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The Polars of a Radical Center

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In [1] , the late mathematician **Cezar Cosnita**, using the barycenter coordinates, proves two theorems which are the subject of this article.

In a remark made after proving the first theorem, C. Cosnita suggests an elementary proof by employing the **concept of polar**.

In the following, we prove the theorems based on the indicated path, and state that the second theorem is a particular case of the former. Also, we highlight other particular cases of these theorems.

1st Theorem.

Let ABC be a given triangle; through the pairs of points (B, C), (C, A) and (A, B) we take three circles such that their radical center is on the outside.

The polar lines of the radical center of these circles in relation to each of them cut the sides BC, CA and AB respectively in three collinear points.

Proof.

We denote by D, P, F the second point of intersection of the pairs of circles passing through (A, B) and (B, C); (B, C) and (A, C), (B, C) and (A, B) respectively (see *Figure 1*).

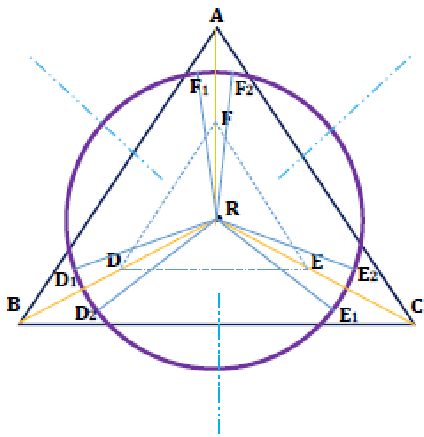

Figure 1

Let R be the radical center of those circles. In fact, $\{R\} = AF \cap BD \cap CE$.

We take from R the tangents $RD_1 = RD_2$ to the circle (B, C), $RE_1 = RE_2$ to the circle (A, C) and $RF_1 = RF_2$ to the circle passing through (A, B). Actually, we build the radical circle $\mathcal{C}(R, RD_1)$ of the given circles.

The polar lines of R to these circles are the lines D_1D_2, E_1E_2, F_1F_2. These three lines cut BC, AC and AB in the points X, Y and Z, and these lines are respectively the polar lines of R in respect to the circles passing through $(B,C),(C,A)$ and (A,B). The polar lines are the radical axis of the radical circle with each of the circles passing through $(B,C),(C,A),(A,B)$, respectively. The points belong to the radical axis having equal powers to those circles, thereby $XD_1 \cdot XD_2 = XC \cdot XB$.

This relationship shows that the point X has equal powers relative to the radical circle and to the circle circumscribed to the triangle ABC; analogically, the point Y has equal powers relative to the radical circle and to the circle circumscribed to the triangle ABC; and, likewise, the point Z has equal powers relative to the radical circle and to the circle circumscribed to the triangle ABC.

Because the locus of the points having equal powers to two circles is generally a line, i.e. their radical axis, we get that the points X, Y and Z are collinear, belonging to the radical axis of the radical circle and to the circle circumscribed to the given triangle.

2nd Theorem.

If M is a point in the plane of the triangle ABC and the tangents in this point to the circles circumscribed to triangles C, MAC, MAB, respectively, cut BC, CA and AB, respectively, in the points X, Y, Z, then these points are collinear.

Proof.

The point M is the radical center for the circles (MBC), (MAC), and (MAB), and the tangents in M to these circles are the polar lines to M in relation to these circles.

If X, Y, Z are the intersections of these tangents (polar lines) with BC, CA, AB, then they belong to the radical axis of the circumscribed circle to the triangle ABC and to the circle "reduced" to the point M ($XM^2 = XB \cdot XC$, etc.).

Being located on the radical axis of the two circles, the points X, Y, Z are collinear.

Remarks.

1. Another elementary proof of this theorem is to be found in [3].
2. If the circles from the *1st theorem* are adjoint circles of the triangle ABC, then they intersect in Ω (the Brocard's point). Therefore, we get that the tangents taken in Ω to the adjoin circles cut the sides BC, CA and AB in collinear points.

References.

[1] C. Coșniță: *Coordonées baricentrique* [Barycentric Coordinates], Bucarest – Paris: Librairie Vuibert, 1941.

[3] I. Pătrașcu: *Axe și centre radicale ale cercului adjuncte unui triunghi* [Axis and radical centers of the adjoin circle of a triangle], in "Recreații matematice", year XII, no. 1, 2010.

[3] C. Mihalescu: *Geometria elementelor remarcabile* [The Geometry of Outstanding Elements], Bucharest: Editura Tehnica, 1957.

[4] F. Smarandache, I. Patrascu: *Geometry of Homological Triangle*, Columbus: The Educational Publisher Inc., 2012.