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Nonsplit Geodetic Number of a Graph

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Abstract: Let G be a graph. If $u, v \in V(G)$, a $u - v$ geodesic of G is the shortest path between u and v. The closed interval I[u, v] consists of all vertices lying in some u - v geodesic of G. For $S \subseteq V(G)$ the set I[S] is the union of all sets I[u, v] for $u, v \in S$. A set S is a geodetic set of G if $I(S) = V(G)$. The cardinality of a minimum geodetic set of G is the geodetic number of G, denoted by g(G). In this paper, we study the nonsplit geodetic number of a graph $g_{ns}(G)$. The set $S \subseteq V(G)$ is a nonsplit geodetic set in G if S is a geodetic set and $\langle V(G) - S\rangle$ is connected, nonsplit geodetic number $g_{ns}(G)$ of G is the minimum cardinality of a nonsplit geodetic set of G. We investigate the relationship between nonsplit geodetic number and geodetic number. We also obtain the nonsplit geodetic number in the cartesian product of graphs.

Key Words: Cartesian products, distance, edge covering number, Smarandachely k-geodetic set, geodetic number, vertex covering number.

AMS(2010): 05C05, 05C12.

§1. Introduction

As usual $n = |V|$ and $m = |E|$ denote the number of vertices and edges of a graph G respectively. The graphs considered here are finite, undirected,simple and connected. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. It is well known that this distance is a metric on the vertex set $V(G)$. For a vertex v of G, the eccentricity $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is radius, rad G and the maximum eccentricity is the diameter, diam G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic. We define I[u, v] to the set (interval) of all vertices lying on some $u - v$ geodesic of G and for a nonempty subset S of V (G), $I[S] = \cup_{u,v\in S} I[u, v]$. A set S of vertices of G is called a geodetic set in G if $I[S] = V(G)$, and a geodetic set of minimum cardinality is a minimum geodetic set, and generally, if there is a k-subset T of $V(G)$ such that $I(S) \bigcup T = V(G)$, where $0 \leq k < |G| - |S|$, then S is called a *Smarandachely k-geodetic set* of G. The cardinality of a minimum geodetic set in G is called

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the geodetic number and is denoted by $g(G)$. The concept of geodetic number of a graph was introduced in [1, 4, 7], further studied in [2, 3], and the split geodetic number of a graph was introduced in [10]. It was shown in [7] that determining the geodetic number of a graph is an NP-hard problem.

A set of vertices S in a graph G is a nonsplit geodetic set if S is a geodetic set and the subgraph $G[V-S]$ induced by $\langle V(G)-S\rangle$ is connected. The minimum cardinality of a nonsplit geodetic set, denoted $g_{ns}(G)$, is called the nonsplit geodetic number of G.

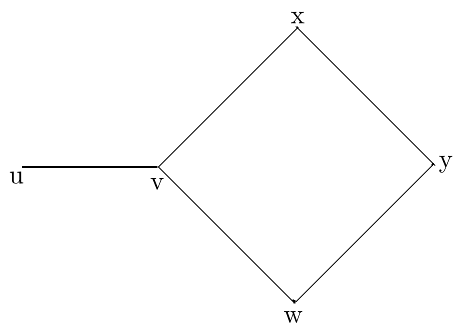

Figure 1.1

Consider the graph G of Figure 1.1. For the vertices u and y in G $d(u,y)=3$ and every vertex of G lies on an $u-y$ geodesic in G. Thus $S=\{u,y\}$ is the geodetic set of G and so $g(G)$. Here the induced subgraph $\langle V(G)-S\rangle$ is connected. So that S is a minimum nonsplit geodetic set of G. Therefore nonsplit geodetic number $g_{ns}(G)=2$.

A vertex v is an extreme vertex in a graph G, if the subgraph induced by its neighbours is complete. A vertex cover in a graph G is a set of vertices that covers all edges of G. The minimum number of vertices in a vertex cover of G is the vertex covering number $\alpha_0(G)$ of G. An edge cover of a graph G without isolated vertices is a set of edges of G that covers all the vertices of G. The edge covering number $\alpha_1(G)$ of a graph G is the minimum cardinality of an edge cover of G. For any undefined term in this paper, see [1, 6]

§2. Preliminary Notes

We need the following results to prove our results.

