Some series associated with Catalan's constant

Edgar Valdebenito

abstract

In this note show series for the Catalan's constant:

$$G = \int_0^1 \frac{\tan^{-1} x}{x} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.915965594177 \ldots$$

Keywords: Catalan constant , series.

1. Series for Catalan's constant

1. Sea $-\pi/2 < x < \pi/2$, se tiene :

$$G = (\cos x) \ln\left(\frac{1 - \cos x + \sqrt{1 + (\cos x)^2}}{1 + \cos x - \sqrt{1 + (\cos x)^2}}\right) + \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{\cos((2n-1)x)}{2n-1} - \frac{\cos((2n+1)x)}{2n+1}\right) f_n \tag{1}$$

donde

$$f_n = \ln\left(\frac{1 - \cos x + \sqrt{1 + (\cos x)^2}}{1 + \cos x - \sqrt{1 + (\cos x)^2}}\right) - 2 \sum_{k=1}^{n} \frac{\left(\sqrt{1 + (\cos x)^2} - \cos x\right)^{2k-1}}{2k-1} \tag{2}$$

2. Sea $0 < x < \pi$, se tiene :

$$G = (\operatorname{sen} x) \ln\left(\frac{1 - \operatorname{sen} x + \sqrt{1 + (\operatorname{sen} x)^2}}{1 + \operatorname{sen} x - \sqrt{1 + (\operatorname{sen} x)^2}}\right) + \sum_{n=1}^{\infty} \left(\frac{\operatorname{sen}((2n-1)x)}{2n-1} + \frac{\operatorname{sen}((2n+1)x)}{2n+1}\right) f_n \tag{3}$$

donde

$$f_n = \ln\left(\frac{1 - \operatorname{sen} x + \sqrt{1 + (\operatorname{sen} x)^2}}{1 + \operatorname{sen} x - \sqrt{1 + (\operatorname{sen} x)^2}}\right) - 2 \sum_{k=1}^{n} \frac{\left(\sqrt{1 + (\operatorname{sen} x)^2} - \operatorname{sen} x\right)^{2k-1}}{2k-1} \tag{4}$$

3. Sea $0 < x < \pi/2$, se tiene :

$$G = (\tan x) \ln(1 + \cot x) + \sum_{n=2}^{\infty} \frac{\text{sen}(n x)}{n} (\cos x)^{-n} f_n \tag{5}$$

donde

$$f_n = \ln(1 + \cot x) + \sum_{k=0}^{n-2} \binom{n-1}{k} \frac{(-1)^{n-k-1} \left((1 + \cot x)^{k-n+1} - 1\right)}{k - n + 1} \tag{6}$$

2. Examples

$$G = \ln\left(\sqrt{2} + 1\right) + \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) f_n \tag{7}$$

$$f_n = \ln\left(\sqrt{2} + 1\right) - 2 \sum_{k=1}^{n} \frac{\left(\sqrt{2} - 1\right)^{2k-1}}{2k-1} \tag{8}$$

$$G = \frac{1}{\sqrt{2}} \ln\left(\sqrt{2} + \sqrt{3}\right) + \sqrt{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n^2 - 1} \left(\cos\left(\frac{n\pi}{2}\right) + 2n \,\text{sen}\left(\frac{n\pi}{2}\right)\right) f_n \tag{9}$$

$$f_n = \ln\left(\sqrt{2} + \sqrt{3}\right) - 2 \sum_{k=1}^{n} \frac{1}{2k-1} \left(\frac{\sqrt{3} - 1}{\sqrt{2}}\right)^{2k-1} \tag{10}$$

$$G = \frac{1}{2} \ln\left(2 + \sqrt{5}\right) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n^2 - 1} \left(\cos\left(\frac{2n\pi}{3}\right) + 2\sqrt{3}\, n \,\text{sen}\left(\frac{2n\pi}{3}\right)\right) f_n \tag{11}$$

$$f_n = \ln\left(2 + \sqrt{5}\right) - 2 \sum_{k=1}^{n} \frac{1}{2k-1} \left(\frac{\sqrt{5} - 1}{2}\right)^{2k-1} \tag{12}$$

$$G = \frac{\sqrt{3}}{4} \ln\left(\frac{11 + 4\sqrt{7}}{3}\right) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n^2 - 1} \left(\sqrt{3} \cos\left(\frac{n\pi}{3}\right) + 2n \,\text{sen}\left(\frac{n\pi}{3}\right)\right) f_n \tag{13}$$

