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A Function in the Number Theory

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A FUNCTION IN THE NUMBER THEORY

Summary

In this paper I shall construct a function η having the following properties:

$$\forall \eta \in Z \quad n \neq 0 \quad (\eta(n))! = M \cdot n, \tag{1}$$

$$\eta(n) \quad \text{is the smallest natural number with the property(1).} \tag{2}$$

We consider: $N = \{0, 1, 2, 3, \ldots\}$ and $N^* = \{1, 2, 3, \ldots\}$.

Lema 1. $\forall k, p \in N^*, p \neq 1, k$ *is uniquely written under the shape:* $k = t_1 a_{n_1}^{(p)} + \ldots + t_l a_{n_l}^{(p)}$ *where* $a_{n_1}^{(p)} = \frac{p^{n_i - 1}}{p-1}$, $i = \overline{1,l}$, $n_1 > n_2 > \ldots > n_l > 0$ *and* $1 \leq t_j \leq p-1$, $j = \overline{1, l-1}$, $1 \leq t_l \leq p$, $n_j, t_j \in N$, $i = \overline{1,l}$ $l \in N^*$.

Proof. The string $(a_n^{(p)})_{n \in N^*}$ consists of strictly increasing infinite natural numbers and $a_{n+1}^{(p)} - 1 = p \cdot a_n^{(p)}$, $;\forall n \in N^*, p$ is fixed,

$$a_1^{(p)} = 1, a_2^{(p)} = 1 + p, a_3^{(p)} = 1 + p + p^2, \ldots \Rightarrow N^* = \bigcup_{n \in N^*} ([a_n^{(p)}, a_{n+1}^{(p)}) \cap N^*)$$

where $[a_n^{(p)}, a_{n+1}^{(p)}) \cap [a_{n+1}^{(p)}, a_{n+2}^{(p)}) = \emptyset$ because $a_n^{(p)} < a_{n+1}^{(p)} < a_{n+2}^{(p)}$.

Let $k \in N^*$, $N^* = \bigcup_{n \in N^*} ([a_n^{(p)}, a_{n+1}^{(p)}) \cap N^*) \Rightarrow \exists! n_1 \in N^* : k \in ([a_{n_1}^{(p)}, a_{n_1+1}^{(p)}) \Rightarrow k)$ is uniquely written under the shape $k = \left[\frac{k}{a_{n_1}^{(p)}}\right] a_{n_1}^{(p)} + r_1$ (integer division theorem). We note $k = \left[\frac{k}{a_{n_1}^{(p)}}\right] = t_1 \Rightarrow k = t_1 a_{n_1}^{(p)} + r_1$, $r_1 < a_{n_1}^{(p)}$.

If $r_1 = 0$, as $a_{n_1}^{(p)} \leq k \leq a_{n_1+1}^{(p)} - 1 \Rightarrow 1 \leq t_1 \leq p$ and Lemma 1 is proved.

If $r_1 \neq 0 \Rightarrow \exists! n_2 \in N^* : r_1 \in [a_{n_2}^{(p)}, a_{n_2+1}^{(p)})$; $a_{n_1}^{(p)} > r_1 \Rightarrow n_1 > n_2$, $r_1 \neq 0$ and $a_{n_1}^{(p)} \leq k \leq$ $\leq a_{n_1+1}^{(p)} - 1 \Rightarrow 1 \leq t_1 \leq p-1$ because we have $t_1 \leq (a_{n_1+1}^{(p)} - 1 - r_1) : a_n^{(p)} < p_1$.

The procedure continues similarly. After a finite number of steps l, we achieve $r_l = 0$, as k =finite, $k \in N^*$ and $k > r_1 > r_2 \ldots > r_l = 0$ and between 0 and k there is only a finite number of distinct natural numbers.

Thus:

k is uniquely written: $k = t_1 a_{n_1}^{(p)} + r_1$, $1 \leq t_1 \leq p-1$, r is uniquely written: $r_1 = t_2 a_{n_2}^{(p)} + r_2$, $n_2 < n_1$,

$$1 \leq t_2 \leq p-1,$$

r_{l-1} is uniquely written: $r_{l-1} = t_l a_{n_l}^{(p)} + r_l$ and $r_l = 0$,

$$n_l < n_{l-1},\ 1 \le t_l \le p,$$

$\Rightarrow k$ is uniquely written under the shape $k = t_1 a_{n_1}^{(p)} + \ldots + t_l a_{n_l}^{(p)}$ with $n_1 > n_2 > \ldots > n_l$; $n_l > 0$ becuase $n_l \in N^*$, $1 \le t_j \le p-1$, $j = \overline{1, l-1}$, $1 \le t_l \le p$, $l \ge 1$.

