π , G , ζ(n) , γ

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abstract

In this paper we give some formulas related with the numbers: π (pi) , G (catalan) , ζ(n) , γ (Euler-Mascheroni).

Keywords: Number pi , Catalan constant , Euler-Mascheroni constant , Function zeta , Double integrals .

Resumen

En este artículo se muestran algunas fórmulas relacionadas con los números: π , G , ζ(n) , γ.

1 Introducción

En esta nota se muestran fórmulas y relaciones que involucran constantes clásicas como son :

$$\pi = \sum_{n=0}^{\infty} \frac{(-1)^n}{2\,n+1} \tag{1}$$

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2\,n+1)^2} \tag{2}$$

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{3}$$

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} \tag{4}$$

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n\right) \tag{5}$$

En algunas fórmulas aparecen funciones especiales :

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \ , \ \ \mathrm{Re}(s) > 1 \tag{6}$$

la función de Mobius $\mu(n)$, la función de Euler $\phi(n)$, los polinomios de Legendre $P_n(x)$, .., etc.

se muestran algunas integrales dobles que involucran constantes clásicas.

2 El número π , el número G , las funciones ϕ(n) y μ(n)

Una de las funciones aritméticas más importantes en la teoría analítica de los números es la función de Mobius $\mu(n)$ que se define de la siguiente manera :

$$\mu(n)=\begin{cases} 1 & \text{si } n=1 \\ (-1)^k & \text{si } n \text{ es el producto de } k \text{ primos distintos} \\ 0 & \text{si } n \text{ tiene algún divisor cuadrado mayor que 1} \end{cases} \tag{7}$$

la función de Euler $\phi(n)$ se define como el número de enteros positivos primos con n , y menores o iguales que n. la función $\phi(n)$ se puede escribir como :

$$\phi(n)=n\sum_{d|n}\frac{\mu(d)}{d} \tag{8}$$

tres fórmulas que relacionan las constantes π , G , y las funciones $\phi(n)$ y $\mu(n)$ son :

$$\frac{1}{G}=1+\sum_{n=1}^{\infty}\frac{(-1)^n\,\mu(2n+1)}{(2n+1)^2} \tag{9}$$

$$\frac{\pi}{G}=4+4\sum_{n=1}^{\infty}\frac{(-1)^n\,\phi(2n+1)}{(2n+1)^2} \tag{10}$$

$$\frac{\pi}{4}=\frac{1+\sum_{n=1}^{\infty}(-1)^n\,\phi(2n+1)\,(2n+1)^{-2}}{1+\sum_{n=1}^{\infty}(-1)^n\,\mu(2n+1)\,(2n+1)^{-2}} \tag{11}$$

3 La constante G y la función w(x)

Una representación integral para la constante G es :

$$G=\frac{\pi}{4}+\int_{\pi/4}^{1}w(x)\,dx \tag{12}$$

donde $w(x)$ es la función inversa de $y=\frac{\tan^{-1}x}{x}$, $0<x\le 1$, $y(0)=1$.

La función $w(x)$ satisface la ecuación diferencial :

$$\frac{dw}{dx}=\frac{w\,(1+w^2)}{1-\,x\,(1+w^2)}\ ,\ \ w\left(\frac{\pi}{4}\right)=1 \tag{13}$$

La función $w(x)$ se puede representar como :

$$w(x)=\sqrt{\sum_{n=1}^{\infty}a_n(1-x)^n}\ ,\ \ \frac{\pi}{4}\le x\le 1 \tag{14}$$

$$w(x)=\sqrt{3\,(1-x)+\frac{27}{5}\,(1-x)^2+\ldots} \tag{15}$$

$$a_n=\left\{3,\ \frac{27}{5},\ \frac{1377}{175},\ \frac{1809}{175},\ \frac{4\,313\,493}{336\,875},\ \ldots\right\} \tag{16}$$

4 Una serie de Fourier

Recordamos una serie de fourier :

$$\left(1-\frac{1}{L}\right)\frac{\pi}{2}=\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{sen}\left(\frac{n\pi}{L}\right)\ ,\ \ L\in\mathbb{N}-\{1\} \tag{17}$$

la serie (17) se puede escribir como :

$$\left(1-\frac{1}{L}\right)\frac{\pi}{L}=\sum_{m=1}^{L-1}s_m\sum_{k=0}^{N}\frac{1}{2\,k\,L+m}+\sum_{m=L+1}^{2\,L-1}s_m\sum_{k=0}^{N}\frac{1}{2\,k\,L+m}+\sum_{k=N+1}^{\infty}\left(\sum_{m=1}^{L-1}\frac{s_m}{2\,k\,L+m}+\sum_{m=L+1}^{2\,L-1}\frac{s_m}{2\,k\,L+m}\right) \tag{18}$$

donde

$$N\in\mathbb{N}_0\ ,\ \ s_m=\operatorname{sen}\left(\frac{m\,\pi}{L}\right),\ \ m=1,\ 2,\ ...,\ 2\,L,\ \ s_L=s_{2\,L}=0 \tag{19}$$