Theorem 2.1 *Every geodetic set of a graph contains its extreme vertices.*

Theorem 2.2 *For any tree T with k pendant vertices, $g(T)=k$.*

Theorem 2.3 *For any graph G of order n, $\alpha_1(G)+\beta_1(G)=n$.*

Theorem 2.4 *For cycle C_n of order $n \geq 3$,*

$$g(C_n) = \begin{cases} 2 & \text{if } n \text{ even}, \\ 3 & \text{if } n \text{ odd}. \end{cases}$$

Theorem 2.5 *If G is a nontrivial connected graph, then $g(G) \leq g(G \times K_2)$.*

§3. Nonsplit Geodetic Number

Theorem 3.1 *For cycle C_n of order $n \geq 3$,*

$$g_{ns}(C_n) = \begin{cases} \frac{n}{2} + 1 & \text{if } n \text{ is even}, \\ \lfloor \frac{n}{2} \rfloor + 2 & \text{if } n \text{ is odd}. \end{cases}$$

Proof Suppose C_n be cycle with $n \geq 3$, we have the following

Case 1. Let n be even. Consider $C_{2p} = \{v_1, v_2, \cdots, v_{2p}, v_1\}$ be a cycle with 2p vertices. Then v_{p+1} is the antipodal vertex of v_1. Suppose $S = \{v_1, v_{p+1}\}$ be the geodetic set of G. It is clear that $\langle V(G) - S \rangle$ is not connected. Thus S is not a nonsplit geodetic set. But $S' = \{v_1, v_2, \cdots, v_{p+1}\}$ is a nonsplit geodetic set of G. So that $g_{ns}(G) \leq (p+1)$. If S_1 is any set of vertices of G with $|S_1| < |S'|$ then S_1 contains at most p-elements. Hence $V(G) - S_1$ is not connected. This follows that $g_{ns}(G) = p + 1 = \frac{n}{2} + 1$.

Case 2. Let n be odd. Consider $C_{2p+1} = \{v_1, v_2, \cdots, v_{2p+1}, v_1\}$ be a cycle with 2p+1 vertices. Then v_{p+1} and v_{p+2} are the antipodal vertices of v_1. Now consider $S = \{v_1, v_{p+1}, v_{p+2}\}$ be the geodetic set of G and it is clear that $\langle V(G) - S \rangle$ is not connected. Thus S is not a nonsplit geodetic set. But $S' = \{v_1, v_2, \cdots, v_{p+1}, v_{p+2}\}$ is a nonsplit geodetic set of G so that $g_{ns}(G) \leq p + 2$. If S_1 is any set of vertices of G with $|S_1| < |S'|$ then S_1 contains at most p-elements. Hence $\langle V(G) - S_1 \rangle$ is not connected. This follows that

$$g_{ns}(G) = p + 2 = \left\lfloor \frac{n}{2} \right\rfloor + 2. \qquad \square$$

Theorem 3.2 *For any nontrivial tree T with k-pendant-vertices, then $g_{ns}(T) = k$.*

Proof Let $S = \{v_1, v_2, \cdots, v_k\}$ be the set containing pendant vertices of a tree T. By Theorem 2.2, $g(T) \geq |S|$. On the other hand, for an internal vertex v of T there exist pendant vertices x,y of T such that v lies on the unique x-y geodesic in T. Thus, $v \in I[S]$ and $I[S] = V(T)$.Then $g(T) \leq |S|$. Thus S itself a minimum geodetic set of T.Therefore $g(T) = |S| = k$ and $\langle V - S \rangle$ is connected. Hence $g_{ns}(T) = k$. □

Theorem 3.3 *For any integers $r, s \geq 2$,$g_{ns}(K_{r,s}) = r + s - 1$.*

Proof Let G $=K_{r,s}$, such that U $= \{u_1, u_2, \cdots, u_r\}$, W $= \{w_1, w_2, \cdots, w_s\}$ are the partite sets of G, where $r \leq s$ and also $V = U \cup W$.

Consider $S = U \cup W - x$ for any $x \in W$. Every $w_k \in W$, $1 \leq k \leq s-1$ lies on $u_i - u_j$ geodesic for $1 \leq i \neq j \leq r$, so that S is a geodetic set of G. Since $\langle V(G) - S \rangle$ is connected and hence S itself a nonsplit geodetic set of G . Let S' be any set of vertices such that $|S'| < |S|$. If S' is not a subset of U then $\langle V(G) - S' \rangle$ is not connected and so S' is not a nonsplit geodetic set of G. If S' is not a subset of $W - x$, again S' is not a nonsplit geodetic set of G, by a similar argument.If $S' =$ U then S' is a geodetic set but $\langle V(G) - S' \rangle$ is not connected, so S' is not nonsplit geodetic set. If $S' = W - x$ then S' is not a nonsplit geodetic set of G. From the above argument, it is clear that S is a minimum nonsplit geodetic set of G. Hence $g_{ns}(Kr, s) = |S| = r + s - 1$. □

Theorem 3.4 *If G is a star then* $g_{ns}(G) = n - 1$.

Proof Let $V(G) = \{v_1, v_2, \cdots, v_{n-1}, v_n\}$ and let $S = \{v_1, v_2, \cdots, v_{n-1}\}$ be the set of pendant vertices of G and is the geodetic set of G. Clearly, the subgraph induced by $\langle V(G) - S = v_n \rangle$ is connected. Hence $S = \{v_1, v_2, \cdots, v_{n-1}\}$ is a minimum nonsplit geodetic set of G. Therefore $g_{ns}(G) = n - 1$. □

Theorem 3.5 *For any nontrivial connected graph G different from star of order* n *and diameter* d, $g_{ns}(G) \leq n - d + 1$.

