2.3.1.1** simplifies to the following scalar expression:

2.3.1.2:

$$\mathbf{Q} = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|^2}$$

Now let us consider three very useful mappings. Suppose that it is desired to map one of the principle unit vectors **i**, **j**, or **k** onto an arbitrary vector **y**. This can be done fairly easily simply by using **2.3.1.1** directly. Refer to **1.1.2** for the cross products.

For the **i** vector this is:

$$\mathbf{Q}_i = \frac{\mathbf{y}}{\mathbf{i}} = \frac{1}{\|\mathbf{i}\|}(\mathbf{i}\cdot\mathbf{y} + \mathbf{i}\times\mathbf{y})\,;\ \mathbf{Q}_i^* = \frac{\mathbf{y}}{\mathbf{i}} = \frac{1}{\|\mathbf{i}\|}(\mathbf{i}\cdot\mathbf{y} - \mathbf{i}\times\mathbf{y})$$

2.3.2.1:

$$\mathbf{Q}_i = y_i + \left(-y_k\mathbf{j} + y_j\mathbf{k}\right)\,;\ \mathbf{Q}_i^* = y_i - \left(-y_k\mathbf{j} + y_j\mathbf{k}\right)$$

For the **j** vector this is:

$$\mathbf{Q}_j = \frac{\mathbf{y}}{\mathbf{j}} = \frac{1}{\|\mathbf{j}\|}(\mathbf{j}\cdot\mathbf{y} + \mathbf{j}\times\mathbf{y})\,;\ \mathbf{Q}_j^* = \frac{\mathbf{y}}{\mathbf{j}} = \frac{1}{\|\mathbf{j}\|}(\mathbf{j}\cdot\mathbf{y} - \mathbf{j}\times\mathbf{y})$$

2.3.2.2:

$$\mathbf{Q}_j = y_j + (y_k\mathbf{i} - y_i\mathbf{k})\ ;\ \mathbf{Q}_j^* = y_j - (y_k\mathbf{i} - y_i\mathbf{k})$$

For the **k** vector this is:

$$\mathbf{Q}_k = \frac{\mathbf{y}}{\mathbf{k}} = \frac{1}{||\mathbf{k}||}(\mathbf{k}\cdot\mathbf{y} + \mathbf{k}\times\mathbf{y})\ ;\ \mathbf{Q}_i^* = \frac{\mathbf{y}}{\mathbf{k}} = \frac{1}{||\mathbf{k}||}(\mathbf{k}\cdot\mathbf{y} - \mathbf{k}\times\mathbf{y})$$

2.3.2.3:

$$\mathbf{Q}_k = y_k + \left(-y_j\mathbf{i} + y_i\mathbf{j}\right)\ ;\ \mathbf{Q}_k^* = y_k - \left(-y_j\mathbf{i} + y_i\mathbf{j}\right)$$

It is of course possible to reverse these mappings to go from arbitrary vector **y** to one of the unit vectors by using the inverse quaternion.

Now let us solve for the general quaternion **Q** that is the ratio between two arbitrary quaternions.

2.4:

$$\mathbf{Q} = \frac{\mathbf{Y}}{\mathbf{X}} = q_0 + q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k}$$

Both of the following two relationships will satisfy this equation:

2.4.1:

$$\mathbf{QX} = \mathbf{Y}\ ;\mathbf{XQ} = \mathbf{Y}$$

This can be solved by multiplying by the conjugate and then dividing by the square of the length.

$$(\mathbf{QX})\mathbf{X}^* = \mathbf{Y}(\mathbf{X}^*)\ ;\mathbf{X}^*(\mathbf{XQ}) = (\mathbf{X}^*)\mathbf{Y}$$

$$\mathbf{Q}||\mathbf{X}||^2 = \mathbf{Y}(\mathbf{X}^*);\ ||\mathbf{X}||^2\mathbf{Q} = (\mathbf{X}^*)\mathbf{Y}$$

$$\mathbf{Q} = \frac{1}{||\mathbf{X}||^2}\mathbf{Y}(\mathbf{X}^*)\ ;\mathbf{Q} = \frac{1}{||\mathbf{X}||^2}(\mathbf{X}^*)\mathbf{Y}$$

$$\mathbf{Q} = \frac{1}{||\mathbf{X}||^2}[-(\mathbf{Y}\cdot\mathbf{X}^*) + (x_0\mathbf{Y} + y_0\mathbf{X}^* + \mathbf{y}\times\mathbf{x}^*)]\ ;\mathbf{Q} = \frac{1}{||\mathbf{X}||^2}[-(\mathbf{X}^*\cdot\mathbf{Y}) + (y_0\mathbf{X}^* + x_0\mathbf{Y} + \mathbf{x}^*\times\mathbf{y})]$$

2.4.1.1:

$$\mathbf{Q} = \frac{1}{||\mathbf{X}||^2}[-(\mathbf{X}^*\cdot\mathbf{Y}) + (y_0\mathbf{X}^* + x_0\mathbf{Y} + \mathbf{x}\times\mathbf{y})]\ ;\mathbf{Q} = \frac{1}{||\mathbf{X}||^2}[-(\mathbf{X}^*\cdot\mathbf{Y}) + (y_0\mathbf{X}^* + x_0\mathbf{Y} - \mathbf{x}\times\mathbf{y})]$$

In general, these two quaternions are not conjugates. The scalar terms are equal. The cross product terms are opposites. But the vector terms associated with the ($-y_0\mathbf{x} + x_0\mathbf{y}$) term are the same (not opposites) for both quaternions. Therefore, these two quaternions are conjugates only if the term ($-y_0\mathbf{x}$ + $x_0\mathbf{y}$) = 0. This requires that either x_0 and y_0 are both equal to zero or that **x** and **y** are collinear (i.e., $y_0\mathbf{x}$ = $x_0\mathbf{y}$). If **x** and **y** are collinear, then their cross-product is zero and **2.4.1.1** simplifies to the following scalar value:

$$\mathbf{Q} = \frac{1}{\|\mathbf{X}\|^2}[-(\mathbf{X}^* \cdot \mathbf{Y}) + 2x_0y_0]$$

$$\mathbf{Q} = \frac{1}{\|\mathbf{X}\|^2}[-(x_0y_0 - \mathbf{x} \cdot \mathbf{y}) + 2x_0y_0] = \frac{1}{\|\mathbf{X}\|^2}[x_0y_0 + \mathbf{x} \cdot \mathbf{y}]$$

2.4.1.2:

$$\mathbf{Q} = \frac{\mathbf{X} \cdot \mathbf{Y}}{\|\mathbf{X}\|^2}$$

Quaternions exhibit an interesting behavior when repeatedly multiplied by one of the principle vectors **i**, **j**, or **k**. The quaternion will toggle between one of four forms with the coefficients forming pairs that swap positions with each other. Consider the following examples:

$$\mathbf{iQ} = \mathbf{i}(q_0 + q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k})$$

2.5.1.1:

$$\mathbf{iQ} = -q_i + q_0\mathbf{i} - q_k\mathbf{j} + q_j\mathbf{k}$$

Multiply by **i** again.

$$\mathbf{i}^2\mathbf{Q} = \mathbf{i}(-q_i + q_0\mathbf{i} - q_k\mathbf{j} + q_j\mathbf{k})$$

2.5.1.2:

$$\mathbf{i}^2\mathbf{Q} = -q_0 - q_i\mathbf{i} - q_j\mathbf{j} - q_k\mathbf{k}$$

Multiply by **i** again.

$$\mathbf{i}^3\mathbf{Q} = -\mathbf{i}(q_0 + q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k})$$

2.5.1.3:

$$\mathbf{i}^3\mathbf{Q} = q_i - q_0\mathbf{i} + q_k\mathbf{j} - q_j\mathbf{k}$$

Multiplication by **i** a fourth time returns the original quaternion **Q** (i.e., $\mathbf{i}^4 = 1$). In this example, the scalar coefficient and the **i** coefficient have formed a pair. Also, the **j** coefficient and **k** coefficient have formed a pair. The members of each pair swap positions within the quaternion each time the quaternion is multiplied by **i**.

Similar identities can be developed for successive multiplication by **j** and by **k**.

