Free fall through the Earth

Bengt Månsson
Kunskapsskolan Education
Stockholm–Gothenburg, Sweden

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Abstract

Free fall through the Earth, considered as a sphere of radially symmetric mass density, along the axis of rotation, is calculated using a general differential equation in newtonian gravity theory. The passage time is calculated and, further, the shape of the tunnel required if the fall is started at an arbitrary point other than a pole, so that the rotation of the Earth comes into play, is determined. Also a general relativistic case with constant density along the axis is considered. – In this general form the article may also be of some pedagogical value.

Consider a little marble dropped from the surface of Earth into a thin channel through the centre. In the idealization of spherical shape, constant density, no air resistance and disregarding that Earth's rotation (or just assming motion along the Earth's axis of rotation) this results in an oscillatory motion with period time T given by $T = T_0 \equiv 2\pi R\sqrt{R/(GM)}$ where R denotes the radius and M the mass of the Earth and G Newton's gravitational constant. Numerically this equals approximately 84 minutes. Incidentally this equals exactly the orbital period of an artificial satellite in the limit of zero height. This much is well-known.

In this article we will relax the condition of constant density. A general result will derived for radial symmetric density and applied to some particular cases. Next, the effect of Earth's rotation will be considered and, finally, the case of no rotation and constant density will be treated using general relativity. Apart from this last case newtonian theory is used.

1 General formula for period time

Theorem 1 (Equation of motion)**.** *If the density of the sphere is $\rho(r)$ where r is the distance from the centre, then*

$$r^2\ddot{r} + 4\pi G \int_0^r \xi^2 \rho(\xi) \mathrm{d}\xi = 0 \tag{1}$$

Proof: The mass at less distance to the centre of the sphere than r is given by the integral $4\pi \int_0^r \xi^2 \rho(\xi) \mathrm{d}\xi$. If the falling marble has mass m Newton's second law gives

$$m\ddot{r} = -m \cdot G \cdot 4\pi \int_0^r \xi^2 \rho(\xi) \mathrm{d}\xi \cdot \frac{1}{r^2}$$

from which the theorem immediately follows. □

If, in particular, the density is constant, $\rho(\xi) = \rho_0$, the differential equation becomes

$$\ddot{r} + \frac{4\pi G \rho_0}{3} r = 0,$$

so r is a harmonic function of time with angular frequency $\sqrt{4\pi G \rho_0/3}$ corresponding to the period time

$$T = \frac{2\pi}{\sqrt{4\pi G \rho_0 / 3}} = 2\pi R \sqrt{\frac{R}{GM}}$$

as mentioned above.

Theorem 2 (Period time)**.** *With assumptions as above the period time is given by*

$$T = \sqrt{\frac{2}{\pi G}} \cdot \int_0^R \frac{\mathrm{d}\eta}{\sqrt{f(\eta) + g(\eta) - g(R)}} \tag{2}$$

where

$$f(\eta) = \int_\eta^R \xi \rho(\xi)\, \mathrm{d}\xi \tag{3}$$

$$g(\eta) = \frac{1}{\eta} \cdot \int_0^\eta \xi^2 \rho(\xi)\, \mathrm{d}\xi \tag{4}$$

Proof: Multiplying equation (1) with $\dot{r}$ we get

$$\frac{1}{2}\frac{d}{dt}\dot{r}^2 + \frac{4\pi G}{r^2} \cdot \int_0^r \xi^2 \rho(\xi)\, \mathrm{d}\xi \cdot \dot{r} \tag{5}$$

Next, define a function F,

$$F(r) = -\frac{1}{r} \cdot \int_0^r \xi^2 \rho(\xi)\,\mathrm{d}\xi + \int_0^r \xi\rho(\xi)\,\mathrm{d}\xi,$$

with derivative

$$F'(r) = \frac{1}{r^2} \cdot \int_0^r \xi^2 \rho(\xi)\,\mathrm{d}\xi.$$

Since r is a function of time t, $D_t F(r) = \dot{r} \cdot F'(r)$, and equation (5) leads to the separable differential equation

$$D_t \dot{r}^2 + 8\pi G \cdot D_t F(r) = 0\ . \tag{6}$$

From (6)

$$\dot{r}^2 + 8\pi G F(r) = C \tag{7}$$

where C is some constant. If $t = 0$ when the body is released, then $r(0) = R$ and $\dot{r}(0) = 0$, and from (7), $C = 8\pi G F(R)$ so

$$\dot{r}^2 = 8\pi G(F(R) - F(r)) \tag{8}$$

where $F(R) \geq F(r)$. In the time intervall $0 \leq t \leq T/4$, r decreases so (8) gives

$$\dot{r} = -\sqrt{8\pi G} \cdot \sqrt{F(R) - F(r)}$$

and

$$t = \frac{1}{\sqrt{8\pi G}} \int_r^R \frac{\mathrm{d}\eta}{\sqrt{F(R) - F(\eta)}} \tag{9}$$

so the period time T is given by

$$T = \sqrt{\frac{2}{\pi G}} \int_0^R \frac{\mathrm{d}\eta}{\sqrt{F(R) - F(\eta)}}.$$

