Deducing special relativity from Newtonian physics.

The motivation and intended audience

It is widely understood that Newton's laws are a local approximation to general relativity. It is less widely understood that, as this paper and a later paper show, general relativity follows from a reliance on Newton's laws.

I have two audiences in mind. I hope it will help those who, like me, left school decades ago but still like to understand today's physics and are dissatisfied with inexact analogies. The second target audience are those academic physicists who do not realise how readily relativity follows from Newton's laws: I want to be able to reference this argument in later papers rather than having to remake it.

I find[1] that in 1908 Minkowski made effectively the same observation as I am making here. However, he used language that may not easily be understood by non-mathematicians, and anyhow I find that Minkowski's observations have been largely forgotten, so I think this paper still useful.

Background

The key to this paper is understanding what we mean when we say something happens 'at the same time' in two separate places: what exactly does it mean to say distributed clocks are synchronised? There are in fact several different possible meanings to 'at the same time', depending on the context. People may have a gut feeling that the meaning in context is obvious if a little hard to put into words, but that gut feeling can be misleading.

Newtonian laws are symmetrical, so they require that time is unbiased. It's easiest to understand what this means by thinking about a definition of time that is strongly biased. If you fly from London to New York, it takes roughly 3 hours by local time, but the return journey will take roughly 12 hours. The difference is largely accounted for by the bias inherent in local time. We can use symmetry to synchronise two clocks, or to test if two clocks are synchronised. All other things being equal, it should take as long to send a message between the clocks in either direction.[2]

At first sight this looks unhelpful. Movement that is symmetrical about a point which is stationary is not symmetrical about the same point if it is moving, nor is it symmetrical about some other point some distance away. So the symmetry of Newton's laws do not help us decide if two people in relative motion will attach the same meaning to 'at the same time'. It turns out they don't, and in showing that I shall derive special relativity. Similarly Newton's laws do not help us decide if two people who are widely separated in time or space will attach the same meaning to 'at the same time'. General relativity explains why they might not, and will be covered in a later paper.

Newton assumed that time was kept by God's clock, implying everyone everywhere would attach the same meaning to “at the same time”. The assumption was understandable, but wrong and anyhow was unnecessary.

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[1]From https://www.mathpages.com/rr/s1-07/1-07.htm

[2]In practice, 'all other things' are never exactly equal, leading to experimental error. In this paper I am conducting a thought experiment, and the only error possible is a logic error.

Part 1: The derivation of special relativity

I want to describe an area of deep space, sufficiently remote from any galaxy for gravity to be negligible. In order to describe what happens in the space I need a system that allows us to say when and where anything happens. I make a number of arbitrary choices. I choose a reference point as origin, and then choose 3 arbitrary directions that will allow me to say when and where something is relative to that reference point. For example, if at time t something is distance x in front, y to the right, and z above our reference point, I will say it is at **address** {t,x,y,z}. I assume the space is Euclidean.[3] Initially the reference point is at {0,0,0,0}, and after time t it is at {t,0,0,0}. Given this reference system we can apply Newton's laws of motion to the contents of space.[4]

I need a second reference frame against which to compare the first. I choose the same reference point but face in the opposite direction. At first I will choose that each reference frame be stationary with respect to the other, so if the address of a point is {t,x,y,z) in one reference frame, it is {t,-x,-y,z} in the other.[5]

Now consider what happens when one reference frame is moving with respect to the other. To keep things simple, I will always make the reference frames move along the same axis and in the same direction, so that when the relative speed is v, then the address within one reference frame of the origin of the other reference frame will be {t,vt,0,0}.

I want to see how we convert from one reference frame to another, and we shall use M_v to represent the algorithm that achieves that conversion. For example, I showed above what the conversion would be when the reference frames were stationary with respect to each other, meaning that v=0, so we know that $M_0\{t,x,y,z\}=\{t,-x,-y,z\}$.

Traditionally it was assumed that $M_v\{t,x,y,z\}$ should be {t,-x+vt,-y,z}, but that contains an assumption that 'simultaneous' means the same to both laboratories. We do not make that assumption if we take:

$$M_v\{t,x,y,z\} = \{a_v(t-b_vx),c_v(-x+vt),-y,z\}.$$ [6]

This contains three unknown quantities, a_v, b_v, and c_v which we need to evaluate. If we find that $a_v=c_v=1$ and $b_v=0$, then we shall know that Newtonian time is the same for everyone after all. In any case we know that when v is small, both a_v and c_v must be very close to 1, and b_v very close to 0.