2.5.2.1:

$$\mathbf{jQ} = -q_j + q_k\mathbf{i} + q_0\mathbf{j} - q_i\mathbf{k}$$

2.5.2.2:

$$\mathbf{j}^2\mathbf{Q} = -q_0 - q_i\mathbf{i} - q_j\mathbf{j} - q_k\mathbf{k}$$

2.5.2.3:

$$\mathbf{j}^3\mathbf{Q} = q_j - q_k\mathbf{i} - q_0\mathbf{j} + q_i\mathbf{k}$$

2.5.3.1:

$$\mathbf{kQ} = -q_k - q_j\mathbf{i} + q_i\mathbf{j} + q_0\mathbf{k}$$

2.5.3.2:

$$\mathbf{k}^2\mathbf{Q} = -q_0 - q_i\mathbf{i} - q_j\mathbf{j} - q_k\mathbf{k}$$

2.5.3.3:

$$\mathbf{k}^3\mathbf{Q} = q_k + q_j\mathbf{i} - q_i\mathbf{j} - q_0\mathbf{k}$$

3 - Matrices

The quaternion multiplication **AB** can also be written as the following matrix multiplication:

3.1:

$$\mathbf{AB} = \begin{bmatrix} +a_0 & -a_i & -a_j & -a_k \\ +a_i & +a_0 & -a_k & +a_j \\ +a_j & +a_k & +a_0 & -a_i \\ +a_k & -a_j & +a_i & +a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_i \\ b_j \\ b_k \end{bmatrix} = (a_0 b_0 - \mathbf{a} \cdot \mathbf{b}) + (b_0 \mathbf{a} + a_0 \mathbf{b} + \mathbf{a} \times \mathbf{b})$$

The color coding is included to make it easier to recognize how the multiplication produces the result. The scalar and vector terms are the same color with the scalar terms being a lighter shade and the vector terms being either a darker shade or **bold**. The top row of the coefficient matrix produces a scalar value. The term in red gives the a_0b_0 term. The terms in dark red give the negative of the vector dot product. The dark blue terms in column one give b_0 multiplied by the **a** vector. The blue terms along the diagonal produce a_0 multiplied by the **b** vector. The terms in green produce the vector cross product of vector **a** with vector **b**.

Given the interesting simplification using $2a_0b_0$ in **1.2** above, it also seems appropriate to express the coefficient matrix as follows:

$$\begin{bmatrix} +a_0 & -a_i & -a_j & -a_k \\ +a_i & +a_0 & -a_k & +a_j \\ +a_j & +a_k & +a_0 & -a_i \\ +a_k & -a_j & +a_i & +a_0 \end{bmatrix} = \begin{bmatrix} -a_0 & -a_i & -a_j & -a_k \\ 0 & 0 & -a_k & +a_j \\ 0 & +a_k & 0 & -a_i \\ 0 & -a_j & +a_i & 0 \end{bmatrix} + \begin{bmatrix} +a_0 & 0 & 0 & 0 \\ +a_i & 0 & 0 & 0 \\ +a_j & 0 & 0 & 0 \\ +a_k & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} +a_0 & 0 & 0 & 0 \\ 0 & +a_0 & 0 & 0 \\ 0 & 0 & +a_0 & 0 \\ 0 & 0 & 0 & +a_0 \end{bmatrix}$$

The cross product terms could also have been segregated into a separate matrix, or they could have been placed in either of the other two matrices. Each of the cells on the right-hand side is populated with a non-zero value only once, except for the cell at row one and column one. The value a_0 appears in each of the matrices. It seems that a_0b_0 is literally the key to this problem.

Let us take a moment to examine the internal structure of the coefficient matrix in **3.1** carefully. The quaternion characteristics (i.e., 1, **i**, **j**, and **k**) are contained in the column matrix [b]. The coefficient matrix [a] is constructed of four 2x2 scalar matrices as follows:

$$\begin{bmatrix} +[c] & -[d]^T \\ +[d] & +[c] \end{bmatrix} ; [c] = \begin{bmatrix} +a_0 & -a_i \\ +a_i & +a_0 \end{bmatrix} \ and \ [d] = \begin{bmatrix} +a_j & +a_k \\ +a_k & -a_j \end{bmatrix}$$

The superscript T on the –[d] indicates transpose.

Let us also define two additional matrices as follows:

$$[f] = \begin{bmatrix} b_0 \\ b_i \end{bmatrix} \ and \ [g] = \begin{bmatrix} b_j \\ b_k \end{bmatrix}$$

We can now express **3.1** more compactly as:

3.1.1:

$$\mathbf{AB} = \begin{bmatrix} +[c] & -[d]^T \\ +[d] & +[c] \end{bmatrix} \begin{bmatrix} [f] \\ [g] \end{bmatrix} = [c][f] - [d]^T[g] + [d][f] + [c][g]$$

The matrix products on the right-hand side of **3.1.1** do not exactly correspond with the terms on the right-hand side of **3.1** because there are 5 terms for **3.1** but only four terms for **3.1.1**.

For a generic matrix [m], the matrix multiplied by its inverse produces the identity matrix. For the specific case of a 4x4 matrix, this becomes:

3.2.1:

$$[m]^{-1}[m] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The inverse of a quaternion type matrix is easily found by multiplying the quaternion matrix by its transpose matrix. This produces a diagonal matrix with a value along the diagonal that is equal to the sum of the four squares. The inverse matrix is then determined by dividing the transpose matrix by the sum of the four squares. See **2.1.3.2** in the section on **Quaternions**.

$$\begin{bmatrix} +a_0 & +a_i & +a_j & +a_k \\ -a_i & +a_0 & +a_k & -a_j \\ -a_j & -a_k & +a_0 & +a_i \\ -a_k & +a_j & -a_i & +a_0 \end{bmatrix} \begin{bmatrix} +a_0 & -a_i & -a_j & -a_k \\ +a_i & +a_0 & -a_k & +a_j \\ +a_j & +a_k & +a_0 & -a_i \\ +a_k & -a_j & +a_i & +a_0 \end{bmatrix} = \left(a_0^2 + a_i^2 + a_j^2 + a_k^2\right) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3.2.2:

$$\frac{1}{\left(a_0^2 + a_i^2 + a_j^2 + a_k^2\right)} \begin{bmatrix} +a_0 & +a_i & +a_j & +a_k \\ -a_i & +a_0 & +a_k & -a_j \\ -a_j & -a_k & +a_0 & +a_i \\ -a_k & +a_j & -a_i & +a_0 \end{bmatrix} \begin{bmatrix} +a_0 & -a_i & -a_j & -a_k \\ +a_i & +a_0 & -a_k & +a_j \\ +a_j & +a_k & +a_0 & -a_i \\ +a_k & -a_j & +a_i & +a_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Comparison of **3.2.1** with **3.2.2** leads to the conclusion that the inverse matrix is as follows:

3.2.3:

$$\begin{bmatrix} +a_0 & -a_i & -a_j & -a_k \\ +a_i & +a_0 & -a_k & +a_j \\ +a_j & +a_k & +a_0 & -a_i \\ +a_k & -a_j & +a_i & +a_0 \end{bmatrix}^{-1} = \frac{1}{\left(a_0^2 + a_i^2 + a_j^2 + a_k^2\right)} \begin{bmatrix} +a_0 & +a_i & +a_j & +a_k \\ -a_i & +a_0 & +a_k & -a_j \\ -a_j & -a_k & +a_0 & +a_i \\ -a_k & +a_j & -a_i & +a_0 \end{bmatrix}$$

In **3.1.1**, the matrix multiplication was expressed as a group of smaller matrix multiplications. If those same smaller matrices are used, the inverse of the 4x4 coefficient matrix is:

3.2.3.1:

$$\begin{bmatrix} +[c] & -[d]^T \\ +[d] & +[c] \end{bmatrix}^{-1} = \frac{1}{\left(a_0^2 + a_i^2 + a_j^2 + a_k^2\right)} \begin{bmatrix} +[c]^T & +[d]^T \\ -[d] & +[c]^T \end{bmatrix}$$

There are subtleties here that should be mentioned. In the discussion of multiplication by complex conjugates, it was shown that:

$$\mathbf{A}^*\mathbf{A} = \|\mathbf{A}\|^2 = a_0^2 + a_i^2 + a_j^2 + a_k^2$$

Essentially, this means that the conjugate is equivalent to the transpose matrix. If this is treated as a matrix multiplication (see **3.1** above), the result is:

$$\begin{bmatrix} +a_0 & +a_i & +a_j & +a_k \\ -a_i & +a_0 & +a_k & -a_j \\ -a_j & -a_k & +a_0 & +a_i \\ -a_k & +a_j & -a_i & +a_0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_i \\ a_j \\ a_k \end{bmatrix} = \left(a_0^2 + a_i^2 + a_j^2 + a_k^2\right) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \left(a_0^2 + a_i^2 + a_j^2 + a_k^2\right)$$

This is a 4x4 matrix multiplied by a column matrix. The result is a 4x1 column matrix. However, **3.2.2** is based upon the multiplication of a pair of 4x4 matrices. The result is a 4x4 matrix. Yet the operations appear to be equivalent. The resulting 4x1 column matrix appears to be equivalent to the resulting 4x4 square matrix because the non-diagonal terms of the square matrix are zero. The only reason that this works correctly is the internal structure of the 4x4 quaternion matrix. A quaternion can be represented as either a 4x1 column matrix or a 4x4 square matrix. The choice is determined by whether the quaternion is the right term or the left term in **3.1**.