Let $k \in N^*$, $k = t_1 a_{n_1}^{(p)} + \ldots + t_l a_{n_l}^{(p)}$, with $a_{n_i}^{(p)} = \dfrac{p^{n_i}-1}{p-1}$, $i = \overline{1,l}$, $l \ge 1$, $n_i, t_i \in N^*$, $i = \overline{1,l}$, $n_1 > n_2 > \ldots > n_l > 0$, $1 \le t_j \le p-1$, $j = \overline{1, l-1}$, $1 \le t_l \le p$.

I construct the function η_p, $p = \text{prime} > 0$, $\eta_p : N^* \to N^*$ thus:

$$\forall n \in N^* \quad \eta_p(a_n^{(p)}) = p^n,$$

$$\eta_p(t_1 a_{n_1}^{(p)} + \ldots + t_l a_{n_l}^{(p)}) = t_1 \eta_p(a_{n_1}^{(p)}) + \ldots + t_l \eta_p(a_{n_l}^{(p)}).$$

Note 1. The function η_p is well defined for each natural number.

Proof.

Lema 2. $\forall k \in N^* \Rightarrow k$ *is uniquely written as* $k = t_1 a_n^{(p)} + \ldots + t_l a_{n_l}^{(p)}$ *with the conditions from Lemma 1* $\Rightarrow \exists!\ t_1 p^{n_1} + \ldots + t_l p^{n_l} = \eta_p(t_1 a_{n_1}^{(p)} + \ldots + t_l a_{n_l}^{(p)})$ *and* $t_{1p}^{n_1} + t_{lp}^{n_l} \in N^*$.

Lema 3. $\forall k \in N^*$, $\forall p \in N, p = \textit{prime} \Rightarrow k = t_1 a_{n_1}^{(p)} + \ldots t_l a_{n_l}^{(p)}$ *with the conditions from Lemma 2* $\Rightarrow \eta_p(k) = t_1 p^{n_1} + \ldots + t_l p^{n_l}$.

It is known that $\left[\dfrac{a_1 + \ldots + a_n}{b}\right] \ge \left[\dfrac{a_1}{b}\right] + \ldots + \left[\dfrac{a_n}{b}\right]$ $\forall a_i, b \in N^*$ where through $[\alpha]$ we have written the integer side of number α. I shall prove that p's powers sum from the natural numbers make up the result factors $(t_1 p^{n_1} + \ldots + t_l p^{n_l})!$ is $\ge k$;

$$\left[\frac{t_1 p^{n_1} + \ldots + t_l p^{n_l}}{p}\right] \ge \left[\frac{t_1 p^{n_1}}{p}\right] + \ldots + \left[\frac{t_l p^{n_l}}{p}\right] = t_1 p^{n_1 - 1} + \ldots + t_l p^{n_l - 1}$$

$$\vdots$$

$$\left[\frac{t_1 p^{n_1} + \ldots + t_l p^{n_l}}{p^{n}}\right] \ge \left[\frac{t_1 p^{n_1}}{p^{n_l}}\right] + \ldots + \left[\frac{t_l p^{n_l}}{p^{n_l}}\right] = t_1 p^{n_1 - n_l} + \ldots + t_l p^0$$

$$\vdots$$

$$\left[\frac{t_1 p^{n_1} + \ldots + t_l p^{n_1}}{p^{n_1}}\right] \ge \left[\frac{t_1 p^{n_1}}{p^{n_1}}\right] + \ldots + \left[\frac{t_l p^{n_l}}{p^{n_l}}\right] = t_1 p^0 + \ldots + \left[\frac{t_l p^{n_l}}{p^{n_l}}\right].$$

Adding $\Rightarrow$ p's powers sum is $\ge t_1(p^{n_1-1} + \ldots + p^0) + \ldots + t_l(p^{n_l-1} + \ldots + p^0) = t_1 a_{n_1}^{(p)} + \ldots t_l a_{n_l}^{(p)} = k$.

Theorem 1. *The function n_p, p =prime, defined previously, has the following properties:*

(1) $\forall k \in N^*$, $(n_p(k))! = Mp^k$.

(2) $\eta_p(k)$ *is the smallest number with the property (1).*

Proof.

(1) results from Lemma 3.

