A partial proof of the Goldbach conjecture and the twin primes conjecture

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Introduction

This paper presents a "formula" (more or less) for prime numbers in a specific interval. This formula is then used to partially prove the Goldbach conjecture and the twin primes conjecture. The proofs are incomplete however and have not been reviewed by anyone.

A "formula" for primes

Consider an integer $y = 3a_2 + 2a_3$, where a_2 is any integer, positive or negative, which is not divisible by 2, and likewise, a_3 is an integer not divisible by 3. It's pretty straight forward to show that y is not divisible by 2 or 3.

We can continue with this logic to find a formula for an integer which is not divisible by 2, 3 or 5. The formula is $y = 3{\cdot}5a_2 + 2{\cdot}5a_3 + 2{\cdot}3a_5$, where a_2, a_3 and a_5 are not divisible by 2, 3 or 5, respectively.

More generally, we can devise a formula to find all prime numbers in an interval $[p_n, p_n^2]$, where p_n is the nth prime number, if we know all the primes p_1 to p_{n-1}. The equation in all it's glory is:

$$y = \sum_{primes p < p_n} \frac{1}{p} \left(\prod_{primes q < p_n}^{q} q \right) a_p$$

Where a_p is any interger not divisible by p.

Formally, we can prove this formula will never be divisible by any prime p_1 to p_{n-1} by dividing through by each prime p_1 to p_{n-1}. Each term in the summation will give an inerger when divided by some prime except for when the prime is equal to p, in which case the term will be a non-integer, since a_p is not divisible by p. This means that y is not divisible by any prime p_1 to p_{n-1}.

The reason the equation only gives primes in the interval $[p_n, p_n^2]$ is because every integer below p_n^2 is either divisible by one of p_1, p_2, ... , p_{n-1} or is prime. So the formula will continue to give primes up to infinity, but it will also give composite numbers with prime factors of p_n or higher. All the primes below the interval are multiples of themselves, which means the equation shouldn't give these primes at all, except for 1.

Another thing we can do is take the last term of the summation out of the equation, and factor out p_{n-1} to get a recursive formula for prime numbers:

$$y = p_{n-1}k + \left(\prod_{prime q < p_{n-1}} q \right) a_{p_{n-1}}$$

Where k is a number which is not divisible by p_1, p_2, ... , p_{n-2}. If we now define:

$$\pi_n = \prod_{primes q < p_n} q$$

And:

$$\pi_{n,p} = \frac{1}{p} \prod_{primes q < p_n} q$$

The prime number formula is now:

$$y = \sum_{primes p < p_n} \pi_{n,p} \, a_p$$

Take the last term out of the summation and factor out p_{n-1} from what's left over to get:

$$y = p_{n-1}k + \pi_{n-1}a_{n-1}$$

This is the formula used to partially prove the Goldbach conjecture and the twin primes conjecture.

It needs to be shown that this formula will give all primes in the interval $[p_n, p_n^2]$. This can be done by showing that y can be any interger when the a_p's no longer need to be integers not divisible by p (and hence become b_p's.) If this is the case then it follows that the prime number formula will give all the primes greater than p_{n-1}. We can do this by using mathematical induction:

Basis step: Show that for the equation y = $3b_2 + 2b_3$, where b_2 and b_3 can be any integers, there is a solution for every integer y. Since 2 and 3 are co-prime there exist integers f and g such that 3f + 2g = 1. So let b_2 = yf and b_3 = yg and the equation is satisfied.

Inductive step: Assume that k and b_{pn-1} are any intergers, so that:

$$y = p_{n-1}k + \pi_{n-1}b_{p_{n-1}}$$

Becasue p_{n-1} and π_{n-1} are co-prime, there is a solution for any integer y.

This means that when all the b_p's are not divisible by p, y is not divisible by $p_1, p_2, \dots, p_{n-1}$, but if any of the b_p's are divisible by p, then y has a prime factor of p.

Example

Take the equation $y = 3a_2 + 2a_3$ and set $a_2 = 1$, this gives $y = 3 + 2a_3$ where a_3 is any number not divisible by 3, i.e. 1, 2, 4, 5, 7, 8 etc. This means that y is: 5, 7, 11, 13, 17, 19. Essentially all it's doing is skipping every even number, and then skipping every even number which is a multiple of 3. This will give all the primes between 5 and 25 because every number less than 25 is either a multiple of 2 and/or 3, or is prime.

Now take the equation $y = 3{\cdot}5a_2 + 2{\cdot}5a_3 + 2{\cdot}3a_5$ which is $y = 15a_2 + 10a_3 + 6a_5$ and set $a_2 = 1$ and $a_3 = 1$ then y $15 + 10 + 6a_5$, where a_5 is any number which is not divisible by 5, i.e. 1, 2, 3, 4, 6, 7, 8, 9. This gives y as 31, 37, 43, 49, 61, 67, 73, 79. 49 isn't a prime but it's 7^2, so it's not a multiple of 2, 3, or 5, and is outside of the interval. If we set $a_2 = 1$ and $a_3 = 2$ then $y = 15 + 20 + 6a_5$ so y is: 41, 47, 53, 59, 71, 77, 83, 89. 77 isn't a prime either but it's 7 x 11, so isn't a multiple of 2, 3 or 5, and is outside the interval as well.