4 - Octonions

In Part 1 of this work, octonions were briefly mentioned. It was shown that the quaternions could be extended by multiplication by Euler's Equation containing the complex *i*.

4.1:

$$\mathbf{Z} = e^{i\omega}e^{\mathbf{Q}} = e^{i\omega+q_0+\mathbf{q}} = e^{i\omega}e^{q_0}e^{\mathbf{q}} = e^{\mathbf{O}};\ \mathbf{O} = i\omega + \mathbf{Q}$$

Expression **4.1** is a very general wave function. In principle, it can be used as a solution to the various differential equations of QM. It conforms perfectly to the separation of variables method.

It is interesting that this expression cannot be equal to zero. The exponential of *i*ω is never zero because the sine and cosine terms cannot both be zero at the same time. The exponential of q_0 also cannot be zero. It can be as large or as small as desired, but it cannot be zero. Also, the exponential of the vector **q** cannot be zero. This was demonstrated in Part 1 of this work. Therefore, **4.1** can never be zero. Of course, it is possible to add it to its opposite and their sum would be zero.

Also, in Part 1 of this work, it was shown that the exponential of **Q** can be expressed as:

4.2:

$$e^{\mathbf{Q}} = e^{q_0}\begin{bmatrix}\cos\gamma_0\\ \sin\gamma_i\\ \sin\gamma_j\\ \sin\gamma_k\end{bmatrix} = e^{q_0}\begin{bmatrix}\cos q_i & -\sin q_i & 0 & 0\\ \sin q_i & \cos q_i & 0 & 0\\ 0 & 0 & \cos q_i & -\sin q_i\\ 0 & 0 & \sin q_i & \cos q_i\end{bmatrix}\begin{bmatrix}\cos q_j & 0 & -\sin q_j & 0\\ 0 & \cos q_j & 0 & \sin q_j\\ \sin q_j & 0 & \cos q_j & 0\\ 0 & -\sin q_j & 0 & \cos q_j\end{bmatrix}\begin{bmatrix}\cos q_k\\ 0\\ 0\\ \sin q_k\end{bmatrix}$$

where:

4.2.1:

$$\begin{bmatrix}\cos\gamma_0\\ \sin\gamma_i\\ \sin\gamma_j\\ \sin\gamma_k\end{bmatrix} = \cos\gamma_0 + \sin\gamma_i\ \mathbf{i} + \sin\gamma_j\ \mathbf{j} + \sin\gamma_k\ \mathbf{k}$$

Since **4.2** is a quaternion with four terms and the complex *i* form of Euler’s Equation has two terms, it follows that multiplication of **4.2** by the complex *i* form of Euler's Equation in **4.1** will produce 8 terms as follows:

4.3:

$$\mathbf{Z} = e^{i\omega}e^{\mathbf{Q}} = [\cos(\omega) + \sin(\omega)i]e^{q_0}\begin{bmatrix}\cos\gamma_0\\ \sin\gamma_i\\ \sin\gamma_j\\ \sin\gamma_k\end{bmatrix}$$

The author is using red to designate terms associated with the quaternion.

Therefore, the author proposes to represent an octonion as follows:

4.3.1:

$$\mathbf{O} = \mathbf{A} + i\mathbf{B} \,;\, \mathbf{A} = \cos(\omega) e^{q_0} \begin{bmatrix} \cos\gamma_0 \\ \sin\gamma_i \\ \sin\gamma_j \\ \sin\gamma_k \end{bmatrix} \,;\, \mathbf{B} = \sin(\omega) e^{q_0} \begin{bmatrix} \cos\gamma_0 \\ \sin\gamma_i \\ \sin\gamma_j \\ \sin\gamma_k \end{bmatrix}$$

4.3.1.1:

$$\mathbf{A} = a_0 + a_i\mathbf{i} + a_j\mathbf{j} + a_k\mathbf{k} \,;\, \mathbf{B} = b_0 + b_i\mathbf{i} + b_j\mathbf{j} + b_k\mathbf{k}$$

An octonion function based upon **4.3** will be differentiable by the same rules as a quaternion function. The author thinks that **4.3** represents a subset of the generalized octonions because there do not appear to be 8 independent dimensions. Instead, there appear to be only five (i.e., the three unit vectors, the complex *i*, and the scalar q_0). The scalar term is used to adjust the length. In polar coordinates and spherical coordinates, the vector length is counted as a dimension to completely specify the space. Otherwise, the vectors would only map to a surface rather than to a space. In **4.3**, the complex vector portion is linked to the real vector portion by the complex phase angle ω. This accounts for the missing three dimensions (i.e., 8-5=3). The missing dimensions are the complex unit vectors *i***i**, *i***j**, and *i***k**.

In **2.1.3.2** in **Quaternions**, it was shown that multiplication of a quaternion by its conjugate produces a scalar that is equal in value to the square of the length of the quaternion. This was then used as a method of finding the inverse of the quaternion matrix. Something similar can be done based upon **4.3**, but it requires two steps. First, pre-multiply by the conjugate of the complex *i* terms.

$$[\cos(\omega) - \sin(\omega)i][\cos(\omega) + \sin(\omega)i] e^{q_0} \begin{bmatrix} \cos\gamma_0 \\ \sin\gamma_i \\ \sin\gamma_j \\ \sin\gamma_k \end{bmatrix}$$

$$[cos^2(\omega) + sin^2(\omega)] e^{q_0} \begin{bmatrix} \cos\gamma_0 \\ \sin\gamma_i \\ \sin\gamma_j \\ \sin\gamma_k \end{bmatrix}$$

$$[1] e^{q_0} \begin{bmatrix} \cos\gamma_0 \\ \sin\gamma_i \\ \sin\gamma_j \\ \sin\gamma_k \end{bmatrix}$$

Next, post-multiply by the conjugate of the quaternion. The quaternion matrix form must be used.

$$e^{q_0}\begin{bmatrix} +\cos\gamma_0 & -\sin\gamma_i & -\sin\gamma_j & -\sin\gamma_k \\ +\sin\gamma_i & +\cos\gamma_0 & -\sin\gamma_k & +\sin\gamma_j \\ +\sin\gamma_j & +\sin\gamma_k & +\cos\gamma_0 & -\sin\gamma_i \\ +\sin\gamma_k & -\sin\gamma_j & +\sin\gamma_i & +\cos\gamma_0 \end{bmatrix}\begin{bmatrix} +\cos\gamma_0 \\ -\sin\gamma_i \\ -\sin\gamma_j \\ -\sin\gamma_k \end{bmatrix} = e^{q_0}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = e^{q_0}$$

A scalar value is produced by pre-multiplying an octonion in form **4.3.1** by its complex conjugate and by post-multiplying the octonion by its quaternion conjugate.

4.3.2:

$$e^{q_0} = [\cos(\omega) - \sin(\omega)i]\mathbf{Z}\begin{bmatrix} +\cos\gamma_0 \\ -\sin\gamma_i \\ -\sin\gamma_j \\ -\sin\gamma_k \end{bmatrix}$$

The result of multiplication of a pair of octonions is:

4.4:

$$\mathbf{Z} = (\mathbf{A} + i\mathbf{B})(\mathbf{C} + i\mathbf{D}) = (\mathbf{AC} - \mathbf{BD}) + i(\mathbf{BC} + \mathbf{AD})$$

Note: The results presented in **4.4** contain the ***assumption*** that the complex *i* commutes normally with the unit vectors. However, this might not be true.

The multiplication of **4.4** can also be expressed as a matrix multiplication. The author will restrict this discussion to octonions that are based upon **4.3**. It should be possible to determine an inverse coefficient matrix easily since these octonions have “conjugates” as defined by **4.3.2**. For now, the author will ***assume*** that the complex *i* commutes normally.