Proof Let u and v be the vertices of G for which $d(u, v) = d$ and let $u = v_0, v_1, \cdots, v_d = v$ be a $u - v$ path of length d. Now $S = V(G) - \{v_1, v_2, \cdots, v_{d-1}\}$ then $I[S] = V[G]$ and consequently $g_{ns}(G) \leq |S| \leq n - d + 1$. □

Theorem 3.6 *For any tree* T, $g_{ns}(T) + g(T) < 2m$.

Proof Suppose $S = \{v_1, v_2, v_3, \cdots, v_k\}$ be the set of all pendant vertices in T , forms a minimal geodetic set of $I[S] = V(T)$. Further $\{u_1, u_2, u_3, \cdots, u_l\} \subset V(G) - S$ is the set of internal vertices in T. Then $\langle V(G) - S \rangle$ forms a minimal non split geodetic set of T, it follows that $|S| + |S| < 2m$. Hence $g_{ns}(T) + g(T) < 2m$. □

Theorem 3.7 *For any graph G of order* n, $g_{ns}(G) \leq g_s(G)$, *where G is not a cycle..*

Proof Let G be any graph with n vertices. Consider a nonsplit geodetic set $S = \{v_1, v_2, \cdots, v_k\}$ of a graph G. Since $\langle V(G) - S \rangle$ is connected, the set S is not a split geodetic set of G. Now, we consider a set $S' = S \cup \{a, b\}$ for any $a, b \in V(G)$ such that $\langle V(G) - S' \rangle$ is disconnected. Therefore S' is the split geodetic set of G with minimum cardinality. Thus $|S| < |S'|$. Clearly $g_{ns}(G) \leq g_s(G)$. □

Theorem 3.8 *Let G be a cycle of order n then* $g_s(G) \leq g_{ns}(G)$.

Proof Let G be a cycle of order n, we discuss the following cases.

Case 1. Suppose n is even. Let $S = \{v_i, v_j\}$ be the split geodetic set of G where v_i, v_j are the two antipodal vertices of G. The $v_i - v_j$ geodesic includes all the vertices of G and $\langle V(G) - S \rangle$

is disconnected. But $S' = \{v_i, v_{i+1}, \cdots, v_j\}$ is a nonsplit geodetic set of G and the induced subgraph $\langle V(G) - S' \rangle$ is connected. Thus $|S| \le |S'|$. Clearly $g_s(G) \le g_{ns}(G)$.

Case 2. Suppose n is odd. Let $S = \{v_i, v_j, v_k\}$ be the split geodetic set of G. By the Theorem 2.4, no two vertices of S form a non split geodetic set and $\langle V(G) - S \rangle$ is disconnected. But $S' = \{v_i, v_{i+1}, \cdots, v_j, v_k\}$ is a nonsplit geodetic set of G and the induced subgraph $\langle V(G) - S' \rangle$ is connected. Thus $|S| \le |S'|$. Clearly $g_s(G) \le g_{ns}(G)$. □

Theorem 3.9 *For the wheel* $W_n = k_1 + C_{n-1}$ $(n \ge 5)$,

$$g_{ns}(W_n) = \begin{cases} \frac{n}{2} & \text{if n is even} \\ \frac{n-1}{2} & \text{if n is odd} \end{cases}$$

Proof Let $W_n = K_1 + C_{n-1}$ and let $V\ (W_n)$=$\{x, u_1, u_2, \cdots, u_{n-1}\}$, where $deg(x) = n-1 > 3$ and $deg(u_i) = 3$ for each $i \in \{1, 2, \cdots, n-1\}$. We discuss the following cases.

Case 1. Let n be even. Consider geodesic

$P:\ \{u_1, u_2, u_3\}, Q:\ \{u_3, u_4, u_5\}, \cdots,\ R:\ \{u_{2n-1}, u_{2n}, u_{2n+1}, \}$.

It is clear that the vertices $u_2, u_4 \cdots, u_{2n}$ lies on the geodesic P, Q and R. Also $u_1, u_3, u_5, \cdots$, u_{2n-1}, u_{2n+1} is a minimum nonsplit geodetic set such that $\langle V(G) - S \rangle$ is connected and it has $\frac{n}{2}$ vertices. Hence $g_{ns}(W_n) = \frac{n}{2}$.

Case 2. Let n be odd. Consider geodesic

$P:\ \{u_1, u_2, u_3\}, Q:\ \{u_3, u_4, u_5\}, \cdots, R:\ \{u_{2n-1}, u_{2n}, u_{2n+1}, \}$.

It is clear that the vertices $u_2, u_4, \cdots, u_{2n}$ lies on the geodesic P, Q and R. Also u_1, u_3, u_5, $\cdots, u_{2n-1}, u_{2n+1}$ is a minimum nonsplit geodetic set such that $\langle V(G) - S \rangle$ is connected and it has $\frac{n-1}{2}$ vertices. Hence $g_{ns}(W_n) = \frac{n-1}{2}$. □

Theorem 3.10 *Let G be a graph such that both G and* $\overline{G}$ *are connected then* $g_{ns}(G) + g_{ns}(\overline{G}) \le n(n-3) + 2$.