Finally, F is replaced by the functions f and g,

$$\begin{aligned} F(R) - F(\eta) &= -\frac{1}{R} \cdot \int_0^R \xi^2\rho(\xi)\,\mathrm{d}\xi + \int_0^R \xi\rho(\xi)\,\mathrm{d}\xi + \frac{1}{\eta} \cdot \int_0^\eta \xi^2\rho(\xi)\,\mathrm{d}\xi - \int_0^\eta \xi\rho(\xi)\,\mathrm{d}\xi \\ &= -\frac{1}{R} \cdot \int_0^R \xi^2\rho(\xi)\,\mathrm{d}\xi + \int_\eta^R \xi\rho(\xi)\,\mathrm{d}\xi + \frac{1}{\eta} \cdot \int_0^\eta \xi^2\rho(\xi)\,\mathrm{d}\xi \\ &= f(\eta) + g(\eta) - g(R) \end{aligned}$$

and so we get the proposed expression

$$T = \sqrt{\frac{2}{\pi G}} \cdot \int_0^R \frac{\mathrm{d}\eta}{\sqrt{f(\eta) + g(\eta) - g(R)}}. \qquad \square$$

2 Examples

2.1 Constant density

$$f(\eta) = \rho_0 \cdot \frac{1}{2}(R^2 - \eta^2), \qquad g(\eta) = \rho_0 \cdot \frac{1}{3}\eta^2$$

$$f(\eta) + g(\eta) - g(R) = \frac{1}{6}\rho_0 \cdot (R^2 - \eta^2)$$

$$T = \sqrt{\frac{2}{\pi G}} \cdot \int_0^R \frac{\mathrm{d}\eta}{\sqrt{\frac{1}{6}\rho_0 (R^2 - \eta^2)}} = \cdots = 2\pi R \sqrt{\frac{R}{GM}}\ ,$$

the well-known result mentioned in the abstract. We will use this as a comparison in later cases.

2.2 Linear density function I

$$\rho(r) = \rho_0 (1 - r/R)$$

which means that the density is maximal at the centre of the Earth and decreasing to zero at the surface.

$$f(\eta) := \rho_0 \int_\eta^R \xi - \xi^2/R \,\mathrm{d}\xi = \rho_0 \left[\xi^2/2 - \xi^3/(3R)\right]_\eta^R = \rho_0 \left(\frac{1}{6}R^2 - \frac{1}{2}\eta^2 + \frac{1}{3R}\eta^3\right)$$

$$g(\eta) := \frac{1}{\eta} \cdot \rho_0 \int_0^\eta \xi^2 - \xi^3/R \,\mathrm{d}\xi = \frac{1}{\eta} \cdot \rho_0 \left[\xi^3/3 - \xi^4/(4R)\right]_0^\eta = \rho_0 \left(\frac{1}{3}\eta^2 - \frac{1}{4R}\eta^3\right)$$

$$\begin{aligned} f(\eta) + g(\eta) - g(R) &= \\ &= \rho_0 \left(\frac{1}{6}R^2 - \frac{1}{2}\eta^2 + \frac{1}{3R}\eta^3 + \frac{1}{3}\eta^2 - \frac{1}{4R}\eta^3 - \left(\frac{1}{3}R^2 - \frac{1}{4R}R^3\right)\right) \\ &= \frac{1}{12R}\rho_0 (R^3 - 2R\eta^2 + \eta^3) \end{aligned}$$

Inserting this into the integral in equation 2 we get the period time in the form of an elliptic integral,

$$T = \sqrt{\frac{2}{\pi G}} \cdot \int_0^R \frac{\mathrm{d}\eta}{\sqrt{\eta^3 - 2R\eta^2 + R^3}} \cdot \sqrt{\frac{12R}{\rho_0}} = 2R\sqrt{\frac{2R}{GM}} \cdot \int_0^1 \frac{\mathrm{d}x}{\sqrt{x^3 - 2x^2 + 1}}, \tag{10}$$

approximately ($T \approx 0.8475T_0$),
where we have used

$$M = 4\pi \int_0^R \rho(\xi)\, \xi^2\, \mathrm{d}\xi = 4\pi\rho \int_0^R (1 - \xi/R)\, \xi^2\, \mathrm{d}\xi = \pi\rho\, R^3/3\ .$$