A coefficient matrix can be produced by expanding each of the quaternions into four terms and then performing the multiplication and rearranging the terms. A less tedious method is to expand each quaternion into a scalar and vector and then multiply and rearrange. This takes advantage or the natural structure of the system.

$$\mathbf{Z} = [(a_0 + \mathbf{a}) + i(b_0 + \mathbf{b})][(c_0 + \mathbf{c}) + i(d_0 + \mathbf{d})]$$

The result is the sum of the following four groups:

$$(a_0 + \mathbf{a})c_0 + i(b_0 + \mathbf{b})c_0 +$$

$$(a_0 + \mathbf{a})\mathbf{c} + i(b_0 + \mathbf{b})\mathbf{c} +$$

$$(a_0 + \mathbf{a})id_0 + i(b_0 + \mathbf{b})id_0 +$$

$$(a_0 + \mathbf{a})i\mathbf{d} + i(b_0 + \mathbf{b})i\mathbf{d}$$

Since the result is the sum of these terms, they can be added together in any sequence that is desired. The third and fourth groups need to be re-arranged so that their real and complex parts are consistent with groups one and two.

$$(a_0 + \mathbf{a})c_0 + i(b_0 + \mathbf{b})c_0$$

$$(a_0 + \mathbf{a})\mathbf{c} + i(b_0 + \mathbf{b})\mathbf{c}$$

$$i(b_0 + \mathbf{b})id_0 + (a_0 + \mathbf{a})id_0$$

$$i(b_0 + \mathbf{b})i\mathbf{d} + (a_0 + \mathbf{a})i\mathbf{d}$$

Now some terms need to be swapped between row one and row 3 and between row two and row four.

$$(a_0 + \mathbf{a})c_0 + i(b_0 + \mathbf{b})id_0$$

$$(a_0 + \mathbf{a})\mathbf{c} + i(b_0 + \mathbf{b})i\mathbf{d}$$

$$i(b_0 + \mathbf{b})c_0 + (a_0 + \mathbf{a})id_0$$

$$i(b_0 + \mathbf{b})\mathbf{c} + (a_0 + \mathbf{a})i\mathbf{d}$$

This is a useful reference form because these terms do not yet contain any assumption regarding the commutivity of the complex *i*.

Now let us ***assume*** that the complex *i* commutes normally.

$$(a_0 + \mathbf{a})c_0 + -(b_0 + \mathbf{b})d_0$$

$$(a_0 + \mathbf{a})\mathbf{c} + -(b_0 + \mathbf{b})\mathbf{d}$$

$$i(b_0 + \mathbf{b})c_0 + i(a_0 + \mathbf{a})d_0$$

$$i(b_0 + \mathbf{b})\mathbf{c} + i(a_0 + \mathbf{a})\mathbf{d}$$

The terms in red are the quaternion multiplication **AC**. The terms in green are the quaternion multiplication **BD**. The terms in blue are the quaternion multiplication **BC**. The terms in black are the quaternion multiplication **AD**. This agrees with **4.4**.

Refer to **3.2.1** for the coefficient matrix of a quaternion multiplication. The matrix multiplication that results from **4.4** is as follows:

4.4.1:

$$\mathbf{Z} = e^{q_0}\begin{bmatrix} +a_0 & -a_i & -a_j & -a_k & -b_0 & +b_i & +b_j & +b_k \\ +a_i & +a_0 & -a_k & +a_j & -b_i & -b_0 & +b_k & -b_j \\ +a_j & +a_k & +a_0 & -a_i & -b_j & -b_k & -b_0 & +b_i \\ +a_k & -a_j & +a_i & +a_0 & -b_k & +b_j & -b_i & -b_0 \\ +b_0 & -b_i & -b_j & -b_k & +a_0 & -a_i & -a_j & -a_k \\ +b_i & +b_0 & -b_k & +b_j & +a_i & +a_0 & -a_k & +a_j \\ +b_j & +b_k & +b_0 & -b_i & +a_j & +a_k & +a_0 & -a_i \\ +b_k & -b_j & +b_i & +b_0 & +a_k & -a_j & +a_i & +a_0 \end{bmatrix}\begin{bmatrix} +c_0 \\ +c_i \\ +c_j \\ +c_k \\ +d_0 \\ +d_i \\ +d_j \\ +d_k \end{bmatrix}$$

The exponential term has been factored out of the coefficients. The "a" and "b" coefficients in **4.4.1** are now simply combinations of sines and cosines.

Based upon **4.3.2** and the coefficient matrix in **4.4.1**, the inverse of the coefficient matrix should be similar to the following:

$$e^{-q_0}\begin{bmatrix} +a_0 & +a_i & +a_j & +a_k & +b_0 & +b_i & +b_j & +b_k \\ -a_i & +a_0 & +a_k & -a_j & -b_i & +b_0 & +b_k & -b_j \\ -a_j & -a_k & +a_0 & +a_i & -b_j & -b_k & +b_0 & +b_i \\ -a_k & +a_j & -a_i & +a_0 & -b_k & +b_j & -b_i & +b_0 \\ -b_0 & -b_i & -b_j & -b_k & +a_0 & +a_i & +a_j & +a_k \\ +b_i & -b_0 & -b_k & +b_j & -a_i & +a_0 & +a_k & -a_j \\ +b_j & +b_k & -b_0 & -b_i & -a_j & -a_k & +a_0 & +a_i \\ +b_k & -b_j & +b_i & -b_0 & -a_k & +a_j & -a_i & +a_0 \end{bmatrix} \propto [m]^{-1}$$

This is also the transpose. Multiplying them gives (the exponentials cancel each other):

$$\begin{bmatrix} +a_0 & +a_i & +a_j & +a_k & +b_0 & +b_i & +b_j & +b_k \\ -a_i & +a_0 & +a_k & -a_j & -b_i & +b_0 & +b_k & -b_j \\ -a_j & -a_k & +a_0 & +a_i & -b_j & -b_k & +b_0 & +b_i \\ -a_k & +a_j & -a_i & +a_0 & -b_k & +b_j & -b_i & +b_0 \\ -b_0 & -b_i & -b_j & -b_k & +a_0 & +a_i & +a_j & +a_k \\ +b_i & -b_0 & -b_k & +b_j & -a_i & +a_0 & +a_k & -a_j \\ +b_j & +b_k & -b_0 & -b_i & -a_j & -a_k & +a_0 & +a_i \\ +b_k & -b_j & +b_i & -b_0 & -a_k & +a_j & -a_i & +a_0 \end{bmatrix}\begin{bmatrix} +a_0 & -a_i & -a_j & -a_k & -b_0 & +b_i & +b_j & +b_k \\ +a_i & +a_0 & -a_k & +a_j & -b_i & -b_0 & +b_k & -b_j \\ +a_j & +a_k & +a_0 & -a_i & -b_j & -b_k & -b_0 & +b_i \\ +a_k & -a_j & +a_i & +a_0 & -b_k & +b_j & -b_i & -b_0 \\ +b_0 & -b_i & -b_j & -b_k & +a_0 & -a_i & -a_j & -a_k \\ +b_i & +b_0 & -b_k & +b_j & +a_i & +a_0 & -a_k & +a_j \\ +b_j & +b_k & +b_0 & -b_i & +a_j & +a_k & +a_0 & -a_i \\ +b_k & -b_j & +b_i & +b_0 & +a_k & -a_j & +a_i & +a_0 \end{bmatrix}$$

$$= \begin{bmatrix} s & 0 & 0 & 0 & 0 & X & X & X \\ 0 & s & 0 & 0 & X & 0 & X & X \\ 0 & 0 & s & 0 & X & X & 0 & X \\ 0 & 0 & 0 & s & X & X & X & 0 \\ 0 & X & X & X & s & 0 & 0 & 0 \\ X & 0 & X & X & 0 & s & 0 & 0 \\ X & X & 0 & X & 0 & 0 & s & 0 \\ X & X & X & 0 & 0 & 0 & 0 & s \end{bmatrix} ;\ s = \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 ;\ X = nonzero\ element?$$

It "appears" that the matrix inversion has failed. The various X terms are each equal to twice the sum of four mixed sinusoids. In truth, the X terms are equal to zero. However, this is not apparent until the terms are examined closely.

column 1, row 6:

$$X = 2(\cos\omega \cos\gamma_0 \sin\omega \sin\gamma_i - \cos\omega \sin\gamma_i \sin\omega \cos\gamma_0 - \cos\omega \sin\gamma_j \sin\omega \sin\gamma_k + \cos\omega \sin\gamma_k \sin\omega \sin\gamma_j)$$

column 1, row 7:

$$X = 2(\cos\omega \cos\gamma_0 \sin\omega \sin\gamma_j + \cos\omega \sin\gamma_i \sin\omega \sin\gamma_k - \cos\omega \sin\gamma_j \sin\omega \cos\gamma_0 - \cos\omega \sin\gamma_k \sin\omega \sin\gamma_i)$$

column 1, row 8:

$$X = 2(\cos\omega \cos\gamma_0 \sin\omega \sin\gamma_k - \cos\omega \sin\gamma_i \sin\omega \sin\gamma_j + \cos\omega \sin\gamma_j \sin\omega \sin\gamma_i - \cos\omega \sin\gamma_k \sin\omega \cos\gamma_0)$$

column 2, row 7:

$$X = 2(-\cos\omega \sin\gamma_i \sin\omega \sin\gamma_j + \cos\omega \cos\gamma_0 \sin\omega \sin\gamma_k - \cos\omega \sin\gamma_k \sin\omega \cos\gamma_0 + \cos\omega \sin\gamma_j \sin\omega \sin\gamma_i)$$

column 2, row 8:

$$X = 2(-\cos\omega \sin\gamma_i \sin\omega \sin\gamma_k - \cos\omega \cos\gamma_0 \sin\omega \sin\gamma_j + \cos\omega \sin\gamma_k \sin\omega \sin\gamma_i + \cos\omega \sin\gamma_j \sin\omega \cos\gamma_0)$$

column 3, row 8:

$$X = 2(-\cos\omega \sin\gamma_j \sin\omega \sin\gamma_k + \cos\omega \sin\gamma_k \sin\omega \sin\gamma_j + \cos\omega \cos\gamma_0 \sin\omega \sin\gamma_i - \cos\omega \sin\gamma_i \sin\omega \cos\gamma_0)$$

As complicated as these expressions may appear to be, they each sum to zero. The X terms are anti-symmetric about the b_0 diagonals (but they are still equal to zero). Also, by symmetry the X values in the upper right quadrant are also zero.