Proof Since both G and $\overline{G}$ are connected , we have $\Delta(G) \cdot \Delta\overline{(G)} < n-1$.Thus $\beta_0(G), \beta_0\overline{(G)} \ge 2$. Hence,

$$g_{ns} \le n - 1 \Rightarrow g_{ns}(G) \le 2(n-1) - n + 1 \Rightarrow g_{ns}(G) \le 2m - n + 1.$$

Similarly, $g_{ns}(\overline{G}) \le 2\overline{m} - n + 1$. Thus,

$$\begin{aligned} g_{ns}(G) + g_{ns}(\overline{G}) \le 2(m + (\overline{m})) - 2n + 2 \quad &\Rightarrow \quad g_{ns}(G) + g_{ns}(\overline{G}) \le n(n-1) - 2n + 2 \\ &\Rightarrow \quad g_{ns}(G) + g_{ns}(\overline{G}) \le n^2 - n - 2n + 2 \\ &\Rightarrow \quad g_{ns}(G) + g_{ns}(\overline{G}) \le n^2 - 3n + 2 \\ &\Rightarrow \quad g_{ns}(G) + g_{ns}(\overline{G}) \le n(n-3) + 2. \end{aligned}$$ □

Theorem 3.11 *For any nontrivial tree* T, $g_{ns}(T) \ge \alpha_0(T)$.

Proof Let S be a minimum cover set of vertices in T. Then S has at least one vertex and every vertex in S is adjacent to some vertices in $\langle V(G) - S\rangle$. This implies that S is a nonsplit geodetic set of G. Thus $g_{ns}(T) \geq \alpha_0(T)$. □

Theorem 3.12 *For any nontrivial tree T with m edges, $g_{ns}(T) \leq m - \lceil \frac{\alpha_1(T)}{2} \rceil + 2$,where $\alpha_1(T)$ is an edge covering number.*

Proof Suppose $S' = \{e_1, e_2, \cdots, e_i\}$ be the set of all end edges in T and $J \subseteq E(T) - S'$ be the minimal set of edges such that $|S' \cup J| = \alpha_1(T)$. By the Theorem 2.2 S' is the minimal geodetic set of G. Also it follows that $\langle V(G) - S'\rangle$ is connected. Clearly,

$$g_{ns}(T) \leq |E(T)| - \left|\left\lceil \frac{S' \cup J}{2} \right\rceil\right| + 2 \Rightarrow g_{ns}(T) \leq m - \left\lceil \frac{\alpha_1(T)}{2} \right\rceil + 2.$$ □

Theorem 3.13 *For a cycle C_n of order n, $g_{ns}(G) = \alpha_0(C_n) + 1$.*

Proof Consider a cycle C_n of order n. We discuss the following cases.

Case 1. Suppose that n is even and $\alpha_0(C_n)$ is the vertex covering number of C_n. We have by Theorem 3.1, $g_{ns}(G) = \frac{n}{2} + 1$ and also for an even cycle, vertex covering number $\alpha_0(C_n) = \frac{n}{2}$. Hence,

$$g_{ns}(G) = \frac{n}{2} + 1 = \alpha_0(C_n) + 1.$$

Case 2. Suppose that n is odd and $\alpha_0(C_n)$ is the vertex covering number of C_n. We have by Theorem 3.1, $g_{ns}(G) = \lfloor \frac{n}{2} \rfloor + 2$ and also for an odd cycle, vertex covering number $\alpha_0(C_n) = \lfloor \frac{n}{2} \rfloor + 1$. Hence,

$$g_{ns}(G) = \lfloor \frac{n}{2} \rfloor + 2 \Rightarrow g_{ns}(G) = \alpha_0(C_n) + 1.$$ □

Theorem 3.14 *If is a connected noncomplete graph G of order n,$g_{ns} \leq (n - \kappa(G)) + 1$, where $\kappa(G)$ is vertex connectivity.*

Proof Let $\kappa(G) = k$. Since G is connected but not complete $1 \leq \kappa(G) \leq n - 2$. Let $U = \{u_1, u_2, \cdots, u_k\}$ be a minimum cut set of G, let $G_1, G_2, \cdots, G_r (r \geq 2)$ be the components of $G - U$ and let $W = V(G) - (U - 1)$ then every vertex $u_i (1 \leq i \leq k)$ is adjacent to at least one vertex of G_j for every $(i \leq j \leq r)$. Therefore, every vertex u_i belongs to a W geodesic path. Thus

$$g_{ns}(G) = |W| \leq (V(G) - U) + 1 \leq (n - \kappa(G)) + 1.$$ □

§4. Corona Graph

Let G and H be two graphs and let n be the order of G. The corona product $G \circ H$ is defined as the graph obtained from G and H by taking one copy of G and n copies of H and then joining by an edge, all the vertices from the i^{th}-copy of H with the i^{th}-vertex of G.