2.2.1 Transformation of the elliptic integral into standard (Jacobian) form

For a general algorithm, see [1], ch IX. A variable transformation $x = 1 - y^2$ gives

$$I = \int_0^1 \frac{\mathrm{d}x}{\sqrt{x^3 - 2x^2 + 1}} = 2 \cdot \int_0^1 \frac{\mathrm{d}y}{\sqrt{1 + y^2 - y^4}}$$

and then $t^2 = \phi - y^2$ where $\phi = (1 + \sqrt{5})/2$ and $t = \sqrt{\phi}x$ give

$$I = 2 \cdot \int_{\sqrt{\phi - 1}}^{\sqrt{\phi}} \frac{\mathrm{d}t}{\sqrt{\phi - t^2}\sqrt{2\phi - 1 - t^2}}$$

and

$$I = \frac{2}{\sqrt{2\phi - 1}} \cdot \int_{\phi - 1}^1 \frac{\mathrm{d}x}{\sqrt{1 - x^2}\sqrt{1 - k^2x^2}} =$$

$$= \frac{2}{\sqrt[4]{5}} \cdot \int_{\phi - 1}^1 \frac{\mathrm{d}x}{\sqrt{1 - x^2}\sqrt{1 - k^2x^2}}$$

where

$$k = \frac{\sqrt{\phi}}{\sqrt{2\phi - 1}} = \frac{\sqrt{\phi}}{\sqrt[4]{5}}\ (\approx 0.8507\ .$$

2.3 Linear density function II

$$\rho(r) = \rho_0(1 - kr/R), 0 < k < 1$$

With $k = 0$ the density is constant while with $k = 1$ we regain the previous case. In that case the density is zero at the surface, as is usual in star models but with $0 < k < 1$ the surface density is $\rho_0(1 - k) > 0$.

$$T = \sqrt{\frac{2}{\pi G}} \cdot \int_0^R \frac{\mathrm{d}\eta}{\sqrt{\frac{1}{6}\rho_0(R^2-\eta^2) - \frac{k\rho_0}{12R}(R^3-\eta^3)}} = [\eta = R(1-y^2)]$$

$$= 4\sqrt{\frac{6}{\pi G \rho_0 k}} \cdot \int_0^1 \frac{\mathrm{d}y}{\sqrt{(4/k-3)+(3-2/k)y^2-y^4}}$$

The integrand has a singularity at

$$y = \psi \equiv \sqrt{3/2 - 1/k + \sqrt{(1/k-1/2)(1/k+3/2)}} > 3/2$$

and the variable transformation $y^2 = \psi^2 - x^2$ gives

$$T = 4\sqrt{\frac{6}{\pi G \rho_0 k}} \cdot \int_{\psi}^{\sqrt{\psi^2-1}} \frac{-x\,\mathrm{d}x}{\sqrt{\psi^2-x^2}\sqrt{\kappa^2-x2}\sqrt{x^2}} =$$

$$= 4\sqrt{\frac{6}{\pi G \rho_0 k}} \cdot \int_{\sqrt{\psi^2-1}}^{\psi} \frac{\mathrm{d}x}{\sqrt{\psi^2-x^2}\sqrt{\kappa^2-x^2}}$$

where

$$\kappa^2 = 2\psi^2 + 2/k - 3 > \psi^2 \ .$$

Finally, the transformation $x = \psi t$ gives

$$T = 4\sqrt{\frac{6}{\pi G \rho_0 k}} \cdot \int_{\sqrt{1-1/\psi^2}}^{1} \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-\chi^2 t^2}} \cdot \frac{1}{\kappa}$$

where $\chi = \psi/\kappa < 1$.

The mass M is

$$M = 4\pi\rho_0 \int_0^R r^2(1-kr/R)\mathrm{d}r = \frac{4\pi\rho_0 R^3}{3}\left(1 - \frac{3k}{4}\right)$$

and T can be expressed in M and k instead of ρ_0 and k:

$$T = \frac{8R}{\kappa}\sqrt{\frac{2R}{GM}\left(\frac{1}{k} - \frac{3}{4}\right)} \cdot \int_{\sqrt{1-1/\psi^2}}^{1} \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-\chi^2 t^2}} \qquad (11)$$