Therefore, the inverse of the coefficient matrix of **4.4.1** is:

4.4.2:

$$[m]^{-1} = \frac{1}{e^{q_0}(\|\mathbf{A}\|^2 + \|\mathbf{B}\|^2)} \begin{bmatrix} +a_0 & +a_i & +a_j & +a_k & +b_0 & +b_i & +b_j & +b_k \\ -a_i & +a_0 & +a_k & -a_j & -b_i & +b_0 & +b_k & -b_j \\ -a_j & -a_k & +a_0 & +a_i & -b_j & -b_k & +b_0 & +b_i \\ -a_k & +a_j & -a_i & +a_0 & -b_k & +b_j & -b_i & +b_0 \\ -b_0 & -b_i & -b_j & -b_k & +a_0 & +a_i & +a_j & +a_k \\ +b_i & -b_0 & -b_k & +b_j & -a_i & +a_0 & +a_k & -a_j \\ +b_j & +b_k & -b_0 & -b_i & -a_j & -a_k & +a_0 & +a_i \\ +b_k & -b_j & +b_i & -b_0 & -a_k & +a_j & -a_i & +a_0 \end{bmatrix}$$

If the exponential term is factored out of the matrix, then the two quaternions will have unit length and the various "a" and "b" coefficients will be mixed sinusoids only.

4.4.2.1:

$$[m]^{-1} = \frac{e^{-q_0}}{2}\begin{bmatrix} +a_0 & +a_i & +a_j & +a_k & +b_0 & +b_i & +b_j & +b_k \\ -a_i & +a_0 & +a_k & -a_j & -b_i & +b_0 & +b_k & -b_j \\ -a_j & -a_k & +a_0 & +a_i & -b_j & -b_k & +b_0 & +b_i \\ -a_k & +a_j & -a_i & +a_0 & -b_k & +b_j & -b_i & +b_0 \\ -b_0 & -b_i & -b_j & -b_k & +a_0 & +a_i & +a_j & +a_k \\ +b_i & -b_0 & -b_k & +b_j & -a_i & +a_0 & +a_k & -a_j \\ +b_j & +b_k & -b_0 & -b_i & -a_j & -a_k & +a_0 & +a_i \\ +b_k & -b_j & +b_i & -b_0 & -a_k & +a_j & -a_i & +a_0 \end{bmatrix}$$

Note: If the assumption is made that the complex *i* anti-commutes with the unit vectors, a coefficient matrix results which the author has been unable to invert.

The result of multiplication of a pair of octonions expressed in exponential form is:

$$\mathbf{Z} \;=\; e^{i\alpha}e^{\mathbf{A}}e^{i\beta}e^{\mathbf{B}}$$

4.5:

$$\mathbf{Z} \;=\; e^{i\alpha}e^{\mathbf{A}}e^{i\beta}e^{\mathbf{B}} \;=\; e^{i(\alpha+\beta)}e^{(\mathbf{A}+\mathbf{B})}$$

Now let us consider associativity as it applies to this subset of octonions. Is the following statement true?

4.6:

$$[(\mathbf{A}+i\mathbf{B})(\mathbf{C}+i\mathbf{D})](\mathbf{E}+i\mathbf{F}) \;=\; (\mathbf{A}+i\mathbf{B})[(\mathbf{C}+i\mathbf{D})(\mathbf{E}+i\mathbf{F})]\;;??????$$

$$[(\mathbf{AC}-\mathbf{BD})+\;i(\mathbf{BC}+\mathbf{AD})](\mathbf{E}+i\mathbf{F}) = \;(\mathbf{A}+i\mathbf{B})[(\mathbf{CE}-\mathbf{DF})+\;i(\mathbf{DE}+\mathbf{CF})]$$

$$[(\mathbf{AC}-\mathbf{BD})\mathbf{E}-(\mathbf{BC}+\mathbf{AD})\mathbf{F}]+i[(\mathbf{BC}+\mathbf{AD})\mathbf{E}+(\mathbf{AC}-\mathbf{BD})\mathbf{F}] = [\mathbf{A}(\mathbf{CE}-\mathbf{DF})-\mathbf{B}(\mathbf{DE}+\mathbf{CF})]+i[\mathbf{B}(\mathbf{CE}-\mathbf{DF})+\mathbf{A}(\mathbf{DE}+\mathbf{CF})]$$

$$(\mathbf{ACE}-\mathbf{BDE}-\mathbf{BCF}-\mathbf{ADF})+i(\mathbf{BCE}+\mathbf{ADE}+\mathbf{ACF}-\mathbf{BDF}) = (\mathbf{ACE}-\mathbf{ADF}-\mathbf{BDE}-\mathbf{BCF})+i(\mathbf{BCE}-\mathbf{BDF}+\mathbf{ADE}+\mathbf{ACF})$$

Since addition of quaternions is commutative and since multiplication of quaternions is associative, it follows that multiplication of this subset of octonions is also associative. Unfortunately, this is a direct contradiction of what is accepted to be true for octonions in general. Multiplication of octonions is generally considered to be non-associative. An assumption here is that the complex *i* commutes normally with the various quaternions.

Now let us consider a simple multiplication where only the complex phase angle between the octonions differs.

$$\mathbf{Z} = \left\{(\cos\alpha + i\sin\alpha)e^{q_0}\begin{bmatrix}\cos\gamma_0\\ \sin\gamma_i\\ \sin\gamma_j\\ \sin\gamma_k\end{bmatrix}\right\}\left\{(\cos\beta + i\sin\beta)e^{q_0}\begin{bmatrix}\cos\gamma_0\\ \sin\gamma_i\\ \sin\gamma_j\\ \sin\gamma_k\end{bmatrix}\right\}$$

$$\mathbf{Z} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)e^{2q_0}\begin{bmatrix}\cos\gamma_0\\ \sin\gamma_i\\ \sin\gamma_j\\ \sin\gamma_k\end{bmatrix}^2$$

$$\mathbf{Z} = (\cos\alpha\cos\beta - \sin\alpha\sin\beta + i\cos\alpha\sin\beta + i\sin\alpha\cos\beta)e^{2q_0}\begin{bmatrix}\cos\gamma_0\\ \sin\gamma_i\\ \sin\gamma_j\\ \sin\gamma_k\end{bmatrix}^2$$

4.7:

$$\mathbf{Z} = [\cos(\alpha + \beta) + i\sin(\alpha + \beta)]e^{2q_0}\begin{bmatrix}\cos\gamma_0\\ \sin\gamma_i\\ \sin\gamma_j\\ \sin\gamma_k\end{bmatrix}^2$$

Now let us consider the octonion problem as a natural logarithm rather than as an exponential. The first relationship presented in this section was **4.1**:

$$\mathbf{Z} = e^{i\omega}e^{\mathbf{Q}} = e^{i\omega + q_0 + \mathbf{q}} = e^{i\omega}e^{q_0}e^{\mathbf{q}} = e^{\mathbf{O}};\ \mathbf{O} = i\omega + \mathbf{Q}$$

Take the natural logarithm of **Z**:

4.8:

$$\ln(\mathbf{Z}) = \mathbf{O} = i\omega + \mathbf{Q} = i\omega + q_0 + \mathbf{q}$$

It is very tempting to compare this to a four-vector from Special Relativity by equating ω with ct and setting q_0 equal to 0. However, that would be incorrect because the units for ω and for the coefficients of **Q** must be radians since they were in the exponential. Instead, a four-vector can be produced as follows:

4.8.1:

$$\frac{n\lambda}{2\pi}[\ln(\mathbf{Z}) - q_0] = \frac{n\lambda}{2\pi}(i\omega + \mathbf{q}) = ict + (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\,;\frac{n\lambda}{2\pi}\omega = ct\,;\frac{n\lambda}{2\pi}\mathbf{q} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

or

4.8.2:

$$\ln(\mathbf{Z}) = i\omega + \mathbf{Q} = \frac{2\pi}{n\lambda}[ict + (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})] + q_0$$

The value λ is a length that is used to convert between the octonion form and the four-vector form. The λ value is essentially a wavelength. The “n” term is simply the number of wavelengths. Therefore, the quantity nλ is the length associated with one cycle of 2π radians. In principle, "n" should be an integer. However, there is no rigid mathematical requirement that this be true. The q_0 term must be added to the angular form of the four-vector to produce an object or structure that fits into the octonion format. Therefore, to be compatible with the wave function presented in **4.1**, both SR and GR should be written using a four-vector combined with a scalar term. Of course, the scalar term can be zero. The author will speculate that the scalar term is related to the vacuum energy and/or the cosmological constant. The author will also speculate that it is possible for the scalar term to be a function of time. For example, consider the confusion that would result if the following speculation were true:

$$let\ q_0 = \frac{2\pi}{n\lambda}ct$$

The concepts of scalar time and complex time would be completely confused!!! Nature would never be so devious - or would it?