$$f_n = \frac{1}{2} \ln\left(\frac{11 + 4\sqrt{7}}{3}\right) - 2 \sum_{k=1}^{n} \frac{1}{2k-1} \left(\frac{\sqrt{7} - \sqrt{3}}{2}\right)^{2k-1} \tag{14}$$

$$G = \left(\frac{1 + \sqrt{5}}{4}\right) \ln\left(-1 + \sqrt{5} + \sqrt{7 - 2\sqrt{5}}\right) + \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{\cos\left((2n-1)\frac{\pi}{5}\right)}{2n-1} - \frac{\cos\left((2n+1)\frac{\pi}{5}\right)}{2n+1}\right) f_n \tag{15}$$

$$f_n = \ln\left(-1 + \sqrt{5} + \sqrt{7 - 2\sqrt{5}}\right) - 2 \sum_{k=1}^{n} \frac{1}{2k-1} \left(\frac{-1 - \sqrt{5} + \sqrt{22 + 2\sqrt{5}}}{4}\right)^{2k-1} \tag{16}$$

$$G = \frac{\sqrt{3}}{4} \ln\left(\frac{11 + 4\sqrt{7}}{3}\right) - \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \left(\sqrt{3} \cos\left(\frac{2n\pi}{3}\right) - 2n \,\text{sen}\left(\frac{2n\pi}{3}\right)\right) f_n \tag{17}$$

$$f_n = \frac{1}{2} \ln\left(\frac{11 + 4\sqrt{7}}{3}\right) - 2 \sum_{k=1}^{n} \frac{1}{2k-1} \left(\frac{\sqrt{7} - \sqrt{3}}{2}\right)^{2k-1} \tag{18}$$

$$G = \frac{1}{2}\ln\left(2+\sqrt{5}\right) - \sum_{n=1}^{\infty}\frac{1}{4n^2-1}\left(\cos\left(\frac{n\pi}{3}\right) - 2\sqrt{3}\,n\,\mathrm{sen}\left(\frac{n\pi}{3}\right)\right)f_n \tag{19}$$

$$f_n = \ln\left(2+\sqrt{5}\right) - 2\sum_{k=1}^{n}\frac{1}{2k-1}\left(\frac{\sqrt{5}-1}{2}\right)^{2k-1} \tag{20}$$

$$G = \sqrt{3}\,\ln\left(1+\frac{1}{\sqrt{3}}\right) + \sum_{n=2}^{\infty}\frac{2^n}{n}\,\mathrm{sen}\left(\frac{n\pi}{3}\right)f_n \tag{21}$$

$$f_n = \ln\left(1+\frac{1}{\sqrt{3}}\right) - \sum_{k=0}^{n-2}\frac{(-1)^{n-k-1}}{n-k-1}\binom{n-1}{k}\left(\left(1+\frac{1}{\sqrt{3}}\right)^{k-n+1} - 1\right) \tag{22}$$

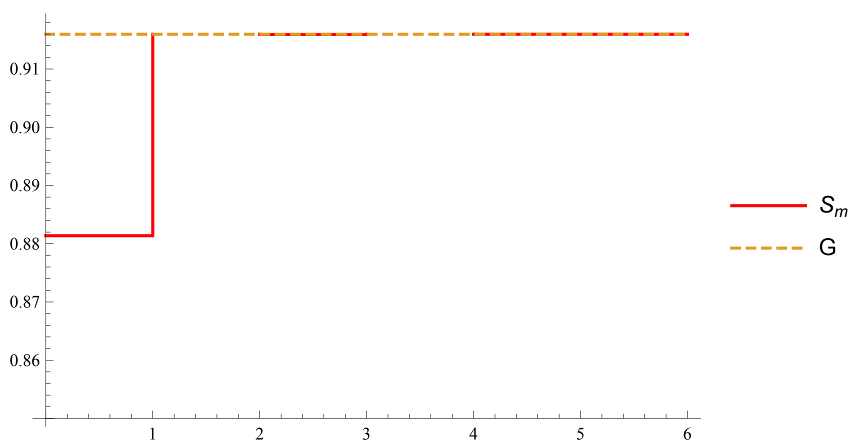
$$G = \ln 2 + \sum_{n=2}^{\infty}\frac{\left(\sqrt{2}\right)^n}{n}\,\mathrm{sen}\left(\frac{n\pi}{4}\right)f_n \tag{23}$$

$$f_n = \ln 2 - \sum_{k=0}^{n-2}\frac{(-1)^{n-k-1}}{n-k-1}\binom{n-1}{k}\left(2^{k-n+1} - 1\right) \tag{24}$$