5 La constante G

Algunas series *y* relaciones que involucran *a* la constante de catalan :

$$G^2+\frac{\pi^4}{96}=2\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^{n+m}}{((2\,n+1)\,(2\,m+1))^2} \tag{20}$$

$$G^2-\frac{\pi^4}{96}=2\sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty}\frac{(-1)^{n+m}}{((2\,n+1)\,(2\,m+1))^2} \tag{21}$$

$$G=2\sum_{k=1}^{m}\sum_{n=1}^{\infty}\frac{(-1)^n\,n}{(2\,n+1)^{k+2}}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2\,n+1)^{m+2}}\ ,\ \ m\in\mathbb{N} \tag{22}$$

La función beta de Dirichlet se define por :

$$\beta(s)=\sum_{k=0}^{\infty}(-1)^k\,(2\,k+1)^{-s}\ ,\ \ \operatorname{Re}(s)>0 \tag{23}$$

de (23) se tiene :

$$\beta(2)=G \tag{24}$$

La ecuación (22) se puede escribir como :

$$G=2\sum_{n=1}^{\infty}(-1)^n\,n\sum_{k=1}^{m}\frac{1}{(2\,k+1)^{k+2}}+\beta(m+2)\ ,\ \ m\in\mathbb{N} \tag{25}$$

Algunas series :

$$G=8\sum_{n=0}^{\infty}\frac{2\,n+1}{((4\,n+1)\,(4\,n+3))^2} \tag{26}$$

$$G=2\sum_{n=0}^{\infty}\frac{1}{(4\,n+1)^2\,(4\,n+3)}+2\sum_{n=0}^{\infty}\frac{1}{(4\,n+1)\,(4\,n+3)^2} \tag{27}$$

$$G = 8\sum_{n=1}^{\infty}\left(\frac{n}{4n^2-1}\right)^2\sum_{m=1}^{n}\frac{(-1)^{m-1}}{m} - \sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}\sum_{m=1}^{n}\frac{(-1)^{m-1}}{m} \tag{28}$$

$$G = 4\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{n}{4n^2-1}\right)^2 + 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{(4n^2-1)^2} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(4n^2-1)^2} \tag{29}$$

$$G = \sum_{n=1}^{\infty}\frac{4n(n+1)-1}{(4n^2-1)^2}\sum_{m=1}^{n}\frac{(-1)^{m-1}}{m} \tag{30}$$

$$G = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\sum_{m=1}^{n}\frac{(-1)^{m-1}}{m} - 2\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}\sum_{m=1}^{n}\frac{(-1)^{m-1}}{m} \tag{31}$$

Otra representación para G es :

$$G = \sum_{k=0}^{m}2^k\binom{m}{k}H(m,k)\ ,\ m \in \mathbb{N}_0 \tag{32}$$

donde

$$H(m,k) = \sum_{n=0}^{\infty}\frac{(-1)^n n^k}{(2n+1)^{m+2}}\ ,\ m \in \mathbb{N}_0\ ,\ 0^0 \equiv 1 \tag{33}$$

$$G = 1 + 4\int_0^1\left(\sum_{n=1}^{\infty}\frac{(-1)^n\ n\,x}{(2n+x)^3}\right)dx \tag{34}$$

Algunas desigualdades :

$$\tan^{-1}x + (1-x)\frac{\pi}{4} < G < x + \left(\frac{1-x}{x}\right)\tan^{-1}x\ ,\ 0 \le x \le 1 \tag{35}$$

$$\frac{\pi}{4n} + \sum_{k=1}^{n-1}\frac{1}{k}\tan^{-1}\left(\frac{k}{n}\right) < G < \frac{1}{n} + \sum_{k=1}^{n-1}\frac{1}{k}\tan^{-1}\left(\frac{k}{n}\right)\ ,\ n \in \mathbb{N} \tag{36}$$

$$\frac{\pi}{8}\left(\frac{2+3x-x^2}{1+x}\right) < G < \frac{3x-x^2}{2-2x} + \frac{1-x}{2x}\tan^{-1}x + \frac{x}{1-x}\tan^{-1}\left(\frac{1-x}{1+x}\right) \tag{37}$$

$$0 < x < 1$$

6 Número π ,polinomios de Legendre , integral doble

Los polinomios de Legendre $P_n(x)$ se definen por la fórmula :

$$P_n(x) = \frac{1}{2^n\,n!}\frac{d^n}{dx^n}\left((x^2-1)^n\right)\ ,\ n \in \mathbb{N}_0 \tag{38}$$

Una integral doble para π es :

$$\pi = 4(-1)^n\int_0^1\int_0^1 x^{y^2}\left(\frac{((3+y^2)/2)_n}{(-y^2/2)_n}\right)P_n(x)\,dx\,dy\ ,\ n \in \mathbb{N}_0 \tag{39}$$

donde

$$(x)_0 = 1\ ,\ (x)_n = x(x+1)\ldots(x+n-1) \tag{40}$$

7 Número π , suma de radicales

Para $m \in \mathbb{N}$, se tiene :

$$\frac{\pi}{4}\left(1+\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}+\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}}+\ldots+\frac{\overbrace{\sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}^{m-\text{radicales}}}{\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{m-\text{radicales}}}\right)=\sum_{n=0}^{\infty}\sum_{k=1}^{m}\frac{2^k}{\left(2^k\,(2\,n+1)\right)^2-1} \tag{41}$$

$$\frac{\pi}{4}\left(1-\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}+\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}}-\ldots+(-1)^{m-1}\frac{\overbrace{\sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}^{m-\text{radicales}}}{\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{m-\text{radicales}}}\right)=\sum_{n=0}^{\infty}\sum_{k=1}^{m}\frac{(-1)^{k-1}\,2^k}{\left(2^k\,(2\,n+1)\right)^2-1} \tag{42}$$