Now, let us consider the octonion multiplication problem as the sum of four-vectors. Begin with **4.5** and apply **4.8.2**:

$$\mathbf{Z} = e^{i\alpha}e^{\mathbf{A}}e^{i\beta}e^{\mathbf{B}} = e^{i(\alpha+\beta)}e^{(\mathbf{A}+\mathbf{B})}$$

4.9:

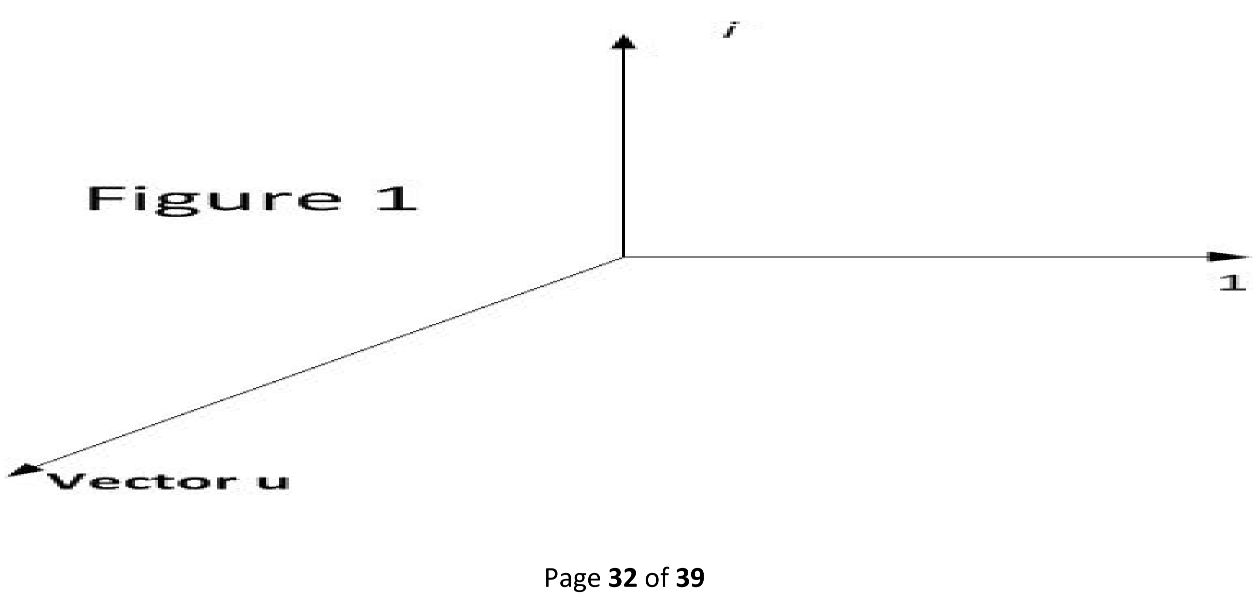
$$\ln(\mathbf{Z}) = i(\alpha + \beta) + (\mathbf{A} + \mathbf{B}) = \frac{2\pi}{n\lambda}[ict + (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})] + (a_0 + b_0)$$

5 - Pentuples

The concepts in this section are very radical. Thus far in the discussion of octonions, the author has argued that multiplication of the unit vectors by the complex *i* creates a complex vector space. This is a fairly conventional way of thinking. The author has also argued that this subset of octonions is a five dimensional space with four of these dimension having direction (the unit vectors and the complex *i*) and the fifth dimension being a scalar with no direction. These two arguments seem to be in conflict. Specifically, in what direction do the complex vectors point if there are only four directions from which to choose?

As an example, let us consider the complex *i* and the unit vector **i**. **Here is the first radical concept**. **The complex vector *i*i points back into regular vector space**. There is no other place for it to point. For the case given immediately above, *i***i** must be somewhere in the **j**-**k** plane since it must be perpendicular to both the complex *i* and the unit vector **i**. It follows that the other complex vectors must also point back into regular vector space. The difficulty with this concept is that these complex vectors could be anywhere in the plane that is perpendicular to the real vector. For the example here, *i***i** could be anywhere in the **j**-**k** plane. These "vector products" are not unique. They are not vector products in the same sense as is presented in section **1 – Vectors**.

Prior to developing these concepts further, let us attempt to visualize the space created by these five dimensions. The five unit dimensions are the scalar value one, the complex *i*, and the three unit vectors **i**, **j**, and **k**. To visualize the space created by these dimensions, it is necessary to go back several hundred years to when mathematicians first attempted to represent a two-dimensional plane. They placed scalar values along the x-axis. They then associated the y-axis with the complex *i* and thereby created the complex plane. The geometry presented here places an arbitrary unit vector **u** at the origin of the complex plane. This arbitrary unit vector is oriented perpendicular to the complex plane in accordance with the right-hand rule. These five axes now constitute a five-dimensional space. This is represented as Figure 1 below.



Looking at Figure 1 above, it is easy to visualize a 3-D space with the **i** axis at the location of the scalar axis, the **j** axis at the location of the complex *i* axis, and the **k** axis at the location of the **u** axis. Therefore, it seems reasonable to wonder if there is an identity similar to **jk** = **i** such as *i***u** = 1. Therefore, let us set the problem up as follows:

$$i\mathbf{u} = i(u_i\mathbf{i} + u_j\mathbf{j} + u_k\mathbf{k}) = 1\,; proposed\ identity$$

$$i\mathbf{i} = a_0 + a_j\mathbf{j} + a_k\mathbf{k}$$

$$i\mathbf{j} = b_0 + b_i\mathbf{i} + b_k\mathbf{k}$$

$$i\mathbf{k} = c_0 + c_i\mathbf{i} + c_j\mathbf{j}$$

Combining these produces the following:

$$scalar: u_i a_0 + u_j b_0 + u_k c_0 = 1$$

$$vector\ \mathbf{i}: u_j b_i + u_k c_i = 0$$

$$vector\ \mathbf{j}: u_i a_j + u_k c_j = 0$$

$$vector\ \mathbf{k}: u_i a_k + u_j b_k = 0$$

This system has four equations and 12 unknown coefficients. Therefore, there are eight degrees of freedom. In the author's opinion, the simplest way to resolve this system is first to specify the three terms in the unit vector **u**. At least one of the **u** coefficients must be non-zero. Next, specify two of the remaining three terms in the scalar relationship. The third term in the scalar relationship will then be determined by the equality. Lastly, specify one of the remaining two coefficients in each of the three vector relationships. The other term in each vector relationship will be determined from the equalities. Obviously, the various coefficients must be selected such that it is possible to satisfy the equalities.

As an example let us consider **u** = **i**. It is certain that $u_i = 1$, and that $u_j = u_k = 0$. Now let us consider the scalar equation. Since both u_j and u_k are zero, if b_0 and c_0 are real numbers then both $u_j b_0$ and $u_k c_0$ are zero. Therefore, a_0 must be one. Next let us consider the vector **i** equation. Since u_j and u_k are both zero, it follows that b_i and c_i can each have any real value. From the vector **j** and vector **k** equations, it follows that a_j and a_k are both zero and that c_j and b_k can each have any real value.

Therefore:

$$i\mathbf{u} = 1$$

$$i\mathbf{i} = 1$$

$$i\mathbf{j} = b_0 + b_i\mathbf{i} + b_k\mathbf{k}$$

$$i\mathbf{k} = c_0 + c_i\mathbf{i} + c_j\mathbf{j}$$

The author will offer one caveat to the above concepts. There is a trigonometric relationship that could be applied to the above problem:

$$y = \frac{\sin x}{x}\ ;\ \lim_{x\to 0}(y) = \lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$$

This is potentially significant because sin(x)/x is a solution to the spherical wave equation.