Showing that $a_v=c_v$.

If we translate an address from one reference system to the other and then translate back, we should end up with the same address as we started with. So let us try that:

$$M_vM_v\{t,x,y,z\} = M_v\{a_v(t-b_vx),c_v(-x+vt),-y,z\}$$
$$=\{a_v^2(t-b_vx)-a_vb_vc_v(-x+vt),-c_v^2(-x+vt)+c_vva_v(t-b_vx),y,z\}$$

[3] This is indeed an assumption: it is not always absolutely true, which is why special relativity cannot be applied over very large distances.

[4] Implying that the reference point is inertial, the reference frame is not rotating, and that time runs at the same rate everywhere (space/time is flat).

[5] The 'in front' measure and the 'to the right' measure have changed signs because we are facing in the opposite direction, and because we (must have) faced in the opposite direction by turning so that 'above' remains 'above'. In other words, we turned just as you would expect to turn.

[6] This definition of M_v follows from an assumption that M_v is linear, meaning that $M_v\{t_1+t_2,x_1+x_2,y_1+y_2,z_1+z_2\}=M_v\{t_1,x_1,y_1,z_1\}+M_v\{t_2,x_2,y_2,z_2\}$, and that in turn follows from the assumptions listed above, plus the symmetry of Newtonian laws.

$=\{a_v(a_v-b_vc_vv)t+a_vb_v(-a_v+c_v)x, c_v(c_v-va_vb_v)x+c_vv(-c_v+a_v)t, y, z\}$

Comparing that address with $\{t,x,y,z\}$ we get:

$x=c_v(c_v-va_vb_v)x+c_vv(-c_v+a_v)t$

From this, $c_v(c_v-va_vb_v)=1$ and $c_vv(-c_v+a_v)=0$

From the first of these, $c_v\neq0$, so from the second, $c_v=a_v$, and hence from the first again, $a_v^2(1-vb_v)=1$.

I will use $a_v^2(1-vb_v)=1$ later. I will use $c_v=a_v$ now to rewrite the definition of M_v thus:

$$M_v\{t,x,y,z\} = \{a_v(t-b_vx), a_v(-x+vt), -y, z\}.$$

Showing that b_v/v is constant

I shall now to work with several reference frames in order to look at what happens as we change the relative speed of the two frames. M_0 turns the frame round without changing the speed, so if u and v are both positive, $M_uM_0M_v$ is equivalent to M_w where w is some speed greater than either u or v. You might instinctively expect that w=u+v, but let us check that.

$M_w\{t,x,y,z\}=M_uM_0M_v\{t,x,y,z\}$

$= M_uM_0\{a_v(t-b_vx), a_v(-x+vt), -y, z\}$

$= M_u\{a_v(t-b_vx), -a_v(-x+vt), y, z\}$

$=\{a_ua_v(t-b_vx)+a_ub_ua_v(-x+vt), a_ua_v(-x+vt)+a_uua_v(t-b_vx), -y, z\}$

$=\{a_ua_v[(1+b_uv)t-(b_v+b_u)x], -a_ua_v[(1+ub_v)x-(v+u)t], -y, z\}$

Compare this address with

$M_w\{t,x,y,z\}=\{a_w(t-b_wx), a_w(-x+wt), -y, z\}$

and note that $a_ua_v(1+b_uv)=a_w$ and $a_ua_v(1+ub_v)=a_w$ and $a_ua_v(v+u)=a_ww$.

From the second and third of these equations, notice that $(v+u)=(1+ub_v)w$, so w is only equal to u+v if u or b_v is zero.

From the first two equations notice that $b_uv=ub_v$, so b_v/v is the same as b_u/u, and is therefore a constant. Setting $k=b_v/v$ we can rewrite M_v as:

$$M_v\{t,x,y,z\}=\{a_v(t-kvx), a_v(-x+vt), -y, z\},$$

We can also rewrite the relation between u, v, and w as $(v+u)=(1+uvk)w$, and remembering that $a_v^2(1-vb_v)=1$, we can now say $a_v^2=1/(1-v^2k)$.

k has dimensions of $\text{time}^2/\text{distance}^2$. We don't know how big k is, but we do know that if $k>0$ then $1/\sqrt{k}$ is the highest speed to which we can accelerate anything that is affected by time, since above that speed we can no longer calculate the value of a_v from the equation $a_v^2=1/(1-v^2k)$.

