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Lucas's Inner Circles

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In this article, we define the **Lucas's inner circles** and we highlight some of their properties.

1. Definition of the Lucas's Inner Circles

Let ABC be a random triangle; we aim to construct the square inscribed in the triangle ABC, having one side on BC.

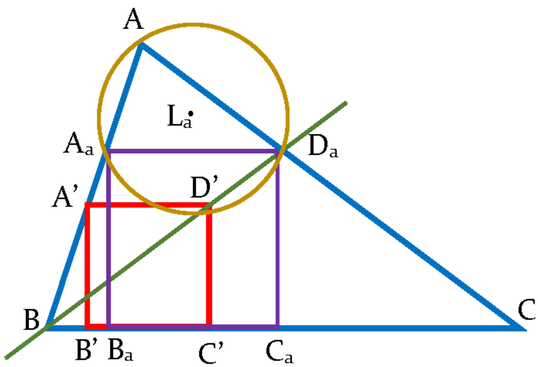

Figure 1.

In order to do this, we construct a square $A'B'C'D'$ with $A' \in (AB)$, $B', C' \in (BC)$ (see *Figure 1*).

We trace the line BD' and we note with D_a its intersection with (AC); through D_a we trace the parallel D_aA_a to BC with $A_a \in (AB)$ and we project onto BC the points A_a, D_a in B_a respectively C_a.

We affirm that the quadrilateral $A_aB_aC_aD_a$ is the required square.

Indeed, $A_aB_aC_aD_a$ is a square, because $\frac{D_aC_a}{D'C'} = \frac{BD_a}{BD'} = \frac{A_aD_a}{A'D'}$ and, as $D'C' = A'D'$, it follows that $A_aD_a = D_aC_a$.

Definition.

It is called A-Lucas's inner circle of the triangle ABC the circle circumscribed to the triangle $AAaDa$.

We will note with L_a the center of the A-Lucas's inner circle and with l_a its radius.

Analogously, we define the B-Lucas's inner circle and the C-Lucas's inner circle of the triangle ABC.

2. Calculation of the Radius of the A-Lucas Inner Circle

We note $A_aD_a = x$, $BC = a$; let h_a be the height from A of the triangle ABC.

The similarity of the triangles AA_aD_a and ABC leads to: $\frac{x}{a} = \frac{h_a{}^{-x}}{h_a}$, therefore $x = \frac{ah_a}{a+h_a}$.

From $\frac{l_a}{R} = \frac{x}{a}$ we obtain $l_a = \frac{R.h_a}{a+h_a}$. (1)

Note.

Relation (1) and the analogues have been deduced by Eduard Lucas (1842-1891) in 1879 and they constitute the "birth certificate of the Lucas's circles".

1st Remark.

If in (1) we replace $h_a = \frac{2S}{a}$ and we also keep into consideration the formula $abc = 4RS$, where R is the radius of the circumscribed circle of the triangle ABC and S represents its area, we obtain:

$l_a = \frac{R}{1+\frac{2aR}{bc}}$ [see Ref. 2].

3. Properties of the Lucas's Inner Circles

1st Theorem.

The Lucas's inner circles of a triangle are inner tangents of the circle circumscribed to the triangle and they are exteriorly tangent pairwise.

Proof.

The triangles AA_aD_a and ABC are homothetic through the homothetic center A and the rapport: $\frac{h_a}{a+h_a}$.

Because $\frac{l_a}{R} = \frac{h_a}{a+h_a}$, it means that the A-Lucas's inner circle and the circle circumscribed to the triangle ABC are inner tangents in A.

Analogously, it follows that the B-Lucas's and C-Lucas's inner circles are inner tangents of the circle circumscribed to ABC.