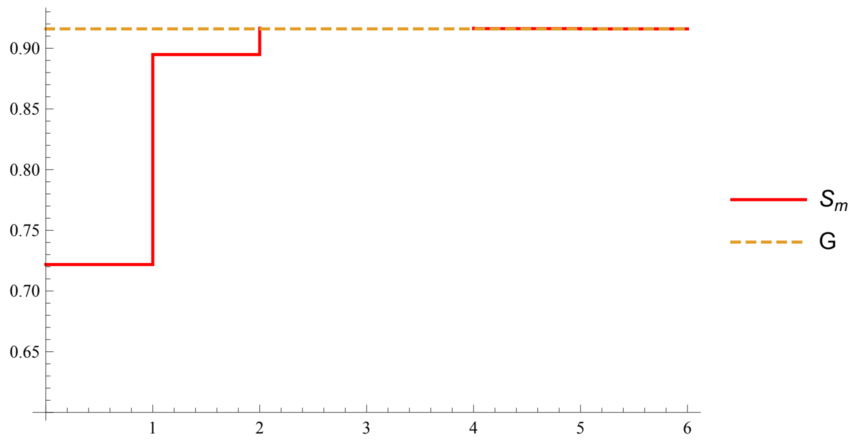
$$G = \frac{1}{\sqrt{3}}\ln\left(1+\sqrt{3}\right) + \sum_{n=2}^{\infty}\frac{1}{n}\left(\frac{2}{\sqrt{3}}\right)^n\mathrm{sen}\left(\frac{n\pi}{6}\right)f_n \tag{25}$$

$$f_n = \ln\left(1+\sqrt{3}\right) - \sum_{k=0}^{n-2}\frac{(-1)^{n-k-1}}{n-k-1}\binom{n-1}{k}\left(\left(1+\sqrt{3}\right)^{k-n+1} - 1\right) \tag{26}$$

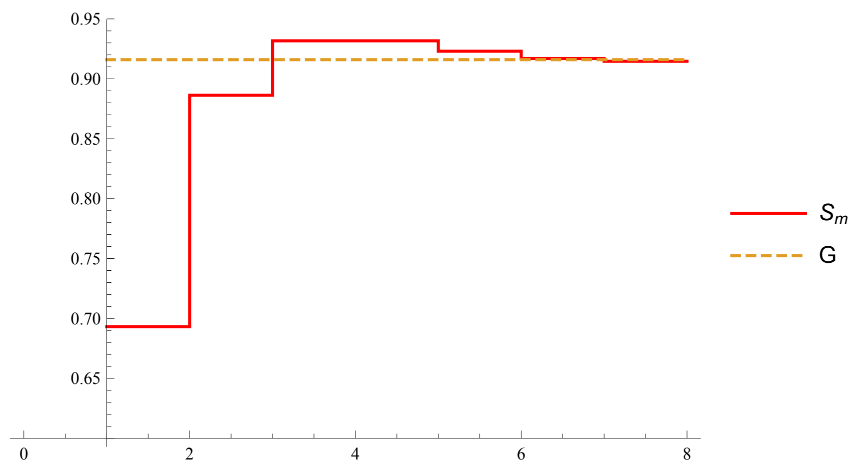
3. Some partial sums



Partial sums formula (7)



Partial sums formula (19)



Partial sums formula (23)

4. Series with hypergeometric function

$$G = \frac{2}{\sqrt{3}} \sum_{n=1}^{\infty} \frac{\left(\sqrt{3}-1\right)^{n-1} F\left(1,\ 1;\ n+1;\ -1\big/\sqrt{3}\right)}{n^2} \operatorname{sen}\left(\frac{n\pi}{3}\right) \tag{27}$$

$$G = \sum_{n=1}^{\infty} \frac{\left(\sqrt{3}-1\right)^{n} F\left(1,\ n;\ n+1;\ \left(\sqrt{3}-1\right)\big/2\right)}{n^2} \operatorname{sen}\left(\frac{n\pi}{3}\right) \tag{28}$$

$$G = \sum_{n=1}^{\infty} \frac{\left(1\big/\sqrt{2}\right)^{n} F(1,\ n;\ n+1;\ 1/2)}{n^2} \operatorname{sen}\left(\frac{n\pi}{4}\right) \tag{29}$$

$$G = 2\sum_{n=1}^{\infty} \frac{\left(1\big/\sqrt{2}\right)^{n} F(1,\ 1;\ n+1;\ -1)}{n^2} \operatorname{sen}\left(\frac{n\pi}{4}\right) \tag{30}$$

$$G = \sum_{n=1}^{\infty} \frac{\left(\sqrt{3}-1\right)^n F\left(1,\, n;\, n+1;\, \left(3-\sqrt{3}\right)/2\right)}{n^2} \operatorname{sen}\left(\frac{n\,\pi}{6}\right) \tag{31}$$