2.3.1 Special case, $k = 2/3$

If $k = 2/3$ then $\psi = \sqrt[4]{3}$, $\kappa = \sqrt{2}\sqrt[4]{3}$, $\chi = 1/\sqrt{2}$ and we get the lower integration limit

$$\sqrt{1 - 1/\psi^2} = \sqrt{1 - 1/\sqrt{3}}\ .$$

Thus

$$T = 4R\sqrt{\frac{R\sqrt{3}}{GM}} \cdot \int_{\sqrt{1-1/\sqrt{3}}}^{1} \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-\frac{1}{2\sqrt{3}}t^2}}\ (\approx 0.9353 T_0)$$

2.4 Quadratic density function

$$\rho(r) = \rho_0(1 - kr^2/R^2), 0 < k < 1$$

$$T = \sqrt{\frac{2}{\pi G}} \int_0^R \frac{\mathrm{d}\eta}{\sqrt{h(R) - h(\eta)}}$$

where

$$\begin{aligned}
h(\eta) &= \int_0^\eta \xi\rho(\xi)\mathrm{d}\xi - \frac{1}{\eta}\int_0^\eta \xi^2\rho(\xi)\mathrm{d}\xi \\
&= \rho_0 \int_0^\eta \xi - \frac{k}{R^2}\xi^3 \mathrm{d}\xi - \rho_0 \frac{1}{\eta}\int_0^\eta \xi^2 - \frac{k}{R^2}\xi^4 \mathrm{d}\xi \\
&= \rho_0 \cdot \left\{ \frac{1}{2}\eta^2 - \frac{k}{4R^2}\eta^4 - \frac{1}{3}\eta^2 + \frac{k}{5R^2}\eta^4 \right\} \\
&= \rho_0 \cdot \left\{ \frac{1}{6}\eta^2 - \frac{k}{20R^2}\eta^4 \right\}
\end{aligned}$$

$$h(R) - h(\eta) = \rho_0 \cdot \left\{ \frac{1}{6}R^2 - \frac{1}{6}\eta^2 - \frac{k}{20R^2}R^4 + \frac{k}{20R^2}\eta^4 \right\}$$

A variable transformation $\eta = Rx$ yields

$$
\begin{aligned}
T &= \sqrt{\frac{2}{\pi G}} \int_0^1 \frac{\mathrm{d}x}{\sqrt{\frac{1}{6}(1-x^2) - \frac{k}{20}(1-x^4)}} \cdot \frac{1}{\sqrt{\rho_0}} \\
&= \sqrt{\frac{2}{\pi G}} \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}\sqrt{\frac{1}{6} - \frac{k}{20} - \frac{k}{20}x^2}} \cdot \frac{1}{\sqrt{\rho_0}} \\
&= \sqrt{\frac{2}{\pi G \rho_0}} \cdot \sqrt{\frac{60}{10-3k}} \cdot \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}\,\sqrt{1-\frac{3k}{10-3k}\,x^2}}
\end{aligned}
$$

Here K denotes a complete jacobian elliptic function of the second kind. Further,

$$
M = 4\pi\rho_0 \int_0^R \xi^2 - \frac{k}{R^2}\xi^4 \,\mathrm{d}\xi = \frac{4\pi R^3 \rho_0}{15}(5-3k)
$$

and so, finally,

$$
T = 4R\sqrt{\frac{2R}{GM}}\sqrt{\frac{5-3k}{10-3k}} \cdot \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}\,\sqrt{1-\frac{3k}{10-3k}\,x^2}} \ .
$$

2.4.1 Special case, k=0

In this case we regain $T = T_0 = 2\pi R\sqrt{R/(GM)}$ since the integral equals $\pi/2$.

2.4.2 Special case, k=1

In this case the mass density decreases to zero at the surface of the sphere as is assumed in star models.

$$
T = 4R\sqrt{\frac{2R}{GM}}\sqrt{\frac{2}{7}} \cdot \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}\,\sqrt{1-\frac{3}{7}\,x^2}} \ (\approx 0.8651 T_0) \ .
$$

2.4.3 Special case, k=2/3

$$
T = 2R\sqrt{\frac{3R}{GM}} \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}\sqrt{1-x^2/4}} \ (\approx 0.9293 T_0) \ .
$$

2.5 Generalization

We will now consider the case $\rho(r) = \rho_0(1 - s(r/R))$ where $s(x)$ is an "arbitrary" function defined for $0 \le x \le 1$ and with $s(0) = s'(0) = 0$ and $s(1) = 1$. Thus, ρ_0 is still the central density and the density is zero at the surface.