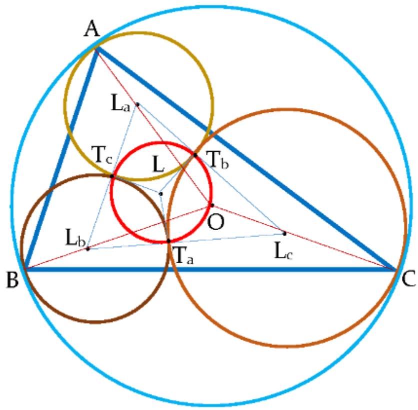

Figure 2.

We will prove that the A-Lucas's and C-Lucas's circles are exterior tangents by verifying

$$L_aL_c = l_a + l_c. \quad (2)$$

We have:

$OL_a = R - l_a;$

$OL_c = R - l_c$

and

$m(\widehat{AOC}) = 2B$

(if $m(\hat{B}) > 90°$ then $m(\widehat{AOC}) = 360° - 2B$).

The theorem of the cosine applied to the triangle OL_aL_c implies, keeping into consideration (2), that:

$$(R-l_a)^2+(R-l_a)^2-2(R-l_a)(R-l_c)cos2B =$$
$$=(l_a+l_c)^2.$$

Because $cos2B = 1-2sin^2B$, it is found that (2) is equivalent to:

$$sin^2B = \frac{l_al_c}{(R-l_a)(R-l_c)}\ . \qquad (3)$$

But we have: $l_al_c = \frac{R^2ab^2c}{(2aR+bc)(2cR+ab)}$,

$l_a + l_c = Rb(\frac{c}{2aR+bc}+\frac{a}{2cR+ab})$.

By replacing in (3), we find that $sin\,^2B = \frac{ab^2c}{4acR^2} = \frac{b^2}{4a^2} \Leftrightarrow \sin B = \frac{b}{2R}$ is true according to the sines theorem. So, the exterior tangent of the A-Lucas's and C-Lucas's circles is proven.

Analogously, we prove the other tangents.

2nd Definition.

It is called an A-Apollonius's circle of the random triangle ABC the circle constructed on the segment determined by the feet of the bisectors of angle A as diameter.

Remark.

Analogously, the B-Apollonius's and C-Apollonius's circles are defined. If ABC is an isosceles triangle with $AB = AC$ then the A-Apollonius's circle isn't defined for ABC, and if ABC is an equilateral triangle, its Apollonius's circle isn't defined.

2nd Theorem.

The A-Apollonius's circle of the random triangle is the geometrical point of the points M from the plane of the triangle with the property: $\frac{MB}{MC} = \frac{c}{b}$.

3rd Definition.

We call a fascicle of circles the bunch of circles that do not have the same radical axis.

a. If the radical axis of the circles' fascicle is exterior to them, we say that the fascicle is of the first type.
b. If the radical axis of the circles' fascicle is secant to the circles, we say that the fascicle is of the second type.
c. If the radical axis of the circles' fascicle is tangent to the circles, we say that the fascicle is of the third type.

3rd Theorem.

The A-Apollonius's circle and the B-Lucas's and C-Lucas's inner circles of the random triangle ABC form a fascicle of the third type.

Proof.

Let $\{O_A\} = L_bL_c \cap BC$ (see *Figure 3*).

Menelaus's theorem applied to the triangle OBC implies that:

$$\frac{O_AB}{O_AC} \cdot \frac{L_bB}{L_bO} \cdot \frac{L_cO}{L_cC} = 1,$$

so:

$$\frac{O_AB}{O_AC} \cdot \frac{l_b}{R-l_b} \cdot \frac{R-l_c}{l_c} = 1$$

and by replacing l_b and l_c, we find that:

$$\frac{O_AB}{O_AC} = \frac{b^2}{c^2}.$$

This relation shows that the point O_A is the foot of the exterior symmedian from A of the triangle ABC (so the tangent in A to the circumscribed circle), namely the center of the A-Apollonius's circle.