8 El producto de Euler para la función zeta ζ(s)

Recordamos el clásico producto de Euler para la función zeta de Riemann :

$$\zeta(x)=\prod_{p}\frac{1}{1-p^{-x}}\ ,\ x>0 \tag{43}$$

la funció zeta satisface la ecuación :

$$\zeta(2\,k)=\frac{2^{2\,k-1}\,\pi^{2\,k}\,B_k}{(2\,k)\,!}\ ,\ k\in\mathbb{N} \tag{44}$$

donde B_k son los números de Bernoulli :

$$B_k=\left\{\frac{1}{6},\ \frac{1}{30},\ \frac{1}{42},\ \frac{1}{30},\ \frac{5}{66},\ \ldots\right\} \tag{45}$$

combinando las fórmulas (43) *y* (44) se tiene :

$$\pi=\frac{1}{2}\left(\frac{2\,(2\,k)\,!}{B_k}\right)^{1/2\,k}\prod_{p}\left(1-p^{-2\,k}\right)^{-1/2\,k}\ ,\ k\in\mathbb{N} \tag{46}$$

El producto de Euler se puede escribir como :

$$\zeta(x)=\prod_{p}\frac{1}{\left(1-p^{-x/2}\right)(1+p^{-x72})}\ ,\ x>0 \tag{47}$$

Si $p(n)$ representa el $n-$ ésimo número primo, entonces se tiene :

$$\zeta_m(x)=\prod_{n=1}^{m}\frac{1}{1-(-1)^{n-1}\left(p\left(\left[\frac{n+1}{2}\right]\right)\right)^{-x/2}}\ ,\ m\in\mathbb{N},\ x>0 \tag{48}$$

donde $[x]$ es la función parte entera , *y* se tiene :

$$\lim_{m\to\infty} \zeta_m(x) = \zeta(x),\ x > 0 \tag{49}$$

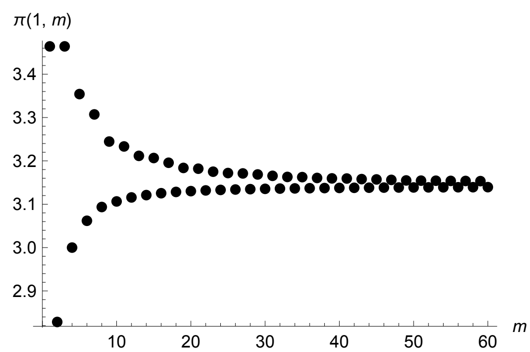
otra fórmula es :

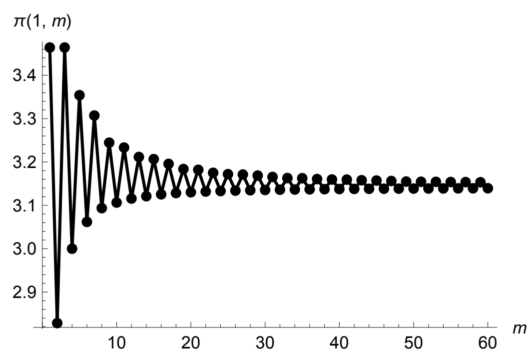
$$\pi(k,\ m) = \frac{1}{2}\left(\frac{2\,(2\,k)\,!}{B_k}\right)^{1/2\,k} (\zeta_m(2\,k))^{1/2\,k}\ \ ,\ \ k,\ m \in \mathbb{N} \tag{50}$$

$$\lim_{m\to\infty} \pi(k,\ m) = \pi\ \ ,\ \ k \in \mathbb{N} \tag{51}$$

para el caso particular $k = 1$, la función $\pi(1,\ m)$ tiene la siguiente representación :

$$\pi(1,\ m) = \sqrt{6}\ \sqrt{\zeta_m(2)}\ \ ,\ \ m \in \mathbb{N} \tag{52}$$





9 Número π , arcotangente , particiones

Si $P(n)$ representa el número de particiones de un entero n , se tiene :

$$\frac{\pi}{6} + \sum_{n=2}^{\infty} (-1)^{n-1} \tan^{-1}\left(\frac{\sqrt{3}}{3^n}\right) = \tan^{-1}\left(\frac{\sqrt{3}\ \sum_{n=1}^{\infty} (-1)^{n-1}\, 3^{-n}\, P(2\,n-1)}{1 + \sum_{n=1}^{\infty} (-1)^n\, 3^{-n}\, P(2\,n)}\right) \tag{53}$$

