International J.Math. Combin. Vol.2(2016), 43-50

Entire Equitable Dominating Graph

B.Basavanagoud[1], V.R.Kulli[2] and Vijay V. Teli*[3]

[1]Department of Mathematics, Karnatak University, Dharwad - 580 003, India

[2]Department of Mathematics, Gulbarga University, Gulbarga - 585 106, India

[3]Department of Mathematics, KLS's, Vishwanathrao Deshpande Rural, Institute of Technology, Haliyal - 581 329, India

E-mail: b.basavanagoud@gmail.com, vrkulli@gmail.com, vijayteli22@gmail.com

Abstract: The entire equitable dominating graph $EE_qD(G)$ of a graph G with vertex set $V \cup S$, where S is the collection of all minimal equitable dominating sets of G and two vertices $u, v \in V \cup S$ are adjacent if u, v are not disjoint minimal equitable dominating sets in S or $u, v \in D$, where D is the minimal equitable dominating set in S or $u \in V$ and v is a minimal equitable dominating set in S containing u. In this paper, we initiate a study of this new graph valued function and also established necessary and sufficient conditions for $EE_qD(G)$ to be connected and complete. Other properties of $EE_qD(G)$ are also obtained.

Key Words: Dominating set, equitable dominating set, entire equitable dominating graph, Smarandachely dominating set.

AMS(2010): 05C69(70).

§1. Introduction

All graphs considered here are finite, undirected with no loops and multiple edges. We denote by p the order(i.e number of vertices) and by q the size (i.e number of edges) of such a graph G. Any undefined term and notation in this paper may be found in Harary [5].

A set of vertices which covers all the edges of a graph G is called *vertex cover* for G. The smallest number of vertices in any vertex cover for G is called its *vertex covering number* and is denoted by $\alpha_0(G)$ or α_0. A set of vertices in G is *independent* if no two of them are adjacent. The largest number of vertices in such a set is called the *vertex independence number* of G and is denoted by $\beta_0(G)$ or β_0. The *connectivity* $\kappa = \kappa(G)$ of a graph G is the minimum number of vertices whose removal results a disconnected or trivial graph. Analogously the *edge-connectivity* $\lambda = \lambda(G)$ is the minimum number of edges whose removal results a disconnected or trivial graph. The *diameter* of a connected graph is the maximum distance between two vertices in G and is denoted by $diam(G)$. If G and H are graphs with the property that the identification of any vertex of G with an arbitrary vertex of H results in a unique graph (up to

[1]Supported by UGC-SAP DRS-II New Delhi, India: for 2010-2015 and the University Grants Commission, New Delhi, India. No.F.4-1/2006(BSR)/7-101/2007(BSR) dated: 20th June, 2012.

[2]Received April 18, 2015, Accepted May 13, 2016.

isomorphism), then we write as $G \bullet H$ for this graph.

A subset D of V is called a *dominating set* of G if every vertex in $V - D$ is adjacent to at least one vertex in D. The *domination number* $\gamma(G)$ of G is the minimum cardinality taken over all minimal dominating sets of G. (See Ore [12]).

A subset D of V is called an *equitable dominating set* if for every $v \in V - D$, there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$. The minimum cardinality of such a dominating set is called the *equitable domination number* of G and is denoted by $\gamma^e(G)$. For more details about graph valued functions, domination number and their related parameters we refer [1-4, 6 - 10, 12]. The opposite of equitable dominating set is the *Smarandachely dominating set* with $|deg(u) - deg(v)| \leq 1$ for $\forall uv \in E(G)$.

The purpose of this paper is to introduce a new graph valued function in the field of domination theory in graphs.

§2. Entire Equitable Dominating Graph

Definition 2.1 *The entire equitable dominating graph $EE_qD(G)$ of a graph G with vertex set $V \cup S$, where S is the collection of all minimal equitable dominating sets of G and two vertices $u, v \in V \cup S$ adjacent if u, v are not disjoint minimal equitable dominating sets in S or $u, v \in D$, where D is the minimal equitable dominating set in S or $u \in V$ and v is a minimal equitable dominating set in S containing u.*

In Fig.1, a graph G and its entire equitable dominating graph $EE_qD(G)$ are shown. Here $D_1 = \{1, 3\}$, $D_2 = \{1, 4\}$, $D_3 = \{2, 3\}$ and $D_4 = \{2, 4\}$ are minimal equitable dominating sets of G.