We use equation 2,

$$T = \sqrt{\frac{2}{\pi G}} \cdot \int_0^R \frac{\mathrm{d}\eta}{\sqrt{f(\eta) + g(\eta) - g(R)}}$$

where

$$f(\eta) = \int_\eta^R \xi \rho(\xi)\,\mathrm{d}\xi \qquad \text{and} \qquad g(\eta) = \frac{1}{\eta} \cdot \int_0^\eta \xi^2 \rho(\xi)\,\mathrm{d}\xi.$$

To get a somewhat simpler form, first let $\xi = Ry$ in the two integrals:

$$f(\eta) = R^2\rho_0 \int_{\eta/R}^1 y(1 - s(y))\,\mathrm{d}y \qquad \text{and}$$

$$g(\eta) = \frac{1}{\eta} \cdot R^3\rho_0 \int_0^{\eta/R} y^2(1 - s(y))\,\mathrm{d}y.$$

Next, let $\eta = Rx$ in the integral for T:

$$T = \sqrt{\frac{2}{\pi G}} \cdot \int_0^1 \frac{R\mathrm{d}x}{\sqrt{f(Rx) + g(Rx) - g(R)}}$$

where

$$\frac{f(Rx)}{R^2\rho_0} = \int_x^1 y(1 - s(y))\,\mathrm{d}y = \frac{1}{2}(1 - x^2) - \int_x^1 ys(y)\,\mathrm{d}y,$$

$$\frac{g(Rx)}{R^2\rho_0} = \frac{1}{x} \cdot \int_0^x y^2(1 - s(y))\,\mathrm{d}y = \frac{1}{3}x^2 - \frac{1}{x} \cdot \int_0^x y^2 s(y)\,\mathrm{d}y, \quad \text{and}$$

$$\frac{g(R)}{R^2\rho_0} = \int_0^1 y^2(1 - s(y))\,\mathrm{d}y = \frac{1}{3} - \int_0^1 y^2 s(y)\,\mathrm{d}y.$$

$$T = 2\sqrt{\frac{3}{\pi G \rho_0}} \cdot \int_0^1 \frac{\mathrm{d}x}{\sqrt{h(x)}} \tag{12}$$

where

$$h(x) = 1 - x^2 + 6\left\{\int_0^1 y^2 s(y)\,\mathrm{d}y - \frac{1}{x} \cdot \int_0^x y^2 s(y)\,\mathrm{d}y - \int_x^1 y s(y)\,\mathrm{d}y\right\}$$

Mass instead of central density:

$$M = 4\pi\rho_0 \int_0^R \xi^2 (1 - s(\xi/R))\,\mathrm{d}\xi = 4\pi\rho_0 R^3 \int_0^1 x^2(1 - s(x))\,\mathrm{d}x$$

$$\frac{1}{\pi\rho_0} = \frac{4R^3}{M} \cdot \int_0^1 x^2(1 - s(x))\,\mathrm{d}x = \frac{4R^3}{M}\left(\frac{1}{3} - \int_0^1 x^2 s(x)\,\mathrm{d}x\right)$$

$$\boxed{T = 4R\sqrt{\frac{R}{GM}}\sqrt{1 - 3\int_0^1 x^2 s(x)\,\mathrm{d}x} \cdot \int_0^1 \frac{\mathrm{d}x}{\sqrt{h(x)}}} \tag{13}$$

2.5.1 Earth model I

$$s(x) = \begin{cases} \dfrac{343}{10} x^2 & \text{if} \quad 0 \le x \le 1/7 \\[2ex] \dfrac{91x - 11}{90x - 10} & \text{if} \quad 1/7 \le x \le 1 \end{cases}$$

Equation (13) gives

$$T \approx 4.79R\sqrt{\frac{R}{GM}} \approx 0.76T_0 \ .$$

2.5.2 Earth model II

$$s(x) = \begin{cases} x^2 & \text{if} \quad 0 \le x < 1/5 \\ x/2 & \text{if} \quad 1/5 < x < 7/13 \\ (5x + 13)/27 & \text{if} \quad 7/13 < x < 19/21 \\ (26x - 6)/27 & \text{if} \quad 19/21 \le x < 1 \end{cases}$$

In this model the density jumps at two points inside the Earth. The calculation is left to the reader.

2.5.3 Earth model III

$$s(x) = x^a$$

where $a > 1$ is some suitable constant. We will just give some results.

Period time (min):

$$T \approx \begin{cases} 71.49 & \text{if} \quad a = 1 \\ 72.97 & \text{if} \quad a = 2 \\ 81.06 & \text{if} \quad a = 25 \\ 70.58 & \text{if} \quad a = 1/2 \\ 70.24 & \text{if} \quad a = 1/3 \\ 69.60 & \text{if} \quad 1/25 \end{cases}$$

If $a \gg 1$ then $T \approx 84$ minutes, i e the same as when $s(x) \equiv 0$. This is to be expected since $s(x) \approx 0$ except when x is very near 1, corresponding to an almost constant mass density.

