On Finding All Solutions to the Lemoine - Levy Problem

Matilda Walter

Abstract

Lemoine - Levy Conjecture, probably the least known of the 'Goldbach Conjectures', states that every positive odd integer > 5 is a sum of a prime and double of a prime. We present a simple sieve procedure for finding all existing solutions to the problem for any given odd number > 5.

The Lemoine - Levy Conjecture states that every odd number greater than 5 can be written as a sum of prime and double of a prime, i.e., for $N > 2,$ we have $2N + 1 = p + 2q$ for some prime numbers p and q. What we refer to as the Lemoine - Levy problem, is a question of finding any, or all, solutions for any given odd number > 5. We write the Lemoine - Levy 'equation', as $2N+1 = 2p + q$, because our sieve will apply to the doubles of p. In what follows, we give a complete solution to this problem.

In [4] we presented a sieve algorithm that finds all solutions to the Goldbach problem for any given $2N > 4$. The solution to the Lemoine - Levy problem, follows a similar pattern, depending on the fact that given a congruence

$$A + B \equiv C \bmod p,$$

the only way that A can be congruent to $C \bmod p$, is if B is congruent to $0 \bmod p$. This, in case of Goldbach problem, is applied to a congruence arising from the identity

$$p + (2N - p) = 2N$$

The sieve does the rest, discarding p if $(2N - p) \equiv 0$ modulo some prime $< (2N)^{1/2}$ and keeping it otherwise. There is a minor difference in the treatment of the two problems that requires an extra step in Lemoine - Levy case and will be explained shortly.

All existing solutions to the Lemoine - Levy problem for $2N+1$, with $N>2$, are found by sieving through the doubles of all primes $< N$, with the residue classes of a system of congruences satisfied by $(2N+1)$, i.e., the system $(2N+1) \bmod 3$, $(2N+1) \bmod 5$,... $(2N+1) \bmod p_k$, with the moduli given by all odd primes p_i, $2 < p_i \leq p_k < (2N+1)^{1/2} < p_{k+1}$. When $(2N+1) - p_k \leq 2p$, so that the difference $(2N+1) - 2p$, if prime, is one of the moduli and if $2p \equiv 2N+1 \bmod p_i$, then we keep, or sieve $2p$ out, according to the difference $(2N+1) - 2p$ being, respectively, equal, or not equal, to p_i. This is the extra step, alluded to earlier and it is necessary because

$$2p \equiv 2N+1 \bmod p_i$$

is equivalent to

$$(2N+1) - 2p \equiv 0 \bmod p_i$$

and the solutions of the latter, being multiples of p_i, include p_i itself.

The claim that the proposed sieve correctly picks out all of the solutions to the Lemoine - Levy problem for any odd number > 5, will now be proved.

Proof: Given $2N+1 > 5$ let p be any prime $< N$ and consider the following identity

$$2p + ((2N+1) - 2p) = 2N+1$$

From the identity we obtain a valid congruence, taken modulo each of the odd primes $p_i < (2N)^{1/2}$

$$2p + ((2N+1) - 2p) \equiv 2N+1 \bmod p_i$$

From the congruence we deduce that $2p$ will be incongruent to $2N+1$, unless

$$(2N+1) - 2p \equiv 0 \bmod p_i$$

for some $p_i < (2N)^{1/2}$.

Therefore, if $(2N+1) - 2p$ is prime[1], it is either $> p_k$, or it will be one of the moduli. In the former case, being bigger than any of the moduli, none of its residues will equal zero. As a result, $2p$ will be incongruent to $(2N+1)$ and will survive the sieve. In the latter case, when

$$2N + 1 - p_k \leq 2p,$$

the difference, $(2N+1) - 2p$, if prime, will be among the moduli. Then, if $2p$ is congruent to $2N+1$ mod p_i, we keep $2p$ if $(2N+1) - 2p = p_i$, and discard it otherwise. Here, $(2N+1) - 2p$ is prime, but being p_i, it causes the condition for sieving $2p$ out, namely, $2p \equiv 2N+1 \bmod p_i$, to be satisfied. Hence, the extra step preventing $2p$ from being erroneously discarded.

If, on the other hand, $(2N+1) - 2p$ is composite, then it has a prime divisor among the moduli. Its residue modulo the divisor will be zero, at which point, $2p$ modulo the same divisor, will be congruent to $2N+1$ and will be sieved out.

For each double of a prime that survives this process, $(2N+1)-2p$ will also be prime, while for the sieved out doubles of primes, $(2N+1)-2p$ will be divisible by the modulus at which the residue of $2N+1$ sieved out $2p$.

Sieving through the doubles of all primes $< N$, applies the sieve to all potential candidates for solutions. The doubles that survive the procedure, each paired to the prime $2N+1-2p$, constitute all solutions to the Lemoine - Levy problem for $2N+1$. *QED*

For example, consider $2N+1=201$. Square root of 201 lies between 14 and 15, so the moduli are all odd primes < 14, namely $3, 5, 7, 11$ and 13. Doubles of primes $< N = 100$ are, respectively, $4, 6, 10, 14, 22, 26, 34, 38, 46, 58, 62, 74, 82, 86, 94, 106, 118, 122, 134, 142, 146, 158, 166, 178$ and 194. The number, 201, satisfies the following congruences with respect to the given moduli:

$$201 \equiv 0 \bmod 3$$
$$201 \equiv 1 \bmod 5$$
$$201 \equiv 5 \bmod 7$$
$$201 \equiv 3 \bmod 11$$
$$201 \equiv 6 \bmod 13$$

[1] We do not need to know whether $(2N + 1 - 2p)$ is prime, or composite (!) We need to assume both, in turn, to show that the behavior of $2p$ with respect to the proposed sieve, reflects which is the actual case..

The first congruence sieves out *6*. The second sieves out *26, 46, 86, 106, 146* and *166* with *6*, which is also congruent to *1 mod 5*, already sieved out. The third sieves out *82* and *194* with *26* and *166*, also congruent to *5 mod 7*, already sieved out, but *194* is greater than $(2N+1) - p_k = 188$ and $(2N + 1) - 2p = 201 - 194 = 7$ and as this is p_i in question, we keep *194*. Fourth congruence sieves out *14* and *58*, with *146* already sieved out and the last congruence sieves out none since *58*, which is congruent to *6* mod *13*, has already been sieved out.

This leaves *4, 10, 22, 34, 38, 62, 74, 94, 118, 122, 134, 142, 158, 178* and *194* as the surviving doubles and to each there corresponds a prime $(2N+1) - 2p$; these are, respectively, *197, 191, 179, 167, 163, 139, 127, 107, 83, 79, 67, 59, 43, 23* and *7*. For each double of a prime that was sieved out, $(2N+1) - 2p$ is composite and is divisible by the modulus at which the residue of $2N+1$ sieved out $2p$.

As we did in the closing paragraph of [4] we would like to point out that the foregoing is not a solution to the Lemoine - Levy Conjecture, as it does not prove, or otherwise imply, existence of solutions for any odd numbers. The sieve presented here, partitions the set of doubles of primes in accordance with the imposed congruence conditions thereby picking out *existing* solutions from all of the candidates. Should a counter-example to the Conjecture exist, the sieve will show that there are no solutions by sieving out all of the doubles that present themselves. In other words, the sieve is neutral, in so far as it does *its* job, regardless of the outcome.

References

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