10 Integrales dobles

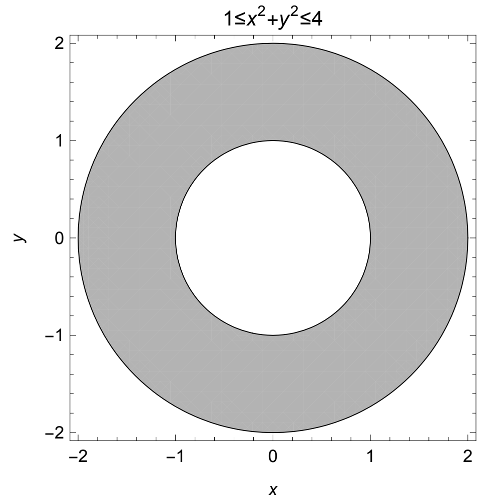
$$\pi = \frac{3}{2\,(b^3 - a^3)} \int\int_{R(a,b)} \sqrt{x^2 + y^2}\ \ dx\,dy \tag{54}$$

$$R(a,\ b) = \left\{(x,\ y) \in \mathbb{R}^2 :\ a^2 \le x^2 + y^2 \le b^2\right\},\ \ 0 \le a < b \tag{55}$$

un caso particular de (54) con $a = 1,\ b = 2,$ es :

$$\pi = \frac{3}{14}\iint_{R(1,2)} \sqrt{x^2+y^2}\ dx\,dy = \frac{3}{14}\iint_{1\le x^2+y^2\le 4} \sqrt{x^2+y^2}\ dx\,dy \tag{56}$$

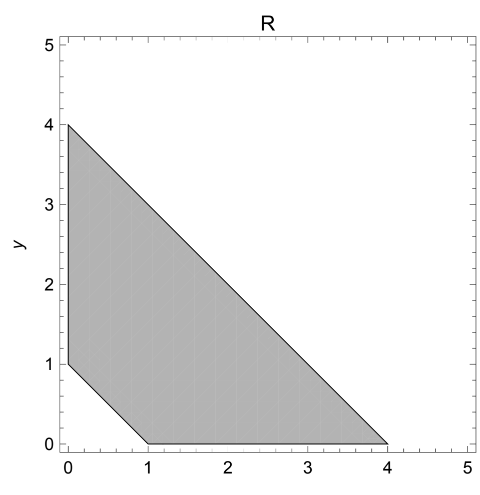
la gráfica de la región $R(1,\ 2)$ es :



de (56) se obtiene :

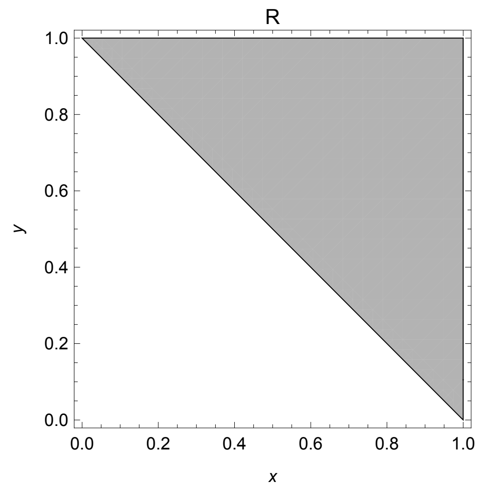
$$\pi = \frac{3}{56}\iint_R \sqrt{\frac{1}{x}+\frac{1}{y}}\ dx\,dy \tag{57}$$

$$R = \left\{(x,\ y) \in \mathbb{R}^2 :\ 1 \le x+y \le 4,\ x > 0,\ y > 0\right\} \tag{58}$$



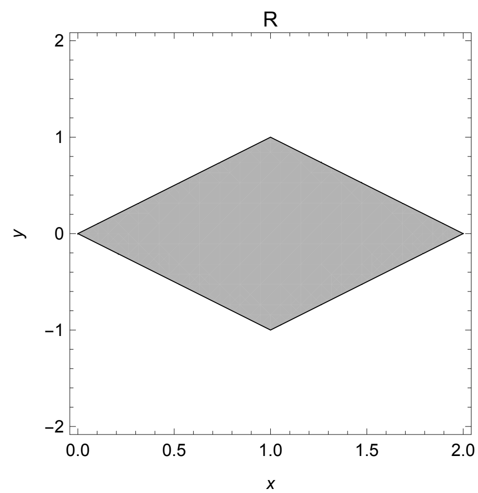
$$\iint_R \frac{-\ln(1-y)}{x\,(1+y)}\ dx\,dy = \frac{7}{4}\zeta(3) - \frac{\pi^2 \ln 2}{6} + \frac{(\ln 2)^3}{3} \tag{59}$$

$$R = \left\{(x,\ y) \in \mathbb{R}^2 :\ 1-x \le y \le 1,\ 0 \le x \le 1\right\} \tag{60}$$