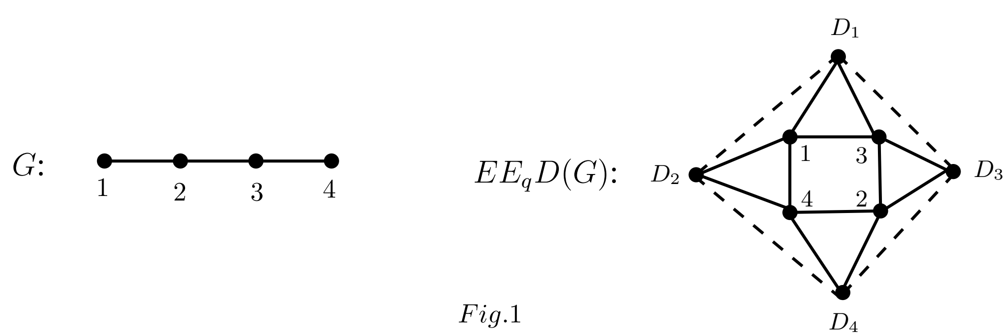

*Fig.*1

§3. Preliminary Results

The following will be useful in the proof of our results.

Theorem 3.1([5]) *For any nontrivial graph G, $\alpha_0 + \beta_0 = p = \alpha_1 + \beta_1$.*

Theorem 3.2([5]) *A connected graph G is Eulerian if and only if every vertex of G has even degree.*

§4. Results

First we obtain a necessary and sufficient condition on a graph G such that the entire equitable dominating graph $EE_qD(G)$ is connected.

Theorem 4.1 *For any graph G with at least three vertices, the entire equitable dominating graph $EE_qD(G)$ is connected if and only if $\Delta(G) < p-1$.*

Proof Let $\Delta(G) < p-1$ and u, v be any two vertices in G. We consider the following cases:

Case 1. If u and v are adjacent vertices in G, then there exist two not disjoint minimal equitable dominating sets D_1 and D_2 containing u and v respectively. Therefore by the definition 2.1, u and v are adjacent in $EE_qD(G)$.

Case 2. Suppose there exist two vertices $u \in D_1$ and $v \in D_2$ such that u and v are not adjacent in G. Then there exists a minimal equitable dominating set D_3 containing both u and v and by definition 2.1, D_1 and D_2 are connected in $EE_qD(G)$.

Conversely, suppose $EE_qD(G)$ is connected. Suppose $\Delta(G) = p-1$ and u is a vertex of degree $p-1$. Then the degree of u in $EE_qD(G)$ is minimum. If every vertex of G has degree $p-1$, then every vertex of G forms a minimal equitable dominating set. Therefore $EE_qD(G)$ has at least two components, a contradiction. Thus $\Delta(G) < p-1$. □

Proposition 4.1 *$EE_qD(G) = pk_2$ if and only if $G = K_p; p \geq 2$.*

Proof Suppose $G = K_p; p \geq 2$. Then clearly each vertex of G will form a minimal equitable dominating set. Hence by definition 2.1, $EE_qD(G) = pK_2$.

Conversely, suppose $EE_qD(G) = pK_2$ and $G \neq K_p$. Then there exists at least one minimal equitable dominating set D containing two vertices of G. Then D will form C_3 in $EE_qD(G)$, a contradiction. Hence $G = K_p; p \geq 2$. □

Theorem 4.2 *For any graph G, $EE_qD(G)$ is either connected or it has at least one component which is K_2.*

Proof If $\Delta(G) < p-1$, then by Theorem 4.1, $EE_qD(G)$ is connected. If G is complete graph $K_p; p \leq 2$ and by Proposition 4.1, then each component of $EE_qD(G)$ is K_2.

Next, we must prove that $\delta(G) < \Delta(G) = p-1$. Let $v_1, v_2, \cdots, v_n$ be the set of vertices in G such that $deg(v_i) = p-1$, then it is clear that $\{v_i\}$ forms a minimal equitable dominating set and which forms a component isomorphic to K_2. Hence $EE_qD(G)$ has at least one component which is K_2. □

In the next theorem, we characterize the graphs G for which $EE_qD(G)$ is complete.

Theorem 4.3 *$EE_qD(G) = K_{p+2}$ if and only if G is $K_{1,p}; p \geq 3$.*

Proof Suppose $G = K_{1,p}; p \geq 3$. Then there exists a minimal equitable dominating set D

contains all the vertices of G i.e $|D| = |\{u, v_1, v_2, v_3, \cdots, v_p\}| = p+1$. Hence $EE_qD(G) = K_{p+2}$.