Some more models:

$s(x) = 5x^6 - 5x^5 - 5x^4 + 6x^3$, $T \approx 73.13$minutes.
$s(x) = 5x^6 - 19x^5 + 24x^4 - 11x^3 + 2x^2$, $T \approx 73.13$ minutes.
$s(x) = 4x(1 - x)$, $T \approx 110.23$ minutes.

3 Tunnel through rotating Earth

Up to now we have not considered the rotation of the Earth. Now we will see what happens if rotation is taken into account. We suppose that the mass density is constant, that the rate of rotation is constant and that the Earth is still a sphere, i e flattening disregarded. According to Newton the gravitational field depends just on the mass distribution at any time[1], so at a distance $r < R$ from the centre of the Earth with radius R and mass M the gravitational acceleration is still GMr/R^3.

[1]The field can vary if the field generating matter is moving but the field does not depend *explicitly* on the motion. In Einsteins theory of gravitation, where the field is determined by the energy-momentum tensor of matter, this is not so.

Since the Earth rotates the falling marble will have horisontal velocity when it is dropped. If the Earth's matter had not been there but the field strength still proportional to r, the marble would have moved in an elliptic orbit with midpoint at the centre of the Earth. This was proved by Newton in his cumbersome though, of course, ingeneous geometrical way [3]. A much easier way is to notice that the motion is harmonic in all space directions [2]. This orbit shows how the tunnel must be drilled!

3.1 Start at the equator

The orbit will be an ellipse as seen from an inertial system S, following the centre of the Earth [3]. The falling marble will move from one end of the major axis of the ellipse to the other end as seen from S and the time elapsed will be like before, $T = T_0 = 2\pi R\sqrt{R/(GM)}$. Since the Earth is rotating the tunnel must end at the equator, but somewhat to the west of the antipode since Earth rotates from west to east. Let the coordinates of the marble be $(\xi(t),\, \eta(t))$ in the equatorial plane in S and $(x(t),\, y(t))$ in the rotating system attached to the Earth.

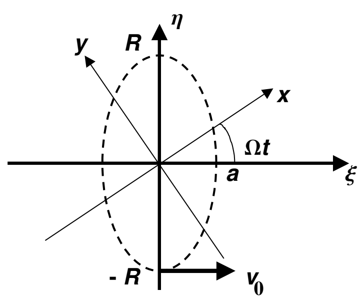

Figure 1: Rotating xy-system

Then

$$\begin{cases} \xi(t) = \dfrac{R\Omega}{\omega}\sin(\omega t) \\[2ex] \eta(t) = -R\cos(\omega t) \end{cases} \tag{14}$$

[2]This was pointed out to me by Tomas Carnstam, Lund University, Lund, Sweden.

[3]We neglect the motion of the Earth in a non-linear orbit. The motion during the free fall period is just a few minutes of arch as seen from the sun.

where $\omega = \sqrt{GM/R^3}$ and Ω is the angular velocity of the Earth.

$$\begin{cases} x(t) = \dfrac{R\Omega}{\omega}\sin(\omega t)\cos(\Omega t) - R\cos(\omega t)\sin(\Omega t) \\[2ex] y(t) = -\dfrac{R\Omega}{\omega}\sin(\omega t)\sin(\Omega t) - R\cos(\omega t)\cos(\Omega t) \end{cases}$$

$$= \frac{R}{2}\begin{cases} (\dfrac{\Omega}{\omega} - 1)\sin[(\omega+\Omega)t] + (\dfrac{\Omega}{\omega}+1)\sin[(\omega-\Omega)t] \\[2ex] -(\dfrac{\Omega}{\omega}+1)\cos[(\omega+\Omega)t] + (\dfrac{\Omega}{\omega}-1)\cos[(\omega+\Omega)t] \end{cases} \tag{15}$$

When $t = T/2$, $(\xi, \eta) = (0, R)$ and then from (14), $\omega \cdot T/2 = \pi$, so $T = T_0 = 2\pi R\sqrt{R/(GM)}$, i e the passage time is independent of the rate of rotation of the Earth.