Theorem 4.1 *Let G be a connected graph of order n and H be any graph of order m then $g_{ns}(G \circ H) = nm$.*

Proof Let S be a nonsplit geodetic set in $G \circ H$, $v_i \in V(G), 1 \leq i \leq n$ and $u_j \in V(H), 1 \leq j \leq m$. For each v_i there is a copy Hv_i which contains u_j vertices. Clearly $V(Hu_j) \cap S$ is a geodetic set of $G \circ H$ and $\langle V(G) - S \rangle$ is connected. Further every $w_k \in (G \circ H)$ lies on the geodesic path in S. Therefore S is the minimum nonsplit geodetic set. Thus, $|S| = g_{ns}(G \circ H) = nm$. □

§5. Adding a Pendant Vertex

An edge $e = (u, v)$ of a graph G with $deg(u) = 1$ and $deg(v) > 1$ is called an *pendant edge* and u an pendant vertex.

Theorem 5.1 *Let G' be the graph obtained by adding an pendant edge (u, v) to a cycle $G = C_n$ of order $n > 3$, with $u \in G$ and $v \notin G$, then*

$$g_{ns}(G') = \begin{cases} 2 & \text{if n is even} \\ 3 & \text{if n is odd} \end{cases}$$

Proof Let $\{u_1, u_2, u_3, \cdots, u_n, u_1\}$ be a cycle with n vertices. Let G' be the graph obtained from $G = C_n$ by adding an pendant edge (u, v) such that $u \in G$ and $v \notin G$. We discuss the following cases.

Case 1. For $G = C_{2n}$, let $S = \{v, u_i\}$ be a non split geodetic set of G' , where v is the pendant vertex of G' and $diam(G') = v - u_i$ path , clearly $I[S] = V[G']$. Also for all $x, y \in V(G') - S$, $\langle V(G') - S \rangle$ is connected. Hence, $g_{ns}(G') = 2$.

Case 2. For $G = C_{2n+1}$, let $S = \{v, a, b\}$ be a non split geodetic set of G' , where v is the pendant vertex of G' and $a, b \in G$ such that $d(v, a) = d(v, b)$. Thus $I[S] = V[G']$ and $\langle V(G') - S \rangle$ is connected. Hence, $g_{ns}(G') = 3$. □

Theorem 5.2 *Let G' be the graph obtained by adding a pendant vertex $(u_i, v_i)$for $i = 1, 2, 3, \cdots, n$ to each vertex of $G = C_n$ such that $u \in G$,$v_i \notin G$, then $g_{ns}(G') = k$.*

Proof Let $G = C_n = \{u_1, u_2, u_3, \cdots, u_n, u_1\}$ be a cycle with n vertices. Let G' be the graph obtained by adding an pendant vertex $\{u_i, v_i\}, i = 1, 2, 3, \cdots, n$ to each vertex of G such that $u_i \in G$ and $v_i \notin G$. Let $S = \{v_1, v_2, v_3, \cdots, v_k\}$ be a non split geodetic set of G'. Clearly $I[X] \neq V(G')$. Also, $x, y \in V(G') - S$ with $V(G') - S$ connected. Thus, $g_{ns}(G') = k$. □

Theorem 5.3 *Let G' be the graph obtained by adding k pendant vertices $\{(u, v_1), \cdots, (u, v_k)\}$ to a cycle $G = C_n$ of order $n > 3$, with $u \in G$ and $\{v_1, v_2, \cdots, v_k\} \notin G$. Then*

$$g_{ns}(G') = \begin{cases} k+1 & \textit{if n is even} \\ k+2 & \textit{if n is odd} \end{cases}$$

Proof Consider $\{u_1, u_2, u_3, \cdots, u_n, u_1\}$ be a cycle with n vertices. Let G' be the graph obtained from $G = C_n$ by adding k pendant edges $\{u_iv_1, u_iv_2, \cdots, u_iv_k\}$ such that u_i a single vertex of G and $\{v_1, v_2, v_3, \cdots, v_k\}$ does not belongs to G. We discuss the following cases.

Case 1. Let $G = C_{2n}$. Consider $X = \{v_1, v_2, v_3, \cdots, v_k\} \cup u_i$, for any vertex u_i of G . Now $S = \{X\}$ be a non split geodetic set , such that $\{v_1, v_2, v_3, \cdots, v_k\}$ are the pendant vertices of G' and u_j is the antipodal vertex of u_i in G. Thus $I[X] = V[G']$. Consider $P = \{v_1, v_2, v_3, \cdots, v_k\}$ as a set of pendant vertices such that $|P| < |S|$ is not a non split geodetic set i.e for some vertex $u_j \in V_{G'}$,$u_j \notin I[P]$. If $P = X$, then P is not nonsplit geodetic set. Thus S is a minimum non split geodetic set of G' and $\langle V(G') - S\rangle$ is connected. Thus, $g_{ns}(G') = k + 1$.