Let N_1 be the contact point of the B-Lucas's and C-Lucas's circles. The radical center of the B-Lucas's, C-Lucas's circles and the circle circumscribed to the triangle ABC is the intersection T_A of the tangents traced in B and in C to the circle circumscribed to the triangle ABC.

It follows that $BT_A = CT_A = N_1T_A$, so N_1 belongs to the circle $\mathcal{C}_A$ that has the center in T_A and orthogonally cuts the circle circumscribed in B and C. The radical axis of the B-Lucas's and C-Lucas's circles is T_AN_1, and O_AN_1 is tangent in N_1 to the circle $\mathcal{C}_A$. Considering the power of the point O_A in relation to $\mathcal{C}_A$, we have:

$$O_AN_1{}^2 = O_AB.O_AC.$$

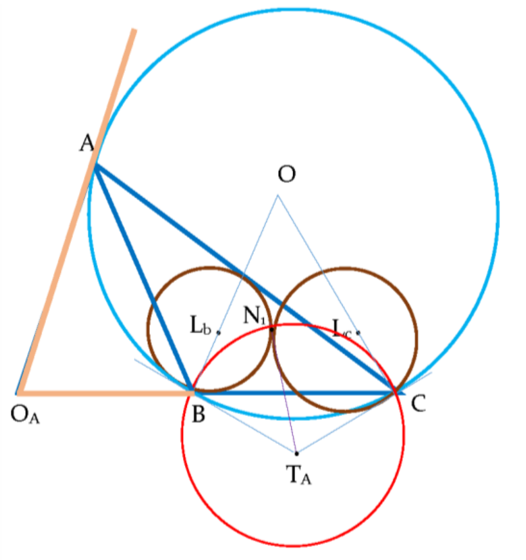

Figure 3.

Also, $O_AO^2 = O_AB \cdot O_AC$; it thus follows that $O_AA = O_AN_1$, which proves that N_1 belongs to the A-Apollonius's circle and is the radical center of the A-Apollonius's, B-Lucas's and C-Lucas's circles.

Remarks.

1. If the triangle ABC is right in A then $L_bL_c||BC$, the radius of the A-Apollonius's circle is equal to: $\frac{abc}{|b^2-c^2|}$. The point N_1 is the foot of the bisector from A. We find that $O_AN_1 = \frac{abc}{|b^2-c^2|}$, so the theorem stands true.

2. The A-Apollonius's and A-Lucas's circles are orthogonal. Indeed, the radius of the A-Apollonius's circle is perpendicular to the radius of the circumscribed circle, OA, so, to the radius of the A-Lucas's circle also.

4th Definition.

The triangle $T_A T_B T_C$ determined by the tangents traced in A, B, C to the circle circumscribed to the triangle ABC is called the tangential triangle of the triangle ABC.

1st Property.

The triangle ABC and the Lucas's triangle $L_a L_b L_c$ are homological.

Proof.

Obviously, AL_a, BL_b, CL_c are concurrent in O, therefore O, the center of the circle circumscribed to the triangle ABC, is the homology center.

We have seen that $\{O_A\} = L_b L_c \cap BC$ and O_A is the center of the A-Apollonius's circle, therefore the homology axis is the Apollonius's line $O_A O_B O_C$ (the line determined by the centers of the Apollonius's circle).

2nd Property.

The tangential triangle and the Lucas's triangle of the triangle ABC are orthogonal triangles.

Proof.

The line $T_A N_1$ is the radical axis of the B-Lucas's inner circle and the C-Lucas's inner circle, therefore it is perpendicular on the line of the centers $L_b L_c$. Analogously, $T_B N_2$ is perpendicular on $L_c L_a$, because the radical axes of the Lucas's circles are concurrent in L, which is the radical center of the Lucas's circles; it follows that $T_A T_B T_C$ and $L_a L_b L_c$ are orthological and L is the center of orthology. The other center of orthology is O the center of the circle circumscribed to ABC.

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