(2) $\forall k \in N^*$, $p \geq 2 \Rightarrow k = t_1 a_{n_1}^{(p)} + \ldots + t_l a_{n_l}^{(p)}$ (by Lemma 2) is uniquely written, where:

$n_i, t_i \in N^*$, $n_1 > n_2 > \ldots > n_l > 0$, $a_{n_i}^{(p)} = \dfrac{p^{n_i} - 1}{p - 1} \in N^*$, $i = \overline{1,l}$, $1 \leq t_j \leq p - 1$, $j = \overline{1, l-1}$, $1 < t_l < p$.

$\Rightarrow \eta_p(k) = t_1 p^{n_1} + \ldots + t_l p^{n_l}$. I note: $z = t_1 p^{n_1} + \ldots t_l p^{n_l}$.

Let us prove the z is the smallest natural number with the property (1). I suppose by the method of reduction ad absurdum that $\exists \gamma \in N$, $\gamma < z$:

$\gamma! = Mp^k$;

$\gamma < z \Rightarrow \gamma \leq z - 1 \Rightarrow (z-1)! = Mp^k$.

$z - 1 = t_1 p^{n_1} + \ldots + t_l p_l^{n_l} - 1$; $n_1 > n_2 > \ldots > n_l \geq 0$ and $n_j \in N$, $j = \overline{1,l}$;

$\left[\dfrac{z-1}{p}\right] = t_1 p^{n_1 - 1} + \ldots + t_{l-1} p^{n_{l-1} - 1} + t_l p^{n_l - 1} - 1$ as $\left[\dfrac{-1}{p}\right] = -1$ because $p \geq 2$,

$\left[\dfrac{z-1}{p^{n_l}}\right] = t_1 p^{n_1 - n_l} + \ldots + t_{l-1} p^{n_{l-1} - n_l} + t_l p^0 - 1$ as $\left[\dfrac{-1}{p^{n_l}}\right] = -1$ as $p \geq 2$, $n_l \geq 1$,

$\left[\dfrac{z-1}{p^{n_l+1}}\right] = t_1 p^{n_1 - n_l - 1} + \ldots + t_{l-1} p^{n_{l-1} - n_l - 1} + \left[\dfrac{t_l p^{n_l} - 1}{p^{n_l+1}}\right] = t_1 p^{n_1 - n_l - 1} + \ldots + t_{l-1} p^{n_{l-1} - n_l - 1}$

because $0 < t_l p^{n_l} - 1 \leq p \cdot p^{n_l} - 1 < p^{n_l + 1}$ as $t_l < p$;

$\left[\dfrac{z-1}{p^{n_{l-1}}}\right] = t_1 p^{n_1 n_{l-1}} + \ldots + t_{l-1} p^0 + \left[\dfrac{t_l p^{n_l} - 1}{p^{n_{l-1}}}\right] = t_1 p^{n_1 - n_{l-1}} + \ldots + t_{l-1} p^0$ as $n_{l-1} > n_l$,

$\left[\dfrac{z-1}{p^{n_1}}\right] = t_1 p^0 + \left[\dfrac{t_2 p^{n_2} + \ldots + t_l p^{n_l} - 1}{p^{n_1}}\right] = t_1 p^0$.

Because $0 < t_2 p^{n_2} + \ldots + t_l p^{n_l} - 1 \leq (p-1)p^{n_2} + \ldots + (p-1)p^{n_{l-1}} + p \cdot p^{n_l} - 1 \leq (p-1) \times$ $\times \sum_{i=n_{l-1}}^{n_2} p^i + p^{n_l+1} - 1 \leq (p-1)\dfrac{p^{n_2+1}}{p-1} = p^{n_2+1} - 1 < p^{n_1} - 1 < p^{n_1} \Rightarrow \left[\dfrac{t_2 p^{n_2} + \ldots t_l p^{n_l} - 1}{p^{n_1}}\right] = 0$

$\left[\dfrac{z-1}{p^{n_1+1}}\right] = \left[\dfrac{t_1 p^{n_1} + \ldots t_l p^{n_l} - 1}{p^{n_1+1}}\right] = 0$

because: $0 < t_1 p^{n_1} + \ldots + t_l p^{n_l} - 1 < p^{n_1+1} - 1 < p^{n_1+1}$ according to a reasoning similar to the previous one.

Adding $\Rightarrow$ p's powers sum in the natural numbers which make up the product factors $(z-1)!$ is:

$t_1(p^{n_1-1}+\ldots+p^0)+\ldots+t_{l-1}(p^{n_{l-1}-1}+\ldots+p^0)+t_l(p^{n_l-1}+\ldots+p^0)-1\cdot n_l = k-n_l < k-1 < k$
because $n_l > 1 \Rightarrow (z-1)! \neq Mp^k$, this contradicts the supposition made.