$$G = 2 \sum_{n=0}^{\infty} \frac{\operatorname{sen}((2\,n+1)\,x)}{2\,n+1} (s(x))^{2\,n+1} f_n \;,\;\; 0 < x < \pi \tag{32}$$

$$f_n = \frac{F(1,\, n+1/2;\, n+3/2;\, (s(x))^2)}{2\,n+1} + \frac{(s(x))^2\, F(1,\, n+3/2;\, n+5/2;\, (s(x))^2)}{2\,n+3} \tag{33}$$

$$s(x) = \sqrt{1+(\operatorname{sen} x)^2} - \operatorname{sen} x \tag{34}$$

$$G = 2 \sum_{n=0}^{\infty} \frac{(-1)^n \cos((2\,n+1)\,x)}{2\,n+1} (c(x))^{2\,n+1} f_n \;,\;\; -\pi/2 < x < \pi/2 \tag{35}$$

$$f_n = \frac{F(1,\, n+1/2;\, n+3/2;\, (c(x))^2)}{2\,n+1} + \frac{(c(x))^2\, F(1,\, n+3/2;\, n+5/2;\, (c(x))^2)}{2\,n+3} \tag{36}$$

$$c(x) = \sqrt{1+(\cos x)^2} - \cos x \tag{37}$$

donde

$$F(a,\, b;\, c;\, x) = \sum_{n=0}^{\infty} \frac{(a)_n\, (b)_n}{(c)_n} \frac{x^n}{n\,!} \;,\;\; -1 < x < 1 \tag{38}$$

5. Other series

$$G = 2 \sum_{n=0}^{\infty} \frac{(-1)^n \left(\sqrt{1+(\cos x)^2} - \cos x\right)^{2\,n+1}}{2\,n+1} \left(\frac{\cos((2\,n+1)\,x)}{2\,n+1} + 2 \sum_{k=1}^{n} \frac{(-1)^k \cos((2\,n-2\,k+1)\,x)}{2\,n-2\,k+1}\right) \tag{39}$$

$$-\pi/2 < x < \pi/2$$

$$G = 2 \sum_{n=0}^{\infty} \frac{\left(\sqrt{1+(\operatorname{sen} x)^2} - \operatorname{sen} x\right)^{2\,n+1}}{2\,n+1} \left(\frac{\operatorname{sen}((2\,n+1)\,x)}{2\,n+1} + 2 \sum_{k=1}^{n} \frac{\operatorname{sen}((2\,n-2\,k+1)\,x)}{2\,n-2\,k+1}\right) \tag{40}$$

$$0 < x < \pi$$

$$G = \sum_{n=0}^{\infty} \frac{(\operatorname{sen} x + \cos x)^{-n-1}}{n+1} \sum_{k=0}^{n} \frac{(\cos x)^{n-k} \operatorname{sen}((k+1)\,x)}{k+1} \;,\;\; 0 < x < \pi/2 \tag{41}$$

$$G = \frac{2}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{\left(-1/\sqrt{3}\right)^n}{n+1} \sum_{k=0}^{n} \binom{n}{n-k} \frac{(-2)^k}{k+1} \operatorname{sen}\left(\frac{(k+1)\,\pi}{3}\right)\Bigg) \tag{42}$$

$$G = \frac{4}{\sqrt{6}+\sqrt{2}} \sum_{n=0}^{\infty} \frac{1}{n+1} \left(-\frac{\left(\sqrt{3}-1\right)^2}{2}\right)^n \sum_{k=0}^{n} \binom{n}{n-k} \frac{\left(-\left(\sqrt{6}+\sqrt{2}\right)\right)^k}{k+1} \operatorname{sen}\left(\frac{5\,(k+1)\,\pi}{12}\right)\Bigg) \tag{43}$$

$$G = \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 \frac{x^{n-1} \operatorname{sen}(n\,x)}{(\operatorname{sen} x + x \cos x)^n}\, dx \tag{44}$$

References

A. Abramowitz, M. and Stegun, I.A. Handbook of Mathematical Functions. Nueva York:Dover,1965.

B. Adamchik, V. Integral and series Representations for Catalan's Constant. http://www-2.cs.cmu.edu/~adamchik/articles/catalan.htm.

C. Adamchik, V. Thirty-Three Representations for Catalan's Constant. http://library.wolfram.com/infocenter/Demos/109/.

D. Gradshteyn, I.S. and Ryzhik, I.M. "Table of Integrals, Series and Products." 5th ed., ed. Alan Jeffrey. Academic Press, 1994.