Curry's non-Paradox and Its False Definition

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Abstract Curry's paradox is generally considered to be one of the hardest paradoxes to solve. It is shown here that the paradox can be arrived in fewer steps and also for a different term of the original biconditional. Further, using different approaches, it is also shown that the conclusion of the paradox must always be false and this is not paradoxical but it is expected to be so. One of the approaches points out that the starting biconditional of the paradox amounts to a false definition or assertion which consequently leads to a false conclusion. Therefore, the solution is trivial and the paradox turns out to be no paradox at all. Despite that fact that verifying the truth value of the first biconditional of the paradox is trivial, mathematicians and logicians have failed to do so and merely assumed that it is true. Taking this into consideration that it is false, the paradox is however dismissed. This conclusion puts to rest an important paradox that preoccupies logicians and points out the importance of verifying one's assumptions.

Curry's Paradox

Curry's paradox, named after its discoverer, Haskell B. Curry,[3, 4] is one of the hardest paradoxes in logic.[2]. For related work on this topic see Beall[1] and Rogerson[7]. The paradox can be presented succinctly as follows:

1. $\neg Y$ (for effect, Y is taken to be false).
2. $Q \longleftrightarrow (Q \longrightarrow Y)$, by any means this can be derived such definition (for example using unrestricted comprehension), assertion or any other way.
3. $Q \longrightarrow (Q \longrightarrow Y)$, derived from step 2 by biconditional elimination.
4. $Q \longrightarrow Y$, derived from step 3 using contraction.
5. Q, derived from step 2 and 4 by modus ponens.
6. Y, derived from step 4 and 5 by modus ponens.

Therefore, the contradiction lies in the fact that we derived that Y is true (step 6) even though we chose Y to be false (step 1). I note however that usually step 2 is the first statement of the paradox and step 1 is only considered a "side statement." But I added this as step 1 in order to make it easier to see the point of this paper.

Classical Solution for the Set-theoretic Version of the Paradox

Curry's paradox is generally considered one of the hardest paradoxes. The paradox has a truth-theoretic version and set-theoretic version and classically, there are different approaches to solving the paradox for each version. We'll discuss here the classical solution to the set theoretic version. This starts with a set defined as follows:

$$S = \{\, x \mid x \in S \longrightarrow Y \,\} \tag{1}$$

(where Y is false). This is equivalent to:

$$x \in S \longleftrightarrow (x \in S \longrightarrow Y)\,. \tag{2}$$

Now, let $Q = (x \in S)$ and we already arrived at step 2 in Curry's paradox above.

It is generally claimed that axiomatic set theory (such as Zermelo-Fraenkel set theory (ZFC)) avoids Curry's paradox by replacing the axiom of unrestricted comprehension by the standard axioms of axiomatic set theory. The solution, the axiomatization of set theory, was put in place in order to avoid Russell's paradox and was extended to Curry's paradox as well. Particularly, it is the axiom of specification (also called the axiom of separation) that excludes large collections (or proper classes) such as the set of all sets from Russell's paradox:

$$\exists S \forall x [x \in S \longleftrightarrow x \in Z \wedge \varphi(x)]\,. \tag{3}$$

The requirement that x belongs to Z makes S a subset of Z and thus it only allows subsets to be constructed.

It is here suggested however that the axioms don't succeed in avoiding the paradox. One reason is that collection S defined in Equation (1) is not a proper class or "large collection" (as it's the case with the set of of all sets from Russell's paradox). Instead, it's the empty set because there is no x that belongs to S for which Y is true. In Equation (2), $x \in S$ (the left side of the biconditional) is true if and only if $x \in S \longrightarrow Y$ (the right side) is true as well. But since Y is always false, $x \in S$ must always be false as well in order for the right side to be true. Therefore, we conclude that, in order for $x \in S$ (the left side) to be true, $x \in S$ (from the right side) must be false. This is contradictory and necessarily always false. Therefore there cannot be any x that belongs to S. Consequently S is the empty set.