$\Rightarrow \eta_p(k)$ is tha smallest natural number with the property $(\eta_p(k))! = Mp^k$.

I construct a new function $\eta : Z \setminus \{0\} \to N$ as follows:

$$\begin{cases} \eta(\pm 1) = 0, \\ \forall n = \epsilon p_1^{\alpha_1} \ldots p_s^{\alpha_s} \text{ with } \epsilon = \pm 1,\ p_i = \text{prime}, \\ p_i \neq p_j \text{ for } i \neq j,\ \alpha_i \geq 1,\ i = \overline{1,s},\ \eta(n) = \max\limits_{i=\overline{1,s}}\{\eta_{p_i}(\alpha_i)\}. \end{cases}$$

Note 2. η is well defined and defined overall.

Proof.

(a) $\forall n \in Z, n \neq 0, n \neq \pm 1, n$ is uniquely written, independent of the order of the factors, under the shape of $n = \epsilon p_1^{\alpha_1} \ldots p_s^{\alpha_s}$ with $\epsilon = \pm 1$ where p_i =prime, $p_i \neq p_j, \alpha_i \geq 1$ (decompose into prime factors in Z =factorial ring).

$\Rightarrow \exists! \eta(n) = \max\limits_{i=\overline{1,s}}\{\eta_{p_i}(\alpha_i)\}$ as s =finite and $\eta_{p_i}(\alpha_i) \in N^*$ and $\exists \max\limits_{i=\overline{1,s}}\{\eta_{p_i}(\alpha_i)\}$

(b) $n = \pm 1 \Rightarrow \exists! \eta(n) = 0$.

Theorem 2. *The function η previously defined has the following properties:*

(1) $(\eta(n))! = Mn$, $\forall n \in Z \setminus \{0\}$;

(2) $\eta(n)$ *is the smallest natural number with this property.*

Proof.

(a) $\eta(n) = \max\limits_{i=\overline{1,s}}\{\eta_{p_i}(\alpha_i)\}$, $n = \epsilon \cdot p_1^{\alpha_1} \ldots p_s^{\alpha_s}$, $(n \neq \pm 1)$; $(\eta_{p_1}(\alpha_1))! = Mp_1^{\alpha_1}, \ldots (n_{p_s}(\alpha_s))! =$
$= Mp_s^{\alpha_s}$.

Supposing $\max\limits_{i=\overline{1,s}}\{\eta_{p_i}(\alpha_1)\} = \eta_{p_{i_0}}(\alpha_{i_0}) \Rightarrow (\eta_{p_{i_0}}(\alpha_{i0}))! = Mp_{i_0}^{\alpha_{i_0}}$, $\eta_{p_{i_0}}(\alpha_{i_0}) \in N^*$ and because $(p_i, p_j) = 1$, $i \neq j$,

$\Rightarrow (\eta_{p_{i_0}}(\alpha_{i_0}))! = Mp_j^{\alpha_j}$, $j = \overline{1,s}$.

$\Rightarrow (\eta_{p_{i_0}}(\alpha_{i_0}))! = Mp_1^{\alpha_1} \ldots p_s^{\alpha_s}$.

(b) $n = \pm 1 \Rightarrow \eta(n) = 0$; $0! = 1, 1 = M\epsilon \cdot 1 = Mn$.

$$(2)\ (a)\ n \neq \pm 1 \Rightarrow n = \epsilon p_1^{\alpha_1} \dots p_s^{\alpha_s} \Rightarrow \eta(n) = \max_{i=\overline{1,s}} \eta_{p_i}.$$

Let $= \max_{i=\overline{1,s}}\{\eta_{p_i}(\alpha_i)\} = \eta_{p_{i_0}}(\alpha_{i_0}),\ 1 \leq i \leq s;$

$\eta_{p_{i_0}}(\alpha_{i_0})$ is the smallest natural number with the property:

$$(\eta_{p_{i_0}}(\alpha_{i_0}))! = Mp_{i_0}^{\alpha_{i_0}} \Rightarrow \forall \gamma \in N,\ \gamma < \eta_{p_{i_0}}(\alpha_{i_0}) \Rightarrow \gamma! \neq Mp_{i_0}^{\alpha_{i_0}} \Rightarrow$$
$$\Rightarrow \gamma! \neq M\epsilon \cdot p_1^{\alpha_i} \dots p_{i_0}^{\alpha_{i_0}} \dots p_s^{\alpha_s} = Mn.$$

$\eta_{p_{i_0}}(\alpha_{i_0})$ is the smallest natural number with the property.