Conversely, $EE_qD(G) = K_{p+2}$, then we prove that G is $K_{1,p}; p \geq 3$. Let us suppose that, $G \neq K_{1,p}; p \geq 3$. Then there exists a minimal equitable dominating set D of cardinality is maximum p i.e $|D| = |\{v_1, v_2, v_3, \cdots, v_p\}| = p$, a contradiction. Therefore G must be $K_{1,p}; p \geq 3$. □

Theorem 4.4 *Let G be a nontrivial connected graph of order p and size q. The entire equitable dominating graph is a graph with order $2p$ and size p if and only if $G = K_p; p \geq 2$.*

Proof Let G be a complete graph with $p \geq 2$, then by Proposition 4.1, $G = K_p;\ p \geq 2$.

Conversely, suppose $EE_qD(G)$ be a $(2p, p)$ graph. Then pK_2 is the only graph with order $2p$ and size q. □

In the next results, we obtain the bounds on the order and size of $EE_qD(G)$.

Theorem 4.5 *For any graph G, $2p \leq p' \leq \frac{p(p-1)}{2} + 1$, where p' denotes the number of vertices in $EE_qD(G)$. Further, the lower bound is attained if and only if G is either P_4 or $K_p; p \geq 2$ and upper bound is attained if and only if G is $K_3 \cup K_2$, $K_3 \bullet K_2$ or $C_4 \cup K_1$.*

Proof The lower bound follows from the fact that the twice the number of vertices in G and the upper bound follows that the maximum number of edges in G.

Suppose the lower bound is attained. Then every vertex of G forms a minimal equitable dominating set or every vertex of G is in exactly two minimal equitable dominating sets. This implies that the necessary condition.

Conversely, suppose G is P_4 or $K_p; p \geq 2$. Then by definition of entire equitable dominating graph, $V(EE_qD(G)) = 2p$. If the upper bound is attained. Then G must be one of the following graphs are $K_3 \cup K_2$, $K_3 \bullet K_2$ or $C_4 \cup K_1$.

If $G = K_3 \cup K_2$, then every vertex of G is in exactly two minimal equitable dominating sets hence

$$V(EE_qD(G)) = \frac{p(p-1)}{2} + 1 = \frac{pq}{2} + 1.$$

Suppose $G = K_3 \bullet K_2$. Then the pendant vertex of G is in all the minimal equitable dominating sets and forms $(p-1)$ minimal equitable dominating sets. Therefore the upper bound holds.

Now if G is $C_4 \cup K_1$. Then every equitable dominating sets contains an isolated vertex and they are not disjoint sets and by definition 2.1. Therefore upper bound holds.

Conversely, suppose G is one of the following graphs $K_3 \cup K_2$, $K_3 \bullet K_2$ or $C_4 \cup K_1$. Then it is obvious that $V(EE_qD(G)) = \frac{p(p-1)}{2} + 1$. □

Theorem 4.6 *For any graph G, $p \leq q' \leq \frac{p(p+1)}{2} + 1$, where q' denotes the number of edges in $EE_qD(G)$. Further, the lower bound is attained if and only $G = K_p \geq 2$ and the upper bound is attained if and only if G is $K_3 \cup K_1$.*

Proof The proof follows from Theorem 4.5. □

In the next result, we find the diameter of $EE_qD(G)$.

Theorem 4.7 *Let G be any graph with $\Delta(G) < p-1$, then diam$(EE_qD(G)) \leq 2$, where diam(G) is the diameter of G.*

Proof Let G be any graph with $\Delta(G) < p-1$, then by Theorem 4.1, $EE_qD(G)$ is connected. Let u, v be any arbitrary vertices in $EE_qD(G)$. We consider the following cases.

Case 1. Suppose $u, v \in V$, u and v are nonadjacent in G. Then there exists a minimal equitable dominating set containing u and v and by definition 2.1, $d_{EE_qD(G)}(u,v) = 1$. If u and v are adjacent in G and there is no minimal equitable dominating set containing u and v, then there exists another vertex $w \in V$ which is not adjacent to both u and v. Let D_1 and D_2 be two minimal equitable dominating sets containing (u, w) and (w, v) respectively. This implies that $d_{EE_qD(G)}(u,v) = 2$.

