Dirichlet's proof of Fermat's Last Theorem for n = 5 is flawed

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Abstract

Dirichlet's using algorithm is not enough for proving FLT of $n = 5$.

1 Dirichlet's proof for n = 5

First, we rewrite a proof in the case z is odd and divisible by 5 (summary only, for details, please see:$x^5 + y^5 = z^5$ (Dirichlet's proof) in [1], [2]), which was proven by Dirichlet as follows:

Lemma. *if the equation $x^5 + y^5 = z^5$ is satisfied in integers, then one of the numbers x, y, and z must be divisible by 5 (corollary of Sophie Germain's theorem)*

Since x and y are both odd, their sum and difference are both even numbers.

$$2p = x + y$$

$$2q = x - y$$

Where the non-zero integers p and q are coprime and have different parity (one is even, the other odd). Since x = p + q and y = p - q, $z = 2^m 5^n z'$ it follows that

$$2^m 5^n z' = x^5 + y^5 = (p + q)^5 + (p - q)^5 = 2p(p^4 + 10p^2q^2 + 5q^4) \quad (1)$$

Since 5 divides $2p(p^4 + 10p^2q^2 + 5q^4)$, then there must be r such that p = 5r
$2p(p^4 + 10p^2q^2 + 5q^4) = 2(5r)[(5r)^4 + 10(5r)^2q^2 + 5q^4] =$
$2.5^2r(125r^4 + 50r^2q^2 + q^4)$
$2.5^2r(q^4 + 50r^2q^2 + 125r^4)$
Define three values u, v, t to be the following:
$t = q^4 + 50r^2q^2 + 125r^4$
$u = q^2 + 25r^2$
$v = 10r^2$
And note that $t = u^2 - 5v^2$
and t is a fifth power since $z^5 = 2.5^2r.t$, two factors 2.5^2r, and t are relatively prime, so t is a fifth power and 2.5^2r is a fifth power.
By using the infinite descent, Dirichlet claimed that if t is a fifth power, then there must be a smaller solution.
Setting:
$u = c(c^4 + 50c^2d^2 + 125d^4)$
$v = 5d(c^4 + 10c^2d^2 + 5d^4)$
now 2.5^2r is a fifth power, so $(2.5^2r)^2$ is a fifth power
$(2.5^2r)^2 = 2.5^3.10r^2 = 2.5^3.v = 2.5^3.5d(c^4 + 10c^2d^2 + 5d^4)$
since $\gcd 2.5^4d, c^4 + 10c^2d^2 + 5d^4 = 1$, then 2.5^4d and $c^4 + 10c^2d^2 + 5d^4$ are fifth power.
In other hand, $c^4 + 10c^2d^2 + 5d^4 = (c + 5d^2)^2 - 5(2d^2)^2 = u'^2 - 5v'^5$
Setting:

$u' = c'(c'^4 + 50c'^2d'^2 + 125d'^4)$
$v' = 5d'(c'^4 + 10c'^2d'^2 + 5d'^4)$
Since 2.5^4d is a fifth power, so $(2.5^4d)^2$ is also a fifth power
$(2.5^4d)^2 = 2.5^82d^2 = 2.5^8v' = 2.5^9d'(c'^4 + 10c'^2d'^2 + 5d'^4)$
So $2.5^9d'$, and $c'^4 + 10c'^2d'^2 + 5d'^4$ are also fifth power. $c'^4 + 10c'^2d'^2 + 5d'^4$ and $c^4 + 10c^2d^2 + 5d^4$ are the same form, and $d' < d$, by infinite descent, the original equation $t = u^2 - 5v^2$ has no solution.

2 Dirichlet's mistake

Dirichlet showed that, there are other ways in which can be a fifth power, but they have the same form as $u_0 = c(c^4 + 50c^2d^2 + 125d^4)$
$v_0 = 5d(c^4 + 10c^2d^2 + 5d^4)$
That means, the other solution will be:
$u_i = c_i(c_i^4 + 50c_i^2d_i^2 + 125d_i^4)$
$v_i = 5d_i(c^4 + 10c_i^2d_i^2 + 5d_i^4)$
Since $t = u^2 - 5v^2 = (c^2 - 5d^2)^5$, he claimed that if $c^2 - 5d^2$ has a prime factor, they are the same form as $c^2 - 5d^2$:
so all solutions must be the same form as u_0, v_0
However, this argument is incorrect as below:
The fact that, if N is not divisible by 5, then $N = e - 5f$
, so $N^5 = (e - 5f)^5 = e(e^2 + 50ef + 125f^2)^2 - 55^2f(e^2 + 10ef + 5f^2)^2$
in other hand,$N^5 = u_0^2 - 5v_0^2$
Select*:$u_0^2 = e(e^2 + 50ef + 125f^2)^2$
and $5v_0^2 = 55^2f(e^2 + 10ef + 5f^2)^2$
then e and f must be square, $e = c^2, f = d^2$
It gives :$u_0 = c(c^4 + 50c^2d^2 + 125d^4)$
$v_0 = 5d(c^4 + 10c^2d^2 + 5d^4)$
and $N = c^2 - 5d^2$
However, select* is the only way? There is no proof.
$N = c^2 - 5d^2$ is from select*, and is not from $N^5 = u_0^2 - 5v_0^2$
Note that: Gives:$A_1 = a_1^2 - 5b_1^2$, $A_2 = a_2^2 - 5b_2^2$, then:
$A = A_1A_2 = (a_1^2 - 5b_1^2)(a_2^2 - 5b_2^2)$
$A = A_1A_2 = (a_1a_2 + 5b_1b_2)^2 - 5(a_1b_2 + 5a_2b_1)^2$
$A = A_1A_2 = (a_1a_2 - 5b_1b_2)^2 - 5(a_1b_2 - 5a_2b_1)^2$
A is the same form as A_1, A_2
but if $A = a^2 - 5b^2 = A_1A_2$,then A_1, A_2 are not always the same form as A .
In Euler's proof of FLT for n = 3, we have seen a similar formula (lemma) such as:

$$a^2 + 3b^2 = (c^2 + 3d^2)^3$$

Here: $a = c(c^2 - 9d^2)$, $b = 3d(c^2 - d^2)$ with gcd(c,d) = 1, and c, d are nonezero.
Euler also used the technique of infinite descent, but by other way in modified version, unfortunately, his proof is also incorrect [3].
The algorithm above (using by Euler and Dirichlet) is the one way to find a solution of FLT for n = 3 and 5, if a solution is not found by this algorithm, it is not enough to conclude that the equation has no solution in integer.

References

[1] Fermat's Last theorem: Proof for n = 5 http://fermatslasttheorem.blogspot.com

[2] Paulo Ribenboim's *Fermat's last theorem for Amateurs*, Springer 1999

[3] Quang N V, Euler's proof of Fermat Last's Theorem for n = 3 is incorrect Vixra:1605.0123v3(NT)

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