From the first part of (15),

$$\begin{cases} x(T/2) = R\sin(\dfrac{\Omega T}{2}) = R\sin(\dfrac{\pi\Omega}{\omega}) \\[2ex] y(T/2) = R\cos(\dfrac{\Omega T}{2}) = R\cos(\dfrac{\pi\Omega}{\omega}) \end{cases}$$

Thus, the location of the end of the tunnel is on the equator at a distance $a \equiv R\pi\Omega/\omega$ from the antipode, as measured along the equator. The minimal distance from the orbit/tunnel to the centre of the Earth can also be found from (15).

$$x^2 + y^2 = \frac{R^2\Omega^2}{\omega^2} + R^2\left(1 - \frac{\Omega^2}{\omega^2}\right)\cos^2(\omega t)$$

so the smallest distance is $d \equiv R\Omega/\omega$. For the Earth $a \approx 1175\,\mathrm{km}$ corresponding to a longitude difference of around $10,6°$ from the antipode, and $d \approx 374\,\mathrm{km}$.

3.2 General case

If the marble is dropped at any point except from a pole the orbit as viewed from the reference frame will still be an ellipse.

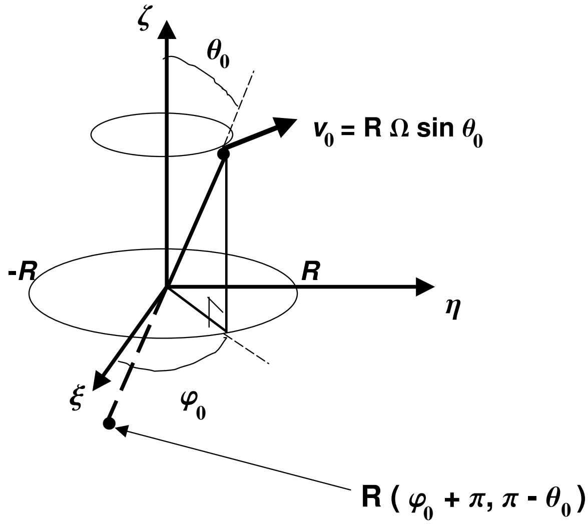

Figure 2: Marble dropped from any point

$$\begin{cases} \xi(t) = \dfrac{R\Omega}{\omega}\sin(\theta_0)\sin(\omega t) \\ \\ \eta(t) = R\sin(\theta_0)\cos(\omega t) \\ \\ \zeta(t) = R\cos(\theta_0)\cos(\omega t) \end{cases} \tag{16}$$

$$\begin{cases} x(t) = \xi(t)\cos(\Omega t) - \eta(t)\sin(\Omega t) = R\sin(\theta_0)[\Omega/\omega\sin(\omega t)\cos(\Omega t) - \cos(\omega t)\sin(\Omega t)] \\ \\ y(t) = -\xi(t)\sin(\Omega t) - \eta(t)\cos(\Omega t) = R\sin(\theta_0)[-\Omega/\omega\sin(\omega t)\sin(\Omega t) - \cos(\omega t)\cos(\Omega t)] \\ \\ z(t) = \zeta(t) = R\cos(\theta_0)\cos(\omega t) \end{cases} \tag{17}$$

At the end of the tunnel $t = T/2 = \pi/\omega$, $\omega t = \pi$ and (17) gives

$$\begin{cases} x(T/2) = R\sin(\theta_0)\sin(\pi\Omega/\omega) \\ \\ y(T/2) = R\sin(\theta_0)\cos(\pi\Omega/\omega) \\ \\ z(T/2) = -R\cos(\theta_0) \end{cases}$$

An alternative form is

$$\begin{cases} x(t) = \dfrac{R}{2}\sin(\theta_0)\left\{\left(\dfrac{\Omega}{\omega}-1\right)\sin[(\omega+\Omega)t] + \left(\dfrac{\Omega}{\omega}+1\right)\sin[(\omega-\Omega)t]\right\} \\ \\ y(t) = \dfrac{R}{2}\sin(\theta_0)\left\{-\left(\dfrac{\Omega}{\omega}+1\right)\cos[(\omega-\Omega)t] + \left(\dfrac{\Omega}{\omega}-1\right)\cos[(\omega+\Omega)t]\right\} \\ \\ z(t) = R\cos(\theta_0)\cos(\omega t) \end{cases} \tag{18}$$

by which we can get the smallest distance from the centre:

$$x^2+y^2+z^2 = \frac{R^2}{2}\sin^2(\theta_0)\cdot\left\{1+\frac{\Omega^2}{\omega^2}+\left(1-\frac{\Omega^2}{\omega^2}\right)\cos(2\omega t)\right\}+R^2\cos^2(\theta_0)\cos^2(\omega t) =$$

$$= \frac{R^2}{2}\left\{1+\frac{\Omega^2}{\omega^2}\sin^2(\theta_0)+\left(1-\frac{\Omega^2}{\omega^2}\sin^2(\theta_0)\right)\cos(2\omega t)\right\}$$

The smallest distance from the centre (corresponding to $\omega t = \pi/2$) is $R\Omega\sin(\theta_0)/\omega$.