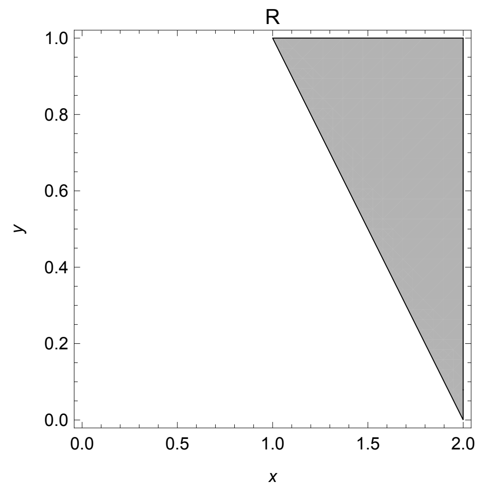
$$\iint_R \frac{36 + x^2 - y^2}{144 - (x^2 - y^2)^2} \, dx \, dy = \frac{\pi^2}{12} - \frac{(\ln 3)^2}{4} = \frac{\zeta(2)}{2} - \frac{(\ln 3)^2}{4} \tag{61}$$

$$R = \left\{(x, y) \in \mathbb{R}^2 : 0 \le x + y \le 2, \; 0 \le x - y \le 2\right\} \tag{62}$$



$$\iint_R \frac{1}{x\,y} \, dx \, dy = \frac{\pi^2}{12} - \frac{(\ln 2)^2}{2} = \frac{\zeta(2)}{2} - \frac{(\ln 2)^2}{2} \tag{63}$$

$$R = \left\{(x, y) \in \mathbb{R}^2 : 0 \le 2 - x \le y, \; 1 \le x \le 2, \; 0 \le y \le 1\right\} \tag{64}$$



sean $a < b$, $c < d$, se tiene :

$$\frac{\pi}{3\sqrt{3}} = \int_c^d \int_a^b \frac{(b-a)^2 (d-c)(y-c)}{((b-a)(d-c))^3 - ((x-a)(y-c))^3} \, dx \, dy \tag{65}$$

$$G = \int_c^d \int_a^b \frac{(b-a)(d-c)}{((b-a)(d-c))^2 + ((x-a)(y-c))^2} \, dx \, dy \tag{66}$$

$$\zeta(2) = \int_c^d \int_a^b \frac{1}{(b-a)(d-c) - (x-a)(y-c)} \, dx \, dy \tag{67}$$

$$2\,\zeta(3) = \int_c^d \int_a^b \frac{\ln((b-a)(d-c)) - \ln((x-a)(y-c))}{(b-a)(d-c) - (x-a)(y-c)} \, dx \, dy \tag{68}$$

$$\zeta(2)\ln((b-a)(d-c)) - 2\,\zeta(3) = \int_c^d \int_a^b \frac{\ln((x-a)(y-c))}{(b-a)(d-c) - (x-a)(y-c)} \, dx \, dy \tag{69}$$

Algunas integrales en la región :

$$R = \left\{(x,\ y) \in \mathbb{R}^2 :\ 1 - x \le y \le 1,\ 0 \le x \le 1\right\} \tag{70}$$

$$\zeta(3) = -\frac{1}{2} \iint_R \frac{\ln(1-y)}{x\,y} \, dx \, dy \tag{71}$$

$$\gamma = -\iint_R \frac{1-x}{x\,y\ln(1-y)} \, dx \, dy \tag{72}$$

$$G = \iint_R \frac{1}{x\,(2 - 2\,y + y^2)} \, dx \, dy \tag{73}$$

$$\zeta(2) = \iint_R \frac{1}{x\,y} \, dx \, dy \tag{74}$$

$$\frac{\pi}{4} = -\iint_R \frac{1}{x\,(1 + (1-y)^2)\ln(1-y)} \, dx \, dy \tag{75}$$

$$\ln\left(\frac{\pi}{4}\right) = \iint_R \frac{1-x}{x\,(2-y)\ln(1-y)} \, dx \, dy \tag{76}$$

11 Número π , integral triple

$$\frac{\pi}{4} = \int_0^1 \zeta\left(2 + x^2\right) dx - \int_0^1 \int_0^1 \int_0^1 \frac{(1-x)\,(-\ln(x\,y))^{z^2}}{(1-x\,y)\,\Gamma(2+z^2)}\, dx\, dy\, dz \tag{77}$$

$$\int_0^1 \zeta\left(2 + x^2\right) dx = 1 + \sum_{n=2}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m\,(\ln n)^m}{m!\,(2\,m+1)\,n^2} \tag{78}$$

12 Número π , serie

$$\frac{1}{\pi} = \frac{9}{32} + \sum_{n=1}^{\infty} \left(2 \binom{2^{n+2}}{2^{n+1}}^2 - 2^{2^{n+2}} \binom{2^{n+1}}{2^n}^2 \right) 2^{n-2^{n+3}} \tag{79}$$

13 La función $\zeta(s)$, números primos , números compuestos

$$\zeta(s) = 1 + \sum_{p \in P} p^{-s} + \sum_{c \in C} c^{-s} \tag{80}$$

donde $P = \{2,\ 3,\ 5,\ 7,\ 11,\ \ldots\}$ es el conjunto de los números primos y $C = \{4,\ 6,\ 8,\ 9,\ 10,\ 12,\ \ldots\}$ es el conjunto de los números compuestos.

$$\zeta(s) = 1 + \sum_{n=1}^{\infty} \frac{1}{p_n^s - 1} + \sum_{n \in A} n^{-s} \tag{81}$$

donde p_n es el $n-$ ésimo número primo y :

$$A = \{n \in \mathbb{N} :\ n \neq p^m,\ \ p \in P,\ m \in \mathbb{N}\} = \{6,\ 10,\ 12,\ 14,\ 15,\ 18,\ 20,\ \ldots\} \tag{82}$$

14 Algunas representaciones integrales

Recordamos un resultado del análisis :