A partial proof of the Goldbach conjecture

For the Goldbach conjecture all that's needed is to add two prime numbers (y_1 and y_2) together:

$$y_1 + y_2 = \sum_{primes p < p_n} \pi_{n,p} \left(a_p + c_p\right)$$

Where a_p and c_p are any two integers which aren't divisible by p. Because a_p and c_p are any two intergers not divisble by p, there are infinite solutions to $a_p + c_p = b_p$, where b_p is any interger. This works for all odd p, for the case when p = 2, $a_2 + c_2$ must be an even integer, this makes sense though because all other $\pi_{n,p}$ have factors of 2, so $\pi_{2,p}$ is the only odd $\pi_{n,p}$.

All that needs to be shown now is:

$$2m = \sum_{primes p < p_n} \pi_{n,p}\, b_p$$

For any interger m. This can be done by using the fact that p_{n-1} and π_{n-1} are co-prime. This would mean that for every even integer there are an infinite number of pairs of numbers which aren't divisible by $p_1, p_2, \dots, p_{n-1}$, and which sum to the integer. We can show that this equation can be satisfied for any m by using mathematical induction:

Basis step: Show that there are solutions to the equation $2m = 3b_2 + 2b_3$, where $b_2 = a_2 + c_2$ and $b_3 = a_3 + c_3$. b_2 must be an even integer, so let $b_2 = 2d$, for any integer d, and b_3 can be any integer at all.

Dividing by 2 gives:

m = 3d + b_3

Which means that for any even number it's possible to find infinite pairs of numbers which aren't divisible by 2 or 3 and add up to the even number.

Inductive step: Assume that for any even number (2e) there exists a pair of numbers k_1 and k_2 which aren't divisible by p_1, p_2, ... , p_{n-2} and add up to the even number. So that:

$$2e = k_1 + k_2 = \sum_{primes p<p_n} \pi_{n-1,p} b_p$$

Now take the equation:

$$2m = \sum_{primes p<p_n} \pi_{n,p}\, b_p$$

And take the last term out of the summation and factor out p_{n-1} from the remaining terms to get:

$$2m = p_{n-1}\left(\sum_{primes p<p_{n-1}} \pi_{n-1,p}\, b_p\right) + \pi_{n-1} b_{p_{n-1}}$$

This gives:

$$2m = p_{n-1}(2e) + \pi_{n-1} b_{p_{n-1}}$$

Dividing by 2:

$$m = p_{n-1}e + (\pi_{n-1}/2) b_{p_{n-1}}$$

Since p_{n-1} and $\pi_{n-1}/2$ share no common prime factors they are coprime intergers. This means that there are integers f and g such that $p_{n-1}f + (\pi_{n-1}/2)g = 1$, this is a result called Bézout's identity. So just let e = mf and b_{pn-1} = mg and the equation is satisfied.

This proves that there are infinite pairs of numbers (y_1 and y_2) which aren't divisible by p_1, p_2, .. , p_{n-1} and add up to any given even number. All that's left to show in order to prove the Goldbach conjecture is that one of these pairs will have both y_1 and y_2 in the interval [p_n, p_n^2] thus making sure they're prime.

<u>Example</u>

let 2m = 38, and $y_1 + y_2 = 15b_2 + 10b_3 + 6b_5$. b_2 is an even integer, so let b_2 = 2d, putting this all together:

$38 = 15\cdot 2d + 10b_3 + 6b_5$
$19 = 15d + 5b_3 + 3b_5$

One solution to this is d = 1, b_3 = 2, b_5 = -2. Splitting up the b's:

b_2 = 2, so let a_2 = 1 and c_2 = 1 (both odd numbers)
b_3 = 2, so let a_3 = 1 and c_3 = 1 (both not divisible by 3)
b_5 = -2, so let a_5 = -1 and c_5 = -1 (both not divisible by 5)

Both prime numbers are:

15(1) + 10(1) + 6(-1) = 19

We can get two other numbers by letting:

a_2 = 1 and c_2 = 1
a_3 = 4 and c_3 = -2
a_5 = -3 and c_3 = 1

To get:

15(1) + 10(4) + 6(-3) = 37
15(1) + 10(-2) + 6(1) = 1

1 isn't in the interval [7, 49], and it's possible to get negative numbers as well.

A partial proof of the twin primes conjecture

Subtracting y_2 from y_1 gives:

$$y_1 - y_2 = p_{n-1}(k_1 - k_2) + \pi_{n-1}B_{p_{n-1}}$$

Where $B_p = a_p – c_p$. There will be solutions for a_p and c_p if B_p is any integer. It's possible to show that k_1 - k_2 can be any even integer using the same logic as with the Goldbach conjecture but with a minus sign instead of a plus sign and B_p's instead of b_p's. Let k_1 - k_2 = 2e, and set y_1 - y_2 = 2, because they're twin primes. We now have:

$$2 = p_{n-1}(2e) + \pi_{n-1}B_{p_{n-1}}$$

And divide by 2:

$$1 = p_{n-1}e + (\pi_{n-1}/2)B_{p_{n-1}}$$

p_{n-1} and $\pi_{n-1}/2$ are co-prime, so there exists an e and B_{pn-1} which satisfies the equation. In fact there are infinite solutions to the equation $B_p = a_p - c_p$ so there are infinite pairs of numbers not divisible by p_1, p_2, ... , p_{n-1} which are only separated by 2.

There are infinite pairs and infinite intervals, and any negative pair of twin primes correspond to a positive pair. However, it still needs to be shown that at least one of these pairs is in each interval, for every interval, to be certain they're prime.

Conclusion

Given the first n-1 primes it's possible to develop a “formula” for all the primes in the interval between the nth prime and is square. Using this formula it's then possible to partially prove the Goldbach conjecture and the twin primes conjecture. The proofs for these conjectures are incomplete, however, and so need to be completed.