Case 2. Let $G = C_{2n+1}$. Consider $S = \{v_1, v_2, v_3, \cdots, v_k, a, b\}$ be a non split geodetic set, where $\{v_1, v_2, \cdots, v_k\} \notin G$ are k pendant vertices of G' not in G and $a, b \in G$ such that $d(u, a) = d(u, b)$. Thus $I[S] = V[G']$. Also $x, y \in V(G') - S$ it follows that $\langle V(G') - S\rangle$ is connected. Therefore, $g_{ns}(G') = k + 2$. □

§6. Cartesian Products

The cartesian product of the graphs H_1 and H_2 written as $H_1 \times H_2$, is the graph with vertex set $V(H_1) \times V(H_2)$, two vertices u_1, u_2 and v_1, v_2 being adjacent in $H_1 \times H_2$ iff either $u_1 = v_1$ and $(u_2, v_2) \in E(H_{2)}$, or $u_2 = v_2$ and $(u_1, v_1) \in E(H_{1)}$.

Theorem 6.1 *Let K_2 and $G = C_n$ be the graphs then*

$$g_{ns}(K_2 \times G) = \begin{cases} 2 & \text{if n is even} \\ 3 & \text{if } n > 5 \text{ is odd} \\ 4 & \text{if n=3} \end{cases}$$

Proof Consider $G = C_n$, let $K_2 \times G$ be graphs formed from two copies G_1 and G_2 of G. Let $V = \{v_1, v_2, \cdots, v_n\}$ be the vertices of G_1 and $W = \{w_1, w_2, \cdots, w_n\}$ be the vertices of G_2 and $U = V \cup W$. We consider the following cases.

Case 1. Let n be even. Consider $S = \{v_i, w_j\}$ be the non split geodetic of $K_2 \times G$ such that v_i to w_j path is equal to $diam(K_2 \times G)$ which includes all the vertices of $K_2 \times G$. Hence $\langle U - S\rangle$ is connected. Therefore, $g_{ns}[K_2 \times G] = 2$.

Case 2. Let n be odd. Consider $S = \{v_i, w_j, v_k\}$ be the non split geodetic set of $K_2 \times G$ such that v_i to w_j path is equal to $diam(K_2 \times G)$ and is equal to $w_j - v_k path$ and also $v_i - w_j \cup w_j - v_k$ path includes all the vertices of $K_2 \times G$. Hence $\langle U - S\rangle$ is connected. Therefore, $g_{ns}[K_2 \times G] = 3$.

Case 3. For $n = 3$, let $S = \{v_i, w_j, v_k\}$ be the geodetic set of $K_2 \times G$, that is $v_i - w_j$ is equal to $diam(K_2 \times G)$ and is equal to $w_j - v_k$ and also $I[S] = U(K_2 \times G)$. But $\langle U - S\rangle$ is not connected. Let $S' = S \cup \{v_n\} = \{v_i, w_j, v_k, v_n\}$ be the non split geodetic set of $K_2 \times G$. Hence, $\langle U - S'\rangle$ is connected. Therefore, $g_{ns}[K_2 \times G] = 4$. □

Theorem 6.2 *For any complete graph K_n of order n, $g_{ns}[K_2 \times K_n] = n + 1$.*

Proof Consider $K_2 \times K_n$ be graph formed from two copies of G_1 and G_2 of G. Now, let us prove the result by mathematical induction,

For $n = 2$, $g_{ns}[k_2 \times K_2] = 3$, since $K_2 \times K_2 = C_4$ by Theorem 3.1 we have $g_{ns}[C_4] = 3$ the result is true.

Let us assume that the result is true for n=m,that is $g_{ns}[K_2 \times K_m] = m + 1$.

Now, we shall prove the result for $n = m+1$. Let $S = \{v_1, v_2, v_3, \cdots, v_{m+2}\}$ be the nonsplit geodetic set formed from some elements in G_1 and the elements which are not corresponds to elements in G_1 of $K_2 \times K_{m+1}$. Clearly $I[S] = V(K_2 \times K_n)$. Let P be any set of vertices such that $|P| < |S|$. Suppose $P = \{v_1, v_2, v_3, \cdots, v_m\}$ which is not non split geodetic set, because $I[P] \neq V[K_2 \times K_{m+1}]$. So S itself a minimum geodetic set of $K_2 \times K_{m+1}$. Hence, $g_{ns}[K_2 \times K_{m+1}] = m + 1 + 1$. Thus, $g_{ns}(K_2 \times K_n) = n + 1$. □

Theorem 6.3 *For any complete graph of order $n \geq 3$, $g_{ns}(K_n \times K_n) = n$.*

Proof We shall prove the result by mathematical induction, For $n \geq 3$, let us assume that the result is true for $n = m$, that is $g_{ns}(K_m \times K_m) = m$.