Sea $a_n > 0$, $n \in \mathbb{N}$ tal que : $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$ y $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{a_n}$ son convergentes, entonces :

$$\pi \sum_{n=1}^{\infty} \frac{1}{a_n} = 2 \int_0^{\infty} \sum_{n=1}^{\infty} \frac{1}{x^2 + a_n^2}\, dx \tag{83}$$

$$\pi \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{a_n} = 2 \int_0^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{x^2 + a_n^2}\, dx \tag{84}$$

Ejemplos :

$$\pi^3 = 16 \int_0^\infty \sum_{n=1}^\infty \frac{1}{x^2 + (2n-1)^4} \, dx \tag{85}$$

$$\pi G = 2 \int_0^\infty \sum_{n=1}^\infty \frac{(-1)^{n-1}}{x^2 + (2n-1)^4} \, dx \tag{86}$$

$$\pi \zeta(s) = 2 \int_0^\infty \sum_{n=1}^\infty \frac{1}{x^2 + n^{2s}} \, dx \ , \ s > 1 \tag{87}$$

$$\pi \zeta(s) \left(1 - 2^{1-s}\right) = 2 \int_0^\infty \sum_{n=1}^\infty \frac{(-1)^{n-1}}{x^2 + n^{2s}} \, dx \ , \ s > 0 \tag{88}$$

$$\pi e = 2 \int_0^\infty \sum_{n=0}^\infty \frac{1}{x^2 + (n!)^2} \, dx \tag{89}$$

$$\frac{\pi}{e} = 2 \int_0^\infty \sum_{n=0}^\infty \frac{(-1)^n}{x^2 + (n!)^2} \, dx \tag{90}$$

15 Número π , función ζ(2n+1) , números de Euler

$$\pi^{2n+1} \zeta(2n+1) E_n = 2^{2n+1} \int_0^\infty \left(x^{2n} \sum_{m=1}^\infty \frac{1}{\cosh(m x)} \right) dx \ , \ n \in \mathbb{N} \tag{91}$$

$$E_n = \{1, 5, 61, 1385, 50\,521, \ldots\} \tag{92}$$

$$E_n = \sum_{k=0}^{2n} (-1)^{n+k} 2^{-k} \binom{2n+1}{k+1} \sum_{m=0}^{k} \binom{k}{m} (k - 2m)^{2n} \ , \ n \in \mathbb{N} \tag{93}$$

16 Número π , función ζ(s) , números de Bernoulli

Sea $m \in \mathbb{N} - \{1\}$ y $m\mathbb{N} = \{m, 2m, 3m, 4m, \ldots\}$, se tiene :

$$\zeta(s)\,(1 - m^{-s}) = \sum_{n \in \mathbb{N} - m\mathbb{N}} n^{-s} \ , \ s > 1 \tag{94}$$

Para $s = 2k, \ k \in \mathbb{N}$ se tiene :

$$\frac{2^{2k-1} B_k \pi^{2k}}{(2k)!} \left(1 - m^{-2k}\right) = \sum_{\mathbb{N} - m\mathbb{N}} n^{-2k} \tag{95}$$

$$B_m = \frac{(-1)^{m-1}}{2^{1-2m} - 1} \sum_{n=0}^{2m} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(k + \frac{1}{2}\right)^{2m} \ , \ m \in \mathbb{N} \tag{96}$$

17 Función ζ(x) , constantes , integrales

$$G\,\zeta(p+s) = \int_0^1 \frac{1}{x} \sum_{n=1}^{\infty} \frac{\tan^{-1}\left(x^{n^s}\right)}{n^p}\, dx \;,\; s>0,\; p>1 \tag{97}$$

$$G\,\zeta(3) = \int_0^1 \frac{1}{x} \sum_{n=1}^{\infty} \frac{\tan^{-1}(x^n)}{n^2}\, dx \tag{98}$$

$$\frac{\pi \ln 2}{2}\,\zeta(p+s) = \int_0^1 \frac{1}{x} \sum_{n=1}^{\infty} \frac{\operatorname{sen}^{-1}\left(x^{n^s}\right)}{n^p}\, dx \;,\; s>0,\; p>1 \tag{99}$$

$$\frac{\pi \ln 2}{2}\,\zeta(3) = \int_0^1 \frac{1}{x} \sum_{n=1}^{\infty} \frac{\operatorname{sen}^{-1}(x^n)}{n^2}\, dx \tag{100}$$

$$\frac{\pi}{2}\,\zeta(p+s) = \int_0^1 \frac{1}{x} \sum_{n=1}^{\infty} \frac{\operatorname{sen}\left(x^{n^s}\right)}{n^p}\, dx \;,\; s>0,\; p>1 \tag{101}$$

$$\frac{\pi}{2}\,\zeta(3) = \int_0^1 \frac{1}{x} \sum_{n=1}^{\infty} \frac{\operatorname{sen}(x^n)}{n^2}\, dx \tag{102}$$

$$\gamma(\zeta(2\,s)-1) = -\int_0^{\infty} \ln x \sum_{n=2}^{\infty} e^{-x^{n^s}}\, x^{n^s-1}\, dx \;,\; s>1/2 \tag{103}$$