Since the empty set is a subset of any set such as Z from Equation (3), then S is also a subset of an existing set and therefore it's a set itself. Further, the axiom of empty set also grants S the status of set under ZFC. Therefore, the axioms of ZFC, while avoiding Russell's paradox, can't avoid Curry's paradox because S, as defined in the set-theoretic version of Curry's paradox, is a perfectly valid set under ZFC.

Identifying the Problem and the Solution

Consider the following syllogism (where "steps" refer to steps in the derivation of Curry's paradox in a previous section):

- Premise 1: In order for Curry's paradox to arise, step 6 must be true. (P1)
- Premise 2: Step 6 is false. (P2)

- Conclusion: The paradox vanishes. If step 6 is false then Y is false which is exactly what is expected since Y was taken to be false at step 1. (C)

The truth of (P1) is pretty obvious. If step 6 is true then Y is true. If Y is true then there is a contradiction since Y was taken to be false at step 1. If (P2) is true then step 6 is false and, consequently, Y is false. But this is exactly what is expected based on step 1 and therefore there is no more contradiction and no more paradox (as it was concluded above (C). However, (P2) is likely to need more explanations which follows. One thing that needs to be pointed out is that it is irrelevant for our conclusion here by which method it is arrived at step 2. Regardless of the specifics of any such method, step 2 is either true or false. If it is true, we have:

- If step 2 is true then step 3 is true. [by derivation]
- If step 3 is true then step 4 is true. [by derivation]
- If step 4 is true and Y is false (step 1), then Q must be false. [by modus ponens]
- If Q is false then step 5 is false. [by identity]
- If Q (that is, step 5) is false and step 4 is true then step 6 is false. [by modus ponens]

Therefore in this case step 6 is false and therefore (P2) is true. If, on the other hand, if step 2 is false, we have:

- If step 2 is false then step 3 is false. [by derivation]
- If step 3 is false then step 4 is false. [by derivation]
- If step 4 is false and Y is false (step 1), then Q must be true. [by modus ponens]
- If Q is true then step 5 is true. [by identity]
- If step 4 is false and Q (that is, step 5) is true then step 6 is false. [by modus ponens]

Therefore, regardless if step 2 is true or false (and, subsequently, regardless of how it was arrived at), step 6 must always be false and, based on the syllogism above, there is no paradox (C).

The approach above allows step 2 to be either true or false and concludes that step 6 is always false. Since it doesn't matter whether step 2 is true or false (the same conclusion, that step 6 is false, is reached either way), it also doesn't matter how step 2 is derived. Any such derivation can only alter the truth value of step 2 and, as shown, that doesn't have any bearing on the truth value of step 6.

The difference between this approach and the classical approach (which leads to the paradox) is in step 4. This approach determines the truth value of step 4 by replacing Y in it with its truth value (false) while the classical approach waits all the way to the last step, step 6, to do the replacement of Y with its assigned value, false. But in step 6 it's too late and the contradiction pops up because by then Y was already derived to be true. Of course, both approaches cannot be true and to find where the problem lies we need to go back to step 4 where the two approaches part ways. The approach above allows step 4 to be either true or false and simply derive Q (step 5) by module ponens from step 4 (whatever it is, either true or false) and Y (which is always false). But the classical approach requires step 4 to be true and, in contrast, step 5 (Q) is derived from step

4 and step 2 without making use of Y's truth value. But step 4 is only true if step 3 is true and step 3 is true only if step 2 is true. Further, step 5 is only true if both step 2 and step 4 are true. Therefore, in order for the classical approach to work and produce the paradox at step 6, steps 1 through 5 must all be true. But are they? Since there is obviously a problem with Y and since we already know what Y's value is (false) we should replace it from the very beginning so that the steps can be rewritten as:

1. $\neg false$
2. $Q \longleftrightarrow (Q \longrightarrow false)$
3. $Q \longrightarrow (Q \longrightarrow false)$
4. $Q \longrightarrow false$
5. Q
6. $false$