An experiment to determine the value of k

We now (at last) have something that we can hope to measure by experiment. If we accelerate something to a high speed, does there come a point at which it is impossible to make it go faster? When Einstein first proposed the theory of relativity, the technology did not exist to perform this experiment, but it has been performed many times since. We now know there is indeed a limiting speed. Places like CERN accelerate a particle to within a whisker of the maximum possible, so we know k is positive, and we know its value very accurately. We find that, within experimental error, the limiting speed is the same as the speed of light in a vacuum.
If something is going at the limiting speed in one frame of reference, it must be moving at the limiting speed in all reference frames,[7] which explains why all attempts to find some variation in the speed of light failed.

[7] This is proved in appendix A.

Relativistic coordinates

For historical reasons we say that nothing can go faster than the speed of light, and it is more convenient to talk about the speed of light than to talk about some abstract limiting speed.

We can choose the units for time and distance that make the speed of light equal to one: for example, light takes a year to travel a distance of one light year. These are known as relativistic units, and when working with relativistic units the speed of light and k can be left out of definitions and formulae. For example, M_v becomes:

$$M_v\{t,x,y,z\} = \{a_v(t-vx),a_v(-x+vt),-y,z\}, \quad \text{where } a_v = 1/\sqrt{(1-v^2)}.$$

It is more usual to use the symbol β rather than a_v, so $\beta=1/\sqrt{(1-v^2)}$. M_v is a special case of the Lorentz transform.[8]

Part 2: Special relativity and Faraday's laws.

It is interesting to see that Faraday's laws arguably allow us to determine the sign and value of k. CERN offers certainty, but Faraday's laws came much earlier. So I will again pretend we know nothing about special relativity in general, and the value of k in particular.

From now on I will use the formula $L_v=M_0M_v$ in order to remove the rotation that M_v on its own introduces, so:

$$\begin{aligned} L_v\{t,x,y,z\} &= M_0M_v\{t,x,y,z\} \\ &= M_0\{a_v(t-kvx),a_v(-x+vt),y,z\}, \text{ where } a_v^2=1/(1-v^2k). \\ &= \{a_v(t-kvx),a_v(x-vt),y,z\}, \text{ since } a_0^2=1. \end{aligned}$$

and I extend the definition of L_v so that if velocity v has components in the three directions of v_x, v_y, and v_z, then $L_v\{1,0,0,0\}=a_v\{1,-v_x,-v_y,-v_z\}$.[9]

The effect on force of changing the reference frame.

Suppose an object has address $A=\{\tau,x,y,z\}$ in its rest frame[10]. τ is called the object's **proper time**. The quantity $dA/d\tau$ is called the **four velocity** of the object: in the rest frame $dA/d\tau = \{1,0,0,0\}$. If m_0 is the mass of the body measured in it rest frame, and **V** is the four velocity, then $\mathbf{P}=m_0\mathbf{V}=\{M,p_x,p_y,p_z\}$ is the **four momentum**, where M is the mass in the current reference frame, and p_x,p_y, and p_z are the momentum of the body in each direction. The **four force** on the body is $d\mathbf{P}/d\tau=\{c_t,c_x,c_y,c_z\}$, so (for example) c_x is proportional to the acceleration of the body in the x direction.

Although the four velocity, momentum, and force are not addresses, we can use the algorithm L_v to convert them between frames.[11]

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[8] M_v represents a Lorentz boost in the first space dimension combined with a rotation through 180 degrees about the third space dimension. A simple Lorentz boost would be:

$$M_0M_v\{t,x,y,z\} = \{\beta(t-vx),\beta(x-vt),y,z\}, \quad \text{where } \beta = 1/\sqrt{(1-v^2)}.$$

[9] I have here taken liberties with the definition of L_v. generalising it to enable us to apply it when v is not a velocity in the x direction. To see that the generalised L_v works in this special case where it is used to convert {1,0,0,0}, rotate the spacial coordinates until v points in the x direction, do the convertion, and then rotate the coordinates back.

[10] The rest frame is that in which the object is not moving. It may be accelerating, but in the rest frame it is accelerating from rest.

[11] since (for example) $L_v(d/d\tau\, m_0\{t,x,y,z\})=m_0 d/d\tau\, L_v\{t,x,y,z\}$.