18 La función G(z)

Definición :

$$G(z) = \int_0^1 \int_0^1 \frac{1}{z^2+x^2\,y^2}\, dx\, dy = \frac{1}{z} \int_0^{1/z} \frac{\tan^{-1} x}{x}\, dx \;,\; z>0 \tag{104}$$

y se tiene : $G(1)=G$ (constante de catalan) . para $0<z<1$, se tiene :

$$G(z) = \frac{G}{z} + \frac{1}{z} \int_1^{1/z} \frac{\tan^{-1} x}{x}\, dx \tag{105}$$

para $z>1$ se tiene :

$$G(z) = \frac{G}{z} - \frac{1}{z} \int_{1/z}^{1} \frac{\tan^{-1} x}{x}\, dx \tag{106}$$

algunas fórmulas :

$$\int_0^1 \int_0^1 \operatorname{sech}\left(\frac{x\,y}{2}\right) dx\, dy = 4\,\pi \sum_{n=0}^{\infty} (-1)^n\,(2\,n+1)\; G((2\,n+1)\,\pi) \tag{107}$$

$$\int_0^1 \int_0^1 \operatorname{sech}(x\,y)\, dx\, dy = \pi \sum_{n=0}^{\infty} (-1)^n\,(2\,n+1)\; G\left(\left(n+\frac{1}{2}\right)\pi\right) \tag{108}$$

$$\int_0^1 \int_0^1 \operatorname{sech}\left(\frac{x\,y\,\pi}{2}\right) dx\, dy = \frac{4}{\pi}\left(G + \sum_{n=1}^{\infty} (-1)^n\,(2\,n+1)\; G(2\,n+1)\right) \tag{109}$$

$$\int_0^1 \int_0^1 \operatorname{sech}(x\,y\,\pi)\, dx\, dy = \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n\,(2\,n+1)\; G\left(n+\frac{1}{2}\right) \tag{110}$$

$$\int_0^1\int_0^1 \frac{1}{x\,y}\tanh\left(\frac{x\,y}{2}\right) dx\,dy = 4\sum_{n=0}^{\infty} G((2\,n+1)\,\pi) \tag{111}$$

$$\int_0^1\int_0^1 \frac{1}{x\,y}\tanh(x\,y)\,dx\,dy = 2\sum_{n=0}^{\infty} G\left(\left(n+\frac{1}{2}\right)\pi\right) \tag{112}$$

$$\int_0^1\int_0^1 \frac{1}{x\,y}\tanh\left(\frac{x\,y\,\pi}{2}\right) dx\,dy = \frac{4}{\pi}\left(G+\sum_{n=1}^{\infty} G(2\,n+1)\right) \tag{113}$$

$$\int_0^1\int_0^1 \frac{1}{x\,y}\tanh(x\,y\,\pi)\,dx\,dy = \frac{2}{\pi}\sum_{n=0}^{\infty} G\left(n+\frac{1}{2}\right) \tag{114}$$

19 Número π , fórmulas

$$\pi = A(j)\prod_{m=0}^{\infty}\left(1-\frac{(-1)^m\,s(j,\,m)}{2\,m+3}\right) \tag{115}$$

donde $A(j)$ y $s(j,\,m)$ se definen como :

$$A(j) = 4\left(\frac{1}{j+1}+\sum_{k=1}^{j}\frac{1}{k^2+k+1}\right),\quad j\in\mathbb{N} \tag{116}$$

$$s(j,\,m) = \frac{(j+1)^{-2\,m-3}+\sum_{k=1}^{j}(k^2+k+1)^{-2\,m-3}}{\sum_{n=0}^{m}(-1)^n\,(2\,n+1)^{-1}\left((j+1)^{-2\,n-1}+\sum_{k=1}^{j}(k^2+k+1)^{-2\,n-1}\right)},\quad j\in\mathbb{N} \tag{117}$$

para $j=1$ se tiene :

$$\pi = \frac{10}{3}\prod_{m=0}^{\infty}\left(1-\frac{(-1)^m\,(2^{-2\,m-3}+3^{-2\,m-3})}{(2\,m+3)\sum_{n=0}^{m}(-1)^n\,(2\,n+1)^{-1}\,(2^{-2\,n-1}+3^{-2\,n-1})}\right) \tag{118}$$

$$\pi = 8\left(\sqrt{2}-1\right)\prod_{m=0}^{\infty}(2\,m+3)\left(P_m+Q_m\sqrt{2}\right) \tag{119}$$

donde

$$P_m = \frac{A_m\,A_{m+1}-2\,B_m\,B_{m+1}}{A_m^2-2\,B_m^2} \tag{120}$$

$$Q_m = \frac{A_m\,B_{m+1}-A_{m+1}\,B_m}{A_m^2-2\,B_m^2} \tag{121}$$

$$A_m = \sum_{n=0}^{m}(-1)^n\,a_n\prod_{0\le k\le m,k\ne n}(2\,k+1) \tag{122}$$

$$B_m = \sum_{n=0}^{m}(-1)^n\,b_n\prod_{0\le k\le m,k\ne n}(2\,k+1) \tag{123}$$

$$a_{n+1} = 3\,a_n-4\,b_n\,,\ \ b_{n+1} = -2\,a_n+3\,b_n\,,\ \ a_0=-1,\ b_0=1 \tag{124}$$