It's obvious that step 4 above is only true if Q is false! That is, even before we get to step 6 we already have a problem that Q must be false when Q could have been taken to be true! The very same reasoning that leads to Curry's paradox about Y actually leads to a similar paradox about Q even before it gets to the Y related paradox! For example, step 1, instead of stating that Y is false it could also state that Q is true and a paradox would be reached one step earlier (at step 5, where Q needs to be false). But the problem goes further than that. Even if we allowed Q to be false, then based on step 2 and the classical reasoning, this requires $Q \longrightarrow false$ (that is, step 4) to be false as well! But step 4 was already derived to be true and it was precisely from this that we derived Q to be false! Going back to step 2, if Q is false then $Q \longrightarrow false$ must be false as well which means that Q must be true! Therefore, given that Y is false, step 4 can only be true if Q is false as well but if Q is false then, based on step 2, Q must be true! Consequently, the problem doesn't really lie with Y at step 6 but shows up earlier and with another term, Q. If steps 1 through 4 are true then Q must be both false and true at the same time!

While the initial approach replaced Y with false at step 4, we can approach this differently and start at step 2 instead. We already did this above but we can go further replace Q with one of the two possibilities (either *true* or *false*). First let's assume that Q is true and use T for *true* and F for *false*:

1. $\neg F$
2. $T \longleftrightarrow (T \longrightarrow F)$ [this leads to $T \longleftrightarrow F$ which is necessarily false]
3. $T \longrightarrow (T \longrightarrow F)$ [this leads to $T \longrightarrow F$ which is necessarily false]
4. $T \longrightarrow F$ [this is false as well]
5. T
6. F

Therefore, if Q is true then steps 2, 3, 4 and 6 are all false and the steps should be rewritten as this:

1. $\neg F$
2. $\neg(T \longleftrightarrow (T \longrightarrow F))$
3. $\neg(T \longrightarrow (T \longrightarrow F))$ [derived from step 2 by biconditional elimination]
4. $\neg(T \longrightarrow F)$ [derived from step 3 using contraction]
5. T [derived from step 2 and 4 by modus ponens]
6. $\neg F$ [derived from step 4 and 5 by modus ponens]

There is now no more paradox with neither Y or Q. In this case, the classical approach failed because it assumed that step 2 was true when in fact it was false (and, derived from it, steps 3 and 4 were also false).

Let's now assume that Q is false (F):

1. $\neg F$
2. $F \longleftrightarrow (F \longrightarrow F)$ [this leads to $F \longleftrightarrow T$ which is necessarily false]
3. $T \longrightarrow (T \longrightarrow F)$ [this leads to $F \longrightarrow F$ which is vacuously true]
4. $F \longrightarrow F$ [this is true]
5. F
6. F

Rewriting the steps to account for the steps which are false, we have:

1. $\neg F$
2. $\neg(F \longleftrightarrow (F \longrightarrow F))$
3. $T \longrightarrow (T \longrightarrow F)$ [derived from step 2 by biconditional elimination]
4. $F \longrightarrow F$ [derived from step 3 using contraction]
5. $\neg F$ [derived from step 2 and 4 by modus ponens]
6. $\neg F$ [derived from step 4 and 5 by modus ponens]

Again, there is no more paradox, neither for Y or Q. What avoids the paradox is the same thing as for the case when Q is true—the fact that step 2 is false. There are different ways to arrive at step 2, for example, by assertion or by definition under set theory (see Equation (2) derived from the definition of S, Equation (1)) or T-schema. Establishing the truth value of the biconditional in step 2 can be deferred to the background theory used but it would be more direct to actually check step 2 itself by plugging in the truth values for Q and Y—which is what we did here. We concluded that, when Y is false, step 2 is necessarily false regardless of how it's arrived at. If step 2 is based on a definition then it *must* be a false definition and therefore invalid! That's obvious when we replace Y and Q with their corresponding truth value. If it's an assertion, it must be a false assertion. We normally expect a definition to be true which may explain why it was treated as exempt from being checked if it's actually true or not. Similarly, an assertion is normally treated as a hypothesis and is taken as true until proved false. The classical contradiction at step 6 does prove it to be false but somehow mathematicians expected to be something else, something deeper and have been looking for the solution in a different place.