4 General relativistic calculation of period time

Using general relativity things become much more complicated. However, for constant mass density (and, as before, no rotation and spherical symmetry) the exact metric is known. We will derive a differential equation, from which the period time of the falling marble is found numerically.

4.1 The metric

$$d\tau^2 = -A(r)\ \mathrm{d}r^2 - r^2(\ \mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\varphi^2) + c^2\,B(r)\ \mathrm{d}t^2$$

and the energy-momemtum tensor

$$T_{\mu\nu} = pg_{\mu\nu} + (p+\rho)u_\mu u_\nu$$

is assumed (perfect fluid). Einsten's field equations give (see [5], ch. 11 or [4]) the following functions $A(r)$ and $B(r)$.

Outside the star, $r \geq R$:

$$B(r) = Ar^{-1} = 1 - \frac{2GM}{r} \quad \text{where} \quad M = 4\pi R^3 \rho/3 \ .$$

Inside the star, $r < R$:

$$\begin{cases} A(r) = \left[1 - \dfrac{2GMr^2}{c^2R^3}\right]^{-1} \\[2ex] B(r) = \dfrac{1}{4}\left\{3 \cdot \sqrt{1 - \dfrac{2GM}{c^2R}} - \sqrt{1 - \dfrac{2GMr^2}{c^2R^3}}\right\}^2 \end{cases}$$

The marble will follow a grodesic with constant θ and φ. We need those ${\Gamma^1}_{ab}$ which are non-zero. These are (prime denotes derivative with respect to r and the coordinates are numbered $(x^1, x^2, x^3, x^4) = (r, \theta, \varphi, t)$)

$$\begin{cases} {\Gamma^1}_{11} = \dfrac{1}{2} g^{11} \cdot \dfrac{\partial g_{11}}{\partial r} = \dfrac{A'}{2A} \\[2ex] {\Gamma^1}_{22} = -\dfrac{1}{2} g^{11} \cdot \dfrac{\partial g_{22}}{\partial r} = -\dfrac{r}{A} \\[2ex] {\Gamma^1}_{33} = -\dfrac{1}{2} g^{11} \cdot \dfrac{\partial g_{33}}{\partial r} = -\dfrac{r}{A} \sin^2\theta \\[2ex] {\Gamma^1}_{44} = -\dfrac{1}{2} g^{11} \cdot \dfrac{\partial g_{44}}{\partial r} = \dfrac{c^2 B'}{2A} \end{cases}$$

The geodesic equations and the metric give (dot denotes derivative with respect to τ)

$$\begin{cases} \ddot{r} + \dfrac{A'}{2A}\dot{r}^2 + \dfrac{c^2B'}{2A}\dot{t}^2 & = & 0 \\ -A\dot{r}^2 + c^2 B\dot{t}^2 & = & 0 \end{cases}$$

Elimination of $\dot{t}^2$ gives

$$\ddot{r} + \frac{A'}{2A}\dot{r}^2 + \frac{B'}{2A} \cdot \frac{1 + A\dot{r}^2}{B} = 0$$

from which

$$\ddot{r} = -\frac{1}{2}\left\{\frac{A'}{A} + \frac{B'}{B}\right\}\dot{r}^2 - \frac{B'}{2AB} \tag{19}$$

This equation has been solved approximately by means of the Runge-Kutta method[4].

For the Earth

$$\begin{cases} M & \approx & 5.972 \cdot 10^{24}\,\mathrm{kg} \\ 2m & \approx & 8.870 \cdot 10^{-3}\,\mathrm{m} \\ R & \approx & 6.371 \cdot 10^{6}\,\mathrm{m} \end{cases}$$

and $G \approx 6.6743 \cdot 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$.

The period time turns out to be $1.99\,\mu$s *shorter* than the newtonian value T_0.

5 Conclusion

We have derived the period time for free fall through the Earth. General expressions have been found with mass density an arbitrary function of the distance from the centre of the Earth, disregarding rotation. For the case of constant density rotation has been considered, the necessary shape of the tunnel and the period time were calculated. Finally, for constant density, a general relativistic calculation of the period time was performed.

For further research it would be interesting to try to take flattening into account and find the orbit of the marble relative the rotating Earth. Also it could be of some interest to apply the general relativistic method to bodies such as neutron stars.

References

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