Suppose that a charged body is moving in an electric field which is aligned with the y axis. If there is no magnetic field, the body will experience four force $\{c_t,0,c_y,0\}$. But look at what we get if we convert to a reference frame moving at speed v in the x direction:

$$L_v\{c_t,0,c_y,0\}=\{a_vc_t,-a_vvc_t,c_y,0\}$$

If $k\neq 0$, the electric field now exerts a force on the charge along the x axis, the direction of the force being dependant on the sign of c_1 and hence on the sign of k. This force is electromagnetism.

If k=0, then mass does not change with velocity so c_1 is zero and Newtonian physics would not predict the magnetic force.

Conservation of momentum and electrical forces.

That k=1 explains magnetism: it also explains how momentum is conserved when an electrical field pushes on an electron. The field pushes on the electron and the electron pushes on the field: action and reaction are equal and opposite.[12] By contrast, if k=0 than there is no predicted reaction, and we must assume the reaction is absorbed by some as yet undetected medium. We make the same sort of assumption when (for example) calculating the acceleration of a car. The car accelerates one way and the earth accelerates in the opposite direction, but the mass of the earth is so enormously greater than that of the car that the acceleration of the earth can be ignored. This supposed medium defines a rest frame relative to which electrical fields and charges, and everything else including the earth, could be said to move. In particular, the speed of light was calculated in relation to the proposed medium, and so was expected to vary in relation to the earth as the earth moved round the sun. It was the failure to detect such variation that finally led to special relativity. However, there are logical problems with an attempt to explain electromagnetism by postulating a stationary medium. For example, it would imply that the mass of the stationary medium would be enormously greater than the mass of any charged particles. This makes it hard to imagine how the relative velocity of the medium could change with time and space. To illustrate the problem, it is hard to generate a sound wave in air that does significant damage, but relative movements of air are the cause of hurricanes, and can do enormous damage. The relative movements of the medium would imply some mechanism that involves changes of momentum and energy on an enormously larger scale and density than can be achieved by electrical forces.

Summary

Explanations of the special theory of relativity usually start from an assumption that the speed of light is a constant for all observers, and use that to work out the Lorentz transform. I have shown that it is not necessary to make any assumption about the speed of light. This is useful because that assumption sometimes misleads people. If they have difficulty accepting the predictions of the special theory, they are tempted to doubt that the speed of light is always constant, and to conclude that we just haven't tried hard enough to find the exceptions. And if we assume that the theory of relativity depends on the constancy of the speed of light, then it invites speculation about what would happen if the speed of light changed.

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[12]This is obvious once a tensor notation is introduced, and I apologise for relying on that. If the energy momentum tensor at a point in space is $T^{ab}=T1^{ab}+T2^{ab}$ where $T1^{ab}$ is the energy momentum tensor for the charge matter and $T2^{ab}$ is the energy momentum tensor for the electromagnetic field, then $T1^{ab}{}_{;b}$ is the four force acting on the charge, and $T2^{ab}{}_{;b}$ is the four force acting on the field, and $T^{ab}{}_{;b}=0$, so $T1^{ab}{}_{;b}=-T2^{ab}{}_{;b}$.

Appendix A: the limiting speed.

If an object moves at the limiting speed in one reference frame, it moves at the limiting speed in all reference frames. To verify this, suppose something moves from {0,0,0,0} to {t,x,y,z}, at speed $s=1/\sqrt{k}$, then $s^2=(x^2+y^2+z^2)/t^2$. Converting to a different reference frame, $M_v\{t,x,y,z\}=\{a_v(t-vx/s^2),a_v(-x+vt),-y,z\}$, where $a_v{}^2=s^2/(s^2-v^2)$, and the square of the speed in the new reference frame will be:

$$(a_v{}^2(-x+vt)^2+y^2+z^2)/a_v{}^2(t-vx/s^2)^2$$

$$= (a_v{}^2(-x+vt)^2+(s^2t^2-x^2))/a_v{}^2(t-vx/s^2)^2$$

$$= s^2(s^2(-x+vt)^2+(s^2-v^2)(s^2t^2-x^2))/(s^2t-vx)^2$$

$$= s^2(s^2x^2-2s^2xvt+s^2v^2t^2+s^4t^2-s^2x^2-v^2s^2t^2+v^2x^2)/(s^2t-vx)^2$$

$$= s^2(-2s^2xvt+s^4t^2+v^2x^2)/(s^2t-vx)^2$$

$$= s^2$$