Now let us repeat the exercises from **4 – Octonions** using only five terms instead of eight. Let us begin by multiplying two of these pentuples. Let us multiply a pentuple and its conjugate. This ***should*** produce a scalar value and will provide some clarity. It is very convenient here to use the more compact vector form of the multiplication. This will also help to provide an understanding of the geometric meaning of this relation several steps below.

$$(a_0 + ai + \mathbf{a})(a_0 - ai - \mathbf{a}) =$$

$$a_0a_0 + aia_0 + \mathbf{a}a_0 +$$

$$-a_0ai - (ai)^2 - \mathbf{a}ai +$$

$$-a_0\mathbf{a} - ai\mathbf{a} - \mathbf{aa}$$

The red terms cancel and the green terms cancel since the scalar value commutes normally. This simplifies to the following:

$$a_0^2 + a^2 - \mathbf{a}^2 - ai\mathbf{a} - \mathbf{a}ai$$

Notice the order of multiplication between the complex *i* and the unit vectors contained within the two blue **a** terms. **The author will now invoke the second radical concept**. **If the complex *i* anti-commutes with the unit vectors, then the two blue terms will sum to zero**! The above expression will then simply be a scalar equal to the sum of the five squares.

The original form presented as a basis for the octonions was:

5.1:

$$\mathbf{Z} = e^{i\omega}e^{\mathbf{Q}} = [\cos(\omega) + \sin(\omega)i]e^{q_0}\begin{bmatrix}\cos\gamma_0\\ \sin\gamma_i\\ \sin\gamma_j\\ \sin\gamma_k\end{bmatrix}$$

Let us re-write this as follows:

5.1.1:

$$\mathbf{Z} = [\cos(\omega) + \sin(\omega)\, i]e^{q_0}[\cos(\gamma_0) + L\mathbf{u}]$$

where

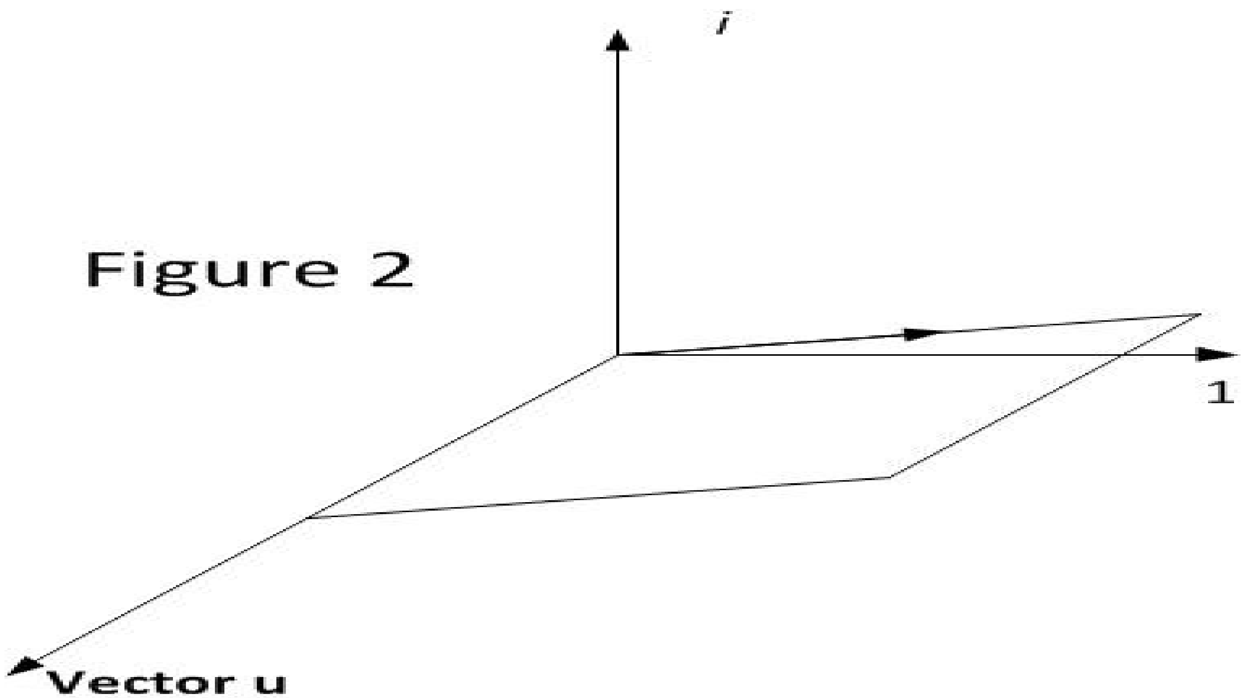
5.1.1.1:

$$L\mathbf{u} = \sin(\gamma_i)\,\mathbf{i} + \sin(\gamma_j)\mathbf{j} + \sin(\gamma_k)\mathbf{k}\ ; L = \sqrt{sin^2(\gamma_i) + sin^2(\gamma_j) + sin^2(\gamma_k)}$$

Here, L is the length of the vector portion of **5.1** and **u** is a unit vector in the direction of the vector portion of **5.1**.

Therefore, **5.1** is the sum of a line segment in the complex plane plus a rectangle in the 5-D space of Figure 1. The line segment is equal to Euler's Equation in the complex plane multiplied by the scalar portion of the quaternion. The rectangle is perpendicular to the complex plane. Its edges are specified by Euler's Equation in the complex plane and by the quaternion's vector portion along the other edge. This is illustrated in Figure 2.

Figure 2



The author will now introduce a new structure that will alter the presentation of this subset of octonions **O**.

$$\mathbf{O} = \mathbf{A} + i\mathbf{B}\,; \mathbf{A} \in \mathbf{Q}, \mathbf{B} \in \mathbf{Q}$$

$$\mathbf{A} = a_0 + a_i\mathbf{i} + a_j\mathbf{j} + a_k\mathbf{k}\,; \mathbf{B} = b_0 + b_i\mathbf{i} + b_j\mathbf{j} + b_k\mathbf{k}$$

$$\mathbf{O} = \left(a_0 + a_i\mathbf{i} + a_j\mathbf{j} + a_k\mathbf{k}\right) + i\left(b_0 + b_i\mathbf{i} + b_j\mathbf{j} + b_k\mathbf{k}\right)$$

$$\mathbf{O} = (a_0 + b_0 i) + (a_i + b_i i)\mathbf{i} + \left(a_j + b_j i\right)\mathbf{j} + (a_k + b_k i)\mathbf{k}$$

5.2:

$$A_0 = \begin{bmatrix} a_0 - \Delta a_0 & \Delta b_0 \\ \Delta a_0 & b_0 - \Delta b_0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} ; A_i = \begin{bmatrix} a_i - \Delta a_i & \Delta b_i \\ \Delta a_i & b_i - \Delta b_i \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$A_j = \begin{bmatrix} a_j - \Delta a_j & \Delta b_j \\ \Delta a_j & b_j - \Delta b_j \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} ; A_k = \begin{bmatrix} a_k - \Delta a_k & \Delta b_k \\ \Delta a_k & b_k - \Delta b_k \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

5.2.1:

$$\mathbf{P_A} = A_0 + A_i\mathbf{i} + A_j\mathbf{j} + A_k\mathbf{k} = \mathbf{O}$$

Equation **5.2.1** contains all of the information of the octonion. The only assumption that is built into **5.2.1** is that the various "a" and "b" scalar values commute normally with the complex *i*. The column matrix composed of [1 + *i*] now represents the complex plane. Multiplication of this column matrix by one of the unit vectors **i**, **j**, or **k** produces a quasi 3-D building block. Therefore, **5.2.1** represents a method of constructing a five-dimensional space using 3 quasi 3-D building blocks (A_i**i**, A_j**j**, A_k**k**) and the complex plane (A_0). **5.2.1** is essentially a Hamilton style quaternion based upon the complex plane rather than real numbers. To be consistent with the form of the wave-function, the various A_X terms must be the following:

5.2.2:

$$a_0 = \cos(\omega)\, e^{q_0} \cos(\gamma_0)\, ;\; a_i = \cos(\omega)\, e^{q_0} \sin(\gamma_i)\, ;\; a_j = \cos(\omega)\, e^{q_0} \sin(\gamma_j)\, ;\; a_k = \cos(\omega)\, e^{q_0} \sin(\gamma_k)$$

$$b_0 = \sin(\omega)\, e^{q_0} \cos(\gamma_0)\, ;\; b_i = \sin(\omega)\, e^{q_0} \sin(\gamma_i)\, ;\; b_j = \sin(\omega)\, e^{q_0} \sin(\gamma_j)\, ;\; b_k = \sin(\omega)\, e^{q_0} \sin(\gamma_k)$$

The obvious next step is to produce a coefficient matrix based upon multiplication of two pentuples as defined by **5.2** and **5.2.1**.