Now, we shall prove the result for $n = m + 1$. Let $S = \{v_1, v_2, v_3, \cdots, v_{m+1}\}$ be the non split geodetic set formed from some elements in G_1 and the elements which are not corresponds to elements in G_1 of $K_{m+1} \times K_{m+1}$. Clearly $I[S] = V(K_2 \times K_n)$. Now, consider P be any set of vertices such that $|P| < |S|$. Suppose $P = \{v_1, v_2, v_3, \cdots, v_m\}$ which is not non split geodetic set, because $I[P] \neq V(K_{m+1} \times K_{m+1})$. So S itself a minimum geodetic set of $K_{m+1} \times K_{m+1}$. Hence, $g_{ns}(K_{m+1} \times K_{m+1}) = m + 1$. Thus $g_{ns}(K_n \times K_n) = n$. □

Theorem 6.4 *Let G and H be graphs then $g_{ns}(G \times H) \geq max\{g(G), g(H)\}$.Equality holds when G,H are complete graphs and $n \geq 3$.*

Proof If S is a minimum geodetic set in $G \times H$ then we have $I[S] = \cup_{a,b \in S} I[a, b] = \cup_{a,b \in S} I[a_1, b_1] \times I[a_2, b_2] \subseteq (\cup_{a_1,b_1 \in S} I[a_1, b_1]) \times (\cup_{a_2,b_2 \in S} I[a_2, b_2]) = I[S_1] \times I[S_2]$,$V(G \times H) = I[S] \subseteq I[S_1] \times I[S_2]$. Therefore S_1 and S_2 geodetic set in G, H respectively, so $g_{ns}(G \times H) = |S| \geq max\{|s_1|, |s_2|\} \geq max\{g(G), g(H)\}$, proving the lower bound.

Consider complete graphs G, H with vertex sets $V(G) = \{u_1, u_2, \cdots, u_p\}$ and $V(H) = \{v_1, v_2, \cdots, v_q\}$ where without loss of generality $p \geq q$. Then $g(G) = p$ and $g(H) = q$. Let $S = \{(u_1, v_2), (u_2, v_2), \cdots, (u_q, v_q), (u_{q+1}, v_q), (u_{q+2}, v_q), \cdots, (u_p, v_q)\}$.

It is straight forward to verify that S is a non split geodetic set for $G \times H$. Hence, $g_{ns}(G \times H) \leq |S| \leq p = max\{g(G), g(H)\} \leq g_{ns}(G \times H)$, so equality holds. □

Theorem 6.5 *Let $G = T$ and $H = K_2$ be the graphs with $g(G) = p \geq g(H) = q \geq 2$ then $g_{ns}(G \times H) \leq pq - q$.*

Proof Set $X = G \times H$. Let $S = \{g_1, g_2, ..., g_p\}$ and $T = \{h_1, h_2, \cdots, h_q\}$ be the geodetic sets of G and H respectively, and $U = \{(S \times T)/ \cup_{i,j=1}^{p,q} \{(g_i, h_j)\}\}$.

We claim that $I_X[U] = V(X)$. Let (g, h) be an arbitrary vertex of X.Then there exists indices i and i' such that $g \in I_G[g_i, g_{i'}]$ and there are indices j and j' such that $h \in I_H[h_j, h_{j'}]$. Since $p, q \geq 2$ we may assume that $i = i'$ and $j = j'$. Indeed ,if say $g = g_i$ then i' to be an arbitrary index from $\{1, 2, \cdots, p\}$ different from i. Set $B = \{(g_i, h_j), (g_i, h_{j'}), (g_{i'}, h_j), (g_{i'}, h_{j'})\}$.

Suppose that one of the vertices from B is not in U. We may without loss of generality assume (g_i, h_j) $\not inU$. This means that $i = j$. Therefore $i' \neq j$ and $i \neq j'$. Then we infer that $(g, h) \in I_X[(g_i, h_{j'}), (g_{i'}, h_j)]$. Otherwise, all vertices from B are in U, then $(g, h) \in I_X[(g_i, h_j), (g_{i'}, h_{j'})]$. Hence, $g_{ns}[G \times H] \leq pq - q$. □

Theorem 6.6 *Let K_2 and T be the graphs then $g_{ns}(K_2 \times T) = g_{ns}(T)$.*

Proof Consider a tree T. Let $K_2 \times T$ be a graph formed from two copies T_1 and T_2 of T and S be a minimum non split geodetic set of $K_2 \times T$. Now, we define S' to be the union of those vertices of S in T_1 and the vertices of T_1 corresponding to vertices of T_2 belonging to S. Let $v \in V(T_1)$ lies on some $x - y$ geodesic,where $x, y \in S$.Since S is a non split geodetic set by Theorem 3.2, i.e., $g_{ns}(T) = k$ at least one of x and y belongs to V_1. If both $x, y \in V_1$ then $x, y \in S'$. Hence, we may assume that $x \in V_1, y \in V_2$. If y corresponds to x then $v = x \in S'$. Hence, we assume that y corresponds to $y' \in S'$,where $y' \neq x$. Since $d(x, y) = d(x, y') + 1$ and the vertex v lies on an $x - y$ geodesic in $K_2 \times G$. Hence, v lies on $x - y$ geodesic in G that is $g_{ns}(G) \leq g_{ns}(K_2 \times G)$.