Case 2. Suppose $u \in V$ and $v \in S$. Then $v = D$ is a minimal equitable dominating set of G. If $u \in S$, then u and v are adjacent in $EE_qD(G)$. Otherwise, there exists another vertex $w \in D$. This implies that

$$d_{EE_qD(G)}(u,v) \leq d_{EE_qD(G)}(u,w) + d_{EE_qD(G)}(w,v) = 2.$$

Case 3. Suppose $u, v \in S$. Then $u \in D_1$ and $v \in D_2$ are two minimal equitable dominating sets of G and by Definition 2.1, $d_{EE_qD(G)}(u,v) = 1$. □

We now characterize graphs G for which $SE_qD(G) = EE_qDG$. A *semientire equitable dominating graph* $SE_qD(G)$ of a graph G is the graph with vertex set $V \cup S$ and two vertices $u, v \in V \cup S$ adjacent if $u, v \in D$, where D is a minimal equitable dominating set or $u \in V$ and $v = D$ is a minimal equitable dominating set containing u ([1]).

Proposition 4.2([3]) *The semientire equitable dominating graph $SE_qD(G)$ is pK_2 if and only if $G = K_p$; $p \geq 2$.*

Remark 4.1([3]) For any graph G, $SE_qD(G)$ is a subgraph of $EE_qD(G)$.

Theorem 4.8 *For any graph G, $SE_qD(G) \subseteq EE_qD(G)$. Further, equality G, $SE_qD(G) = EE_qD(G)$ if and only if G has exactly one minimal equitable dominating set containing all vertices of G.*

Proof By Remark 4.1, $SE_qD(G) \subseteq EE_qD(G)$. Suppose $SE_qD(G) = EE_qD(G)$. Then by Theorem 4.3, D is the only minimal equitable dominating set contains all the vertices of G. Therefore G must be $K_{1,n}; n \geq 3$.

The converse is obvious. □

In the next results, we discuss about α_0 and β_0 of $EE_qD(G)$.

Theorem 4.9 *For any graph G with no isolated vertices,*

(1) $\alpha_0(EE_qD(G)) = |S| + 1$, *where S is the collection of all minimal equitable dominating*

sets of G;

(2) $\beta_0(EE_qD(G)) = \gamma(G)$.

Proof (i) Let G be graph of order p. Let $S = \{s_1, s_2, \cdots s_i\}$ be the set of all minimal equitable dominating sets. Then by definition 2.1 and Theorem **??**. Therefore the minimum number of vertices in $EE_qD(G)$ which covers all the edges. Hence $\alpha_0(EE_qD(G)) = |S| + 1$.

(ii) By definition of $EE_qD(G)$, for any vertex v_i ; $1 \leq i \leq p$ of $EE_qD(G)$ are not adjacent. Hence these vertices forms a maximum independent set of $EE_qD(G)$. Hence (ii) follows. □

In the next two results, we prove the vertex connectivity and edge- connectivity of $EE_qD(G)$.

Theorem 4.10 *For any graph G, $\kappa(EE_qD(G)) = min\{min(deg_{EE_qD(G)_{1\leq i\leq p}} v_i), min_{1\leq j\leq n}|S_j|\}$, where S_j's is the collection of all minimal equitable dominating sets of G.*

Proof Let G be any graph with order p and size q. We consider the following cases.

Case 1. Let $u \in v_i'(EE_qD(G))$ for some i, having the minimum degree among all v_i' in $EE_qD(G)$. If the degree of u is less than any other vertex in $EE_qD(G)$, then by deleting the vertices which are adjacent to u, results a disconnected graph.

Case 2. Let $v \in S_j$ for some j, having the minimum degree among all S_j's in $EE_qD(G)$. If degree of v is less than any other vertex in $EE_qD(G)$, then by deleting all the vertices which are adjacent to v. This results the graph is disconnected. Hence the result follows. □

Theorem 4.11 *For any graph G, $\lambda(EE_qD(G)) = min\{min(deg_{EE_qD(G)_{1\leq i\leq p}} v_i), min_{1\leq j\leq n}|S_j|\}$, where S_j's is the collection of all minimal equitable dominating sets of G.*

Proof Let G be any (p, q) graph. We consider two cases.

Case 1. Let $u \in v_i'(EE_qD(G))$, having minimum degree among all v_i' in $EE_qD(G)$. If the degree of u is less than any other vertex in $EE_qD(G)$, then by deleting those edges of $EE_qD(G)$ which are incident with u, results a disconnected graph.

Case 2. Let $v \in S_j$, having the minimum degree among all vertices of S_j. If degree of v is less than any other vertex in $EE_qD(G)$, then by deleting those edges which are adjacent to v, results in a disconnected. Hence the result follows. □

Next, we prove the necessary and sufficient condition for $EE_qD(G)$ to be Eulerian.