Let us define a second pentuple as follows:

$$\mathbf{O} = \mathbf{C} + i\mathbf{D}$$

$$\mathbf{C} = c_0 + c_i\mathbf{i} + c_j\mathbf{j} + c_k\mathbf{k}\, ;\; \mathbf{D} = d_0 + d_i\mathbf{i} + d_j\mathbf{j} + d_k\mathbf{k}$$

$$C_0 = \begin{bmatrix} c_0 - \Delta c_0 & \Delta d_0 \\ \Delta c_0 & d_0 - \Delta d_0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} ; C_i = \begin{bmatrix} c_i - \Delta c_i & \Delta d_i \\ \Delta c_i & d_i - \Delta d_i \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$C_j = \begin{bmatrix} c_j - \Delta c_j & \Delta d_j \\ \Delta c_j & d_j - \Delta d_j \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} ; C_k = \begin{bmatrix} c_k - \Delta c_k & \Delta d_k \\ \Delta c_k & d_k - \Delta d_k \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\mathbf{P_C} = C_0 + C_i\mathbf{i} + C_j\mathbf{j} + C_k\mathbf{k}$$

Multiplication of two pentuples is therefore:

$$\mathbf{P_A P_C} = \left(A_0 + A_i\mathbf{i} + A_j\mathbf{j} + A_k\mathbf{k}\right)\left(C_0 + C_i\mathbf{i} + C_j\mathbf{j} + C_k\mathbf{k}\right)$$

The coefficient matrix should be similar to that of a quaternion multiplication as presented in section **3 – Matrices**, but there will be differences. It must be remembered that the complex *i* anti-commutes with the unit vectors. Let us *carefully* review each of the 16 terms. In the terms below, C_x* represents the complex conjugate of C_x. For example:

$$\mathrm{C}_0 = \begin{bmatrix} c_0 & 0 \\ 0 & d_0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} \;;\; \mathrm{C}_0^* = \begin{bmatrix} c_0 & 0 \\ 0 & -d_0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} \;; the\ \Delta\ terms\ have\ been\ set\ equal\ to\ zero$$

The A_0 terms are simple:

$$\mathrm{A}_0\mathrm{C}_0 \;=\; \mathrm{A}_0\mathrm{C}_0$$

$$\mathrm{A}_0\mathrm{C}_i\mathbf{i} \;=\; \mathrm{A}_0\mathrm{C}_i\mathbf{i}$$

$$\mathrm{A}_0\mathrm{C}_j\mathbf{j} \;=\; \mathrm{A}_0\mathrm{C}_j\mathbf{j}$$

$$\mathrm{A}_0\mathrm{C}_k\mathbf{k} \;=\; \mathrm{A}_0\mathrm{C}_k\mathbf{k}$$

The $\mathbf{A}_i$, $\mathbf{A}_j$, and $\mathbf{A}_k$ terms are more difficult because of the complex *i* anti-commutation.

$$\mathrm{A}_i\mathbf{i}\mathrm{C}_0 \;=\; +\mathrm{A}_i\mathrm{C}_0^*\mathbf{i}\;;\; \mathrm{C}_0\ becomes\ \mathrm{C}_0^*$$

$$\mathrm{A}_i\mathbf{i}\mathrm{C}_i\mathbf{i} \;=\; -\mathrm{A}_i\mathrm{C}_i^*\;;\; \mathrm{C}_i\ becomes\ \mathrm{C}_i^*$$

$$\mathrm{A}_i\mathbf{i}\mathrm{C}_j\mathbf{j} \;=\; +\mathrm{A}_i\mathrm{C}_j^*\mathbf{k}\;;\; \mathrm{C}_j\ becomes\ \mathrm{C}_j^*$$

$$\mathrm{A}_i\mathbf{i}\mathrm{C}_k\mathbf{k} \;=\; -\mathrm{A}_i\mathrm{C}_k^*\mathbf{j}\;;\; \mathrm{C}_k\ becomes\ \mathrm{C}_k^*$$

$$\mathrm{A}_j\mathbf{j}\mathrm{C}_0 \;=\; +\mathrm{A}_j\mathrm{C}_0^*\mathbf{j}\;;\; \mathrm{C}_0\ becomes\ \mathrm{C}_0^*$$

$$\mathrm{A}_j\mathbf{j}\mathrm{C}_i\mathbf{i} \;=\; -\mathrm{A}_j\mathrm{C}_i^*\mathbf{k}\;;\; \mathrm{C}_i\ becomes\ \mathrm{C}_i^*$$

$$\mathrm{A}_j\mathbf{j}\mathrm{C}_j\mathbf{j} \;=\; -\mathrm{A}_j\mathrm{C}_j^*\;;\; \mathrm{C}_j\ becomes\ \mathrm{C}_j^*$$

$$\mathrm{A}_j\mathbf{j}\mathrm{C}_k\mathbf{k} \;=\; +\mathrm{A}_j\mathrm{C}_k^*\mathbf{i}\;;\; \mathrm{C}_k\ becomes\ \mathrm{C}_k^*$$

$$\mathrm{A}_k\mathbf{k}\mathrm{C}_0 \;=\; +\mathrm{A}_k\mathrm{C}_0^*\mathbf{k}\;;\; \mathrm{C}_0\ becomes\ \mathrm{C}_0^*$$

$$\mathrm{A}_k\mathbf{k}\mathrm{C}_i\mathbf{i} \;=\; +\mathrm{A}_k\mathrm{C}_i^*\mathbf{j}\;;\; \mathrm{C}_i\ becomes\ \mathrm{C}_i^*$$

$$\mathrm{A}_k\mathbf{k}\mathrm{C}_j\mathbf{j} \;=\; -\mathrm{A}_k\mathrm{C}_j^*\mathbf{i}\;;\; \mathrm{C}_j\ becomes\ \mathrm{C}_j^*$$

$$\mathrm{A}_k\mathbf{k}\mathrm{C}_k\mathbf{k} \;=\; -\mathrm{A}_k\mathrm{C}_k^*\;;\; \mathrm{C}_k\ becomes\ \mathrm{C}_k^*$$

Please note that the A_0 terms are multiplied by the C_x terms. However, the other A_x terms are multiplied by the C_x* terms. This is a HUGE problem. It means that a pentuple multiplication with complex *i* anti-commutation cannot be represented by a single matrix multiplication. Instead, the following is proposed:

5.3:

$$\mathbf{P_A P_C} = \begin{bmatrix} +A_0 & 0 & 0 & 0 \\ 0 & +A_0 & 0 & 0 \\ 0 & 0 & +A_0 & 0 \\ 0 & 0 & 0 & +A_0 \end{bmatrix} \begin{bmatrix} C_0 \\ C_i \\ C_j \\ C_k \end{bmatrix} + \begin{bmatrix} 0 & -A_i & -A_j & -A_k \\ +A_i & 0 & -A_k & +A_j \\ +A_j & +A_k & 0 & -A_i \\ +A_k & -A_j & +A_i & 0 \end{bmatrix} \begin{bmatrix} C_0^* \\ C_i^* \\ C_j^* \\ C_k^* \end{bmatrix}$$

After a few moments of consideration, the following identity becomes clear:

5.3.1:

$$\mathbf{P_A P_C} + \mathbf{P_A P_{C^*}} = \begin{bmatrix} +A_0 & -A_i & -A_j & -A_k \\ +A_i & +A_0 & -A_k & +A_j \\ +A_j & +A_k & +A_0 & -A_i \\ +A_k & -A_j & +A_i & +A_0 \end{bmatrix} \begin{bmatrix} C_0 \\ C_i \\ C_j \\ C_k \end{bmatrix} + \begin{bmatrix} +A_0 & -A_i & -A_j & -A_k \\ +A_i & +A_0 & -A_k & +A_j \\ +A_j & +A_k & +A_0 & -A_i \\ +A_k & -A_j & +A_i & +A_0 \end{bmatrix} \begin{bmatrix} C_0^* \\ C_i^* \\ C_j^* \\ C_k^* \end{bmatrix}$$

Furthermore, the complex *i* terms associated with C_x will cancel out in **5.3.1** leaving two copies of pentuple **A** multiplied by the real portion of pentuple **C**.

The author is not yet prepared to state the inverse of this relationship. In section **4 – Octonions**, the author demonstrated that the octonion coefficient matrix could be inverted by pre-multiplying by the complex conjugate and by post-multiplying by the quaternion conjugate. Something similar might be applicable here. The problem is complicated by the anti-commutation of the complex *i* and by the question of the validity of multiplication associativity.

It is noteworthy that if the various Δ terms in **5.2** are zero, then 32 of the 64 cells of these coefficient matrices would be empty (zero valued). It would therefore be conceivable that a second wave-function could be included in this equation. This sparseness gives the ***illusion*** that information has been lost or is missing from the coefficient matrices. The sparseness of the coefficient matrices is actually the result of the fact that the various C_x terms each contain two (or four) coefficients. This is an active area of study for the author.

When the various A_x and C_x terms above are multiplied, it is necessary to use the following relationship:

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = (a+bi)(c+di) = (ac-bd) + (bc+ad)i = \begin{bmatrix} ac & bc \\ -bd & ad \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

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1. Thomas, G. 1972. Calculus and Analytic Geometry - Alternate Edition, Addison-Wesley Publishing Company, Reading Mass., p. 486.