Conversely, let S contains a vertex with the property that every vertex of T_1 lies on $x - w$ geodesic T_1 for some $w \in S$. Let S' consists of x together with those vertices of T_2 corresponding to those $S - \{x\}$. Thus, $|S'| = |S|$. We show that S' is a non split geodetic set of $K_2 \times T$. Hence $g_{ns}(K_2 \times T) \leq g_{ns}(T)$. Thus, $g_{ns}(K_2 \times T) = g_{ns}(T)$. □

Theorem 6.7 *Let K_2 and $G = P_n$ be the two graphs,*

$$g_{ns}(K_2 \times G) = \begin{cases} 2 & ifn \geq 3 \\ 3 & ifn = 2 \end{cases}$$

Proof Consider a trivial graph K_1 as a connected graph. Let G_1 and G_2 be the two copies of G and also $V(G_1) = \{a_1, a_2, \cdots, a_n\}$, $V(G_2) = \{b_1, b_2, \cdots, b_n\}$. Let $S = \{a_1, b_n\}$ be the non split geodetic set of $K_2 \times G$ and also $d(a_1, b_n) = diam(a_1, b_n)$. Thus, $V - S = \{a_2, a_3, \cdots, a_n, b_1, b_2, \cdots, b_{n-1}\}$ is the induced subgraph and it is connected. Hence $g_{ns}[K_2 \times G] = 2$.

Similarly,the result is obvious for $n = 2$ that is $g_{ns}[K_2 \times G] = 3$. □

§7. Block Graphs

A block graph has a subgraph G_1 of G(not a null graph) such that G_1 is non separable and if G_2 is any other graph of G, then $G_1 \cup G_2 = G_1$ or $G_1 \cup G_2$ is separable. For any graph G a complete subgraph of G is called clique of G. The number of vertices in a largest clique of G is called the clique number of G and denoted by $\omega(G)$.

Theorem 7.1 *For any block graph G, $g_{ns}(G) = n - c_i$ where n be the number of vertices and c_i be the number of cut vertices.*

Proof Let $V = \{v_1, v_2, \cdots, v_n\}$ be the number of vertices of G. Consider S be the geodetic set of G and $\langle V(G) - S\rangle$ is connected. Thus S itself a nonsplit geodetic set of G. Since every geodetic set does not contain any cut vertices. Hence, $g_{ns}(G) = n - c_i$. □

Theorem 7.2 *For any block graph G, $g_{ns}(G) \leq \omega(G) + 2c_i$ where $\omega(G)$ be the clique number and c_i be the number of cut vertices.*

Proof Let $V = \{v_1, v_2, \cdots, v_n\}$ be the number of vertices of G. In a block graph, every geodetic set is a nonsplit geodetic set. Consider S be the geodetic set of G and $\langle V(G) - S\rangle$ is connected. Thus S itself a nonsplit geodetic set of G. By the definition, the number of vertices in a largest clique of G is $\omega(G)$ and also every geodetic set does not contain any cut vertices of G. It follows that $g_{ns}(G) \leq \omega(G) + 2c_i$. □

Theorem 7.3 *For any block graph G, $g_{ns}(G) = \alpha_0(G) + 1$ where $\alpha_0(G)$ be the vertex covering number.*

Proof Let G be a block graph of order n. Now, we prove the result by mathematical induction.

For $c_i = 1$, the vertex covering number of G is

$$\alpha_0(G) = n - c_i - 1 \Rightarrow \alpha_0(G) = n - 1 - 1 \Rightarrow \alpha_0(G) + 1 = n - 1,$$

by Theorem 7.1, we have

$$g_{ns}(G) = n - c_i \Rightarrow g_{ns}(G) = n - 1.$$

Therefore, $g_{ns}(G) = \alpha_0(G) + 1$. Thus the result is result is true for $c_i = 1$. Let us assume that the result is true for $c_i = m$ that is $g_{ns}(G) = \alpha_0(G) + 1$.

Now, we shall prove the result for $c_i = m + 1$, where m+1 is the number of cut vertices. Let $S = \{v_1, v_2, \cdots, v_n\}$ be the minimum nonsplit geodetic set of G. Since every geodetic set does not contain any cut vertex, by Theorem 7.1 we have $g_{ns}(G) = n - m - 1$. Therefore,

$$\alpha_0(G) = n - c_i - 1 \Rightarrow \alpha_0(G) = (n - m - 1) - 1 \Rightarrow \alpha_0(G) + 1 = n - m - 1.$$

Thus, $g_{ns}(G) = \alpha_0(G) + 1$. □

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