Theorem 4.12 *For any graph G, $EE_qD(G)$ is Eulerian if and only if one of the following conditions are satisfied:*

(1) *There exists a vertex $u \in V$ is in all minimal equitable dominating sets and cardinality of every minimal equitable dominating set D of G is even;*

(2) *If $v \in V$ is a vertex of odd degree, then it is in odd number of minimal equitable dominating sets, otherwise it is in even number of minimal equitable dominating sets of G.*

Proof Suppose $\Delta < p-1$ and by Theorem 4.1, $EE_qD(G)$ is connected. Suppose $EE_qD(G)$

is Eulerian. on the contrary if condition (i) is not satisfied, then there exists a minimal equitable dominating set contains odd number of vertices and does not contains a vertex of odd degree, a contradiction. Therefore by Theorem 3.2, $EE_qD(G)$ is Eulerian. Hence condition (1) holds.

Suppose (2) does not hold. Then there exists $v \in V$ of even degree which is in odd number of minimal equitable dominating sets, a contradiction. Hence (ii) hold.

Conversely, suppose the conditions (1) and (2) are satisfied. Then every vertex of $EE_qD(G)$ has even degree and hence $EE_qD(G)$ is Eulerian. □

§5. Domination in $EE_qD(G)$

We calculate the domination number of $EE_qD(G)$ of some standard class of graphs.

Theorem 5.1 *For any graph G with no isolated vertices.*

(1) *If $G = K_p; p \geq 2$, then $\gamma(EE_qD(K_p) = p$;*

(2) *If $G = K_{1,p}; p \geq 3$, then $\gamma(EE_qD(K_{1,p}) = 1$;*

(3) *If $G = C_p$, $p \geq 3$, then $\gamma(EE_qD(C_p) = 2$.*

Theorem 5.2 *For any graph G, $\gamma(EE_qD(G)) = 1$, if and only if G is $K_{1,p}; p \geq 3$.*

Proof If G is $K_{1,p}; p \geq 3$, then there exists a minimal equitable dominating set D contains all the vertices of G and by Theorem **??**, it is clear that, $EE_qD(G)$ is complete. Hence $\gamma(EE_qD(G)) = 1$.

Conversely, suppose $\gamma(EE_qD(G)) = 1$ and $G \neq K_{1,p}; p \geq 3$. Then there exists a minimal dominating set D in $EE_qD(G)$ of cardinality greater than or equal to 2, a contradiction. Therefore G must be $K_{1,p}; p \geq 3$. □

We conclude this paper by exploring one open problem on $EE_qD(G)$.

Problem 1. *Give necessary and sufficient condition for a given graph G is entire equitable dominating graph of some graph.*

References

[1] B.Basavanagoud, V.R.Kulli and Vijay V.Teli, Equitable Dominating Graph, *Int. J. of Mathematical Science & Engineering Applications*, 9 (2), (2015), 109 - 114.

[2] B.Basavanagoud, V.R.Kulli and Vijay V.Teli, Equitable Total Domination in Graphs, *Journal of Computer and Mathematical Sciences*, 5 (2), (2014), 235-241.

[3] B.Basavanagoud, V.R.Kulli and Vijay V.Teli, Semientire Equitable Dominating Graph, *Intenational Journal of Mathematical Combinatorics*, 3 (2014), 49-54.

[4] B.Basavanagoud, V.R.Kulli and Vijay V.Teli, Equitable Total Minimal Dominating Graph, *International Research Journal of Pure Algebra*, 3 (10), (2013), 307-310.

[5] F.Harary, *Graph Theory*, Addison-Wesley, Reading, Mass, (1969).

[6] T.W.Haynes, S.T.Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, (1998).

[7] T.W.Haynes, S.T.Hedetniemi and P.J.Slater, *Domination in Graphs - Advanced Topics*, Marcel Dekker, Inc., New York, (1998).

[8] V.R.Kulli, *Theory of Domination in Graphs*, Vishwa International Publications, Gulbarga, India (2010).

[9] V.R.Kulli, *Advances in Domination Theory I*, ed. V.R.Kulli, Vishwa International Publications, Gulbarga, India (2012).

[10] V.R.Kulli, *Advances in Domination Theory II*, ed. V.R.Kulli, Vishwa International Publications, Gulbarga, India (2013).

[11] V.R.Kulli, On Entire Equitable Dominating Transformation Graphs and Fuzzy Transformation Graphs, *Journal of Computer and Mathematical Sciences*, 6(8), (2015) 449-454.

[12] O.Ore, *Theory of Graphs*, Amer. Math. Soc. Colloq. Publ., 38, Providence, (1962).

[13] V.Swaminathan and K.M.Dharmalingam, Degree Equitable Domination on Graphs, *Kragujevak Journal of Mathematics*, 35(1), 191-197 (2011).