It happens that step 2 is also false even when Y is true and Q is false. Then, using the classical approach which assumes step 2 to be right, a contradiction still ensues. In the original demonstration of Curry's paradox, if we replace step 1, which states $\neg Y$ with $\neg Q$, we derive at step 5 that Q is true which would then contradict that Q was taken to be false in step 1. The paradox shows up one step earlier, at step 5 instead of step 6. The only case when step 2 is true is when both Q and Y are true. Finding out the truth value of step 2 is trivially simple, about as hard as solving $q \times (q \times y)$ when $y = -1$ and q is either 1 or -1. Out of all four possibilities for Q and Y, each being either true or false, three of them yield a false step 2 (or, three out of the 4 possibilities for $q \times (q \times y)$ resolve to -1). And the one case where step 2 is true doesn't even fit the scenario of Curry's paradox since it requires Y to be true. Ironically, this trivial observation was overlooked and step 2 was assumed to be true.

Therefore, the solution merely amounts to recognizing there is a problem with the starting point of the paradox which turns out to be false. This realization resolves all versions of the paradox. It should now be no surprise that it leads to false statement. On the contrary, the contradiction and the surprise would be if it didn't!

Analysis of the Problem-Solution

Mathematical entities exist by virtue of being able to construct them. A mathematical definition, in effect, constructs the entity being defined, specifying the conditions under which it exists. The contradiction arises when the entity being asserted to exist is impossible to exist. A definition, by its very nature, is affirmative. It defines what something *is* not what something *isn't*. It defines when somethings exists or how something can be constructed (which means that it mathematically exists) not when something doesn't exist. Thus, a definition is assumed to be true. Defining something that is or should be known to be false makes no mathematical sense. Neither does making an assumption that is or should be known to be false. Just like a definition, an assumption is assumed to be true. But the definition or assertion behind Curry's paradox stipulates that an entity (or relation between entities) exists precisely when it doesn't (when conditions of its existence forbid its existence). In other words, a false statement is assumed or defined which then leads to a false conclusion.

Arthur Prior offered a solution to the liar's paradox by pointing out that every statement includes an implicit assertion of its own truth[6]. Therefore step 2 is equivalent to:

$$\text{It is true that } Q \longleftrightarrow (Q \longrightarrow Y)\,.$$

Because of this implicit assertion of its own truth, the step was taken as true without checking and the corresponding truth value wasn't checked until the very end when Y turned out to be true when it was in fact expected to be false.

Curry started from the Kleene–Rosser paradox[5] and realized that the paradox can be simplified into what is now known as Curry's paradox. But the paradox can be simplified even further and arrived at by mere definition or assertion:

$$Q := \text{ false}\,, \tag{4}$$

from which we derive that Q is false! That is, in fact, the simplified version of Curry's paradox and it only takes one step:

$$Q\,. \tag{5}$$

Prior's implicit assertion applies here as well so that this translates to "The following is true: Q." But this leads to a contradiction because Q was defined to be false! Curry's paradox is reducible to this simplified version, where step 2 is reducible to Equation (5) above. It is often that we complicate things and those complications are in the way of seeing the obvious. The number of steps taken by Curry's paradox just hide that the starting point (step 2) was false. By the last step when Y is replaced with *false* and the apparent contradiction arises there were too many steps to link the result back to step 2.

While the lack of check of the truth value of step 2 ($Q \longleftrightarrow (Q \longrightarrow Y)$) is perplexing given how easy such a check is, Prior's insight about implicit assertions sheds some light on how and why it happened.

Although "it is generally agreed that one of the hardest among the paradoxes is Curry's paradox,"[2] the solution turned out to be trivial and the paradox turned out to be no paradox at all but rather a trick played by mathematicians' assumptions. The result therefore shows the importance of verifying one's assumptions and how long time and how much effort goes wasted on a non-problem when this check is overlooked.

References

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