(b) $n = \pm 1 \Rightarrow \eta(n) = 0$ and it is the smallest natural number $\Rightarrow$ 0 is the smallest natural number with the property $0! = M(\pm 1)$.

Note 3. The functions η_p are increasing, not injective, on $N^* \to \{p^k | k = 1, 2, \dots\}$ they are surjective.

The function η is increasing, not injective, it is surjective on $Z \setminus \{0\} \to N \setminus \{1\}$.

CONSEQUENCE. Let $n \in N^*$, $n > 4$. Then n =prime $\Leftrightarrow \eta(n) = n$.

Proof.

" $\Rightarrow$ " n =prime and $n \geq 5 \Rightarrow \eta(n) = \eta_n(1) = n$.

" $\Leftarrow$ " Let $\eta(n) = n$ and suppose by absurd that $n \neq$prime $\Rightarrow$

(a) or $n = p_1^{\alpha_1} \dots p_s^{\alpha_s}$ with $s \geq 2$, $\alpha_i \in N^*$, $i = \overline{1,s}$,

$$\eta(n) = \max_{i=\overline{1,s}}\{\eta_{p_i}(\alpha_i)\} = \eta_{p_{i_0}}(\alpha_{i_0}) < \alpha_{i_0} p_{i_0} < n$$

contradicts the assumtion; or

(b) $n = p_1^{\alpha_1}$ with $\alpha_1 \geq 2 \Rightarrow \eta(n) = \eta_{p_1}(\alpha_1) \leq p_1 \cdot \alpha_1 < p_1^{\alpha_1} = n$

because $\alpha_1 \geq 2$ and $n > 4$ and it contradicts the hypothesis.

Application

1. Find the smallest natural number with the property: $n! = M(\pm 2^{31} \cdot 3^{27} \cdot 7^{13})$.

Solution

$\eta(\pm 2^{31} \cdot 3^{27} \cdot 7^{13}) = \max\{\eta_2(31), \eta_3(27), \eta_7(13)\}$.

Let us calculate $\eta_2(31)$; we make the string $(a_n^{(2)})_{n \in N^*} = 1, 3, 7, 15, 31, 63, \dots$

$31 = 1 \cdot 31 \Rightarrow \eta_2(31) = \eta_2(1 \cdot 31) = 1 \cdot 2^5 = 32$.

Let's calculate $\eta_3(27)$ making the string $(a_n^{(3)})_{n \in N^*} = 1, 4, 13, 40, \ldots$; $27 = 2 \cdot 13 + 1 \Rightarrow \eta_3^{(27)} = \eta_3(2 \cdot 13 + 1 \cdot 1) = 2 \cdot \eta_3(13) + 1 \cdot \eta_3(1) = 2 \cdot 3^3 + 1 \cdot 3^1 = 54 + 3 = 57$.

Let's calculate $\eta_7(13)$; making the string $(a_n^{(7)})_{n \in N^*} = 1, 8, 57, \ldots$; $13 = 1 \cdot 8 + 5 \cdot 1 \Rightarrow \eta_7(13) = 1 \cdot \eta_7(8) + 5 \cdot \eta_7(1) = 1 \cdot 7^2 + 5 \cdot 7^1 = 49 + 35 = 84 \Rightarrow \eta(\pm 2^{31} \cdot 3^{27} \cdot 7^{13}) = \max\{32, 57, 84\} = 84 \Rightarrow 84! = M(\pm 2^{31} \cdot 3^{27} \cdot 7^{13})$ and 84 is the smallest number with this property.

2. Which are the numbers with the factorial ending in 1000 zeros ?

Solution

$n = 10^{1000}$, $(\eta(n))! = M10^{1000}$ and it is the smallest number with this property.

$\eta(10^{1000}) = \eta(2^{1000} \cdot 5^{1000}) = \max\{\eta_2(1000), \eta_5(1000)\} = \eta_5(1 \cdot 781 + 1 \cdot 156 + 2 \cdot 31 + 1) = 1 \cdot 5^5 + 1 \cdot 5^4 + 2 \cdot 5^3 + 1 \cdot 5^7 = 4005$, 4005 is the smallest number with this property. 4006, 4007, 4008, 4009 verify the property but 4010 does not because $4010! = 4009! 4010$ has 1001 zeros.

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