$$\pi = 8\left(\sqrt{2} - 1\right)\left(\frac{2\sqrt{2}}{3}\right)\left(\frac{51\sqrt{2} - 52}{20}\right)\left(\frac{4\left(3735\sqrt{2} - 3098\right)}{8743}\right)\ldots \tag{125}$$

$$\pi = 2^m \underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{\longleftarrow m-\text{radicales} \longrightarrow} \prod_{n=0}^{\infty} \frac{2}{\underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{\longleftarrow (m+n+1)-\text{radicales} \longrightarrow}} \tag{126}$$

$$m \in \mathbb{N}_0$$

$$\pi = 1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2 - \sqrt{2}}}{2} + \frac{\sqrt{2 - \sqrt{2 + \sqrt{2}}}}{2} + \frac{\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}{2} + \ldots + 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(-1)^{n-1} a_m}{2^{2m} n^2 - 1} \tag{127}$$

$$\frac{\pi}{3} = 1 - \frac{\sqrt{2}}{2} + \frac{\sqrt{2 - \sqrt{2}}}{2} - \frac{\sqrt{2 - \sqrt{2 + \sqrt{2}}}}{2} + \frac{\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}{2} - \ldots + 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(-1)^{n+m} a_m}{2^{2m} n^2 - 1} \tag{128}$$

en las fórmulas (127), (128) , se tiene :

$$a_1 = 1\,,\ a_m = \frac{1}{2}\underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{\longleftarrow (m-1)-\text{radicales} \longrightarrow}\ ,\ m \in \mathbb{N} - \{1\} \tag{129}$$

$$\frac{\pi}{4} = 1 - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{k=1}^{n} \frac{1}{n+k} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{k=1}^{n} \frac{1}{(2k)^3 - 2k} \tag{130}$$

20 Suma de arcotangentes

$$\sum_{n=1}^{\infty} (-1)^{n-1} \tan^{-1}\left(x^{2n-1}\right) = \sum_{n=1}^{\infty} (-1)^{n-1} a_n x^{2n-1}\ ,\ |x| < 1 \tag{131}$$

$$a_n = \sum_{k=1}^{\tau(2n-1)} \frac{(-1)^{d(2n-1,k)-1}}{d(2n-1,\ k)}\ ,\ n \in \mathbb{N} \tag{132}$$

$$\tau(2n-1) = \text{número de divisores de } 2n-1 \tag{133}$$

$$d(2n-1,\ k) = k - \text{ésimo divisor de } 2n-1 \tag{134}$$

algunos valores de a_n son :

$$a_n = \left\{1,\ \frac{4}{3},\ \frac{6}{5},\ \frac{8}{7},\ \frac{13}{9},\ \frac{12}{11},\ \frac{14}{13},\ \frac{8}{5},\ \frac{18}{17},\ \frac{20}{19},\ \frac{32}{21},\ \frac{24}{23},\ \frac{31}{25},\ \frac{40}{27},\ \frac{30}{29},\ \ldots\right\} \tag{135}$$

ejemplos particulares :

$$\frac{\pi}{6\sqrt{3}} + \frac{1}{\sqrt{3}} \sum_{n=2}^{\infty} (-1)^{n-1} \tan^{-1}\left(\frac{\sqrt{3}}{3^n}\right) = \sum_{n=1}^{\infty} (-1)^{n-1}\, a_n\, 3^{-n} \tag{136}$$

$$\frac{\pi}{4} + \sum_{n=2}^{\infty} (-1)^{n-1} \left(\tan^{-1}\left(\frac{1}{2^{2n-1}}\right) + \tan^{-1}\left(\frac{1}{3^{2n-1}}\right)\right) = \sum_{n=1}^{\infty} (-1)^{n-1}\, a_n \left(\frac{1}{2^{2n-1}} + \frac{1}{3^{2n-1}}\right) \tag{137}$$

$$\frac{\pi\sqrt{3}}{6} + \sqrt{3} \sum_{n=2}^{\infty} (-1)^{n-1} \left(\tan^{-1}\left(\left(\frac{\sqrt{3}}{4}\right)^{2n-1}\right) + \tan^{-1}\left(\left(\frac{\sqrt{3}}{15}\right)^{2n-1}\right)\right) = \sum_{n=1}^{\infty} (-1)^{n-1}\, a_n \left(4\left(\frac{3}{16}\right)^n + 15\left(\frac{3}{225}\right)^n\right) \tag{138}$$

$$\frac{\pi\sqrt{3}}{6} + \sqrt{3} \sum_{n=2}^{\infty} (-1)^{n-1} \left(\tan^{-1}\left(\left(\frac{\sqrt{3}}{2}\right)^{2n-1}\right) - \tan^{-1}\left(\left(\frac{\sqrt{3}}{9}\right)^{2n-1}\right)\right) = \sum_{n=1}^{\infty} (-1)^{n-1}\, a_n \left(2\left(\frac{3}{4}\right)^n - 9\left(\frac{1}{27}\right)^n\right) \tag{139}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \tan^{-1}\left(x^{2n-1}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, x^{2n-1}}{(2n-1)\,(1 + x^{4n-2})}\ ,\ \ |x| < 1 \tag{140}$$

$$\frac{\pi\sqrt{3}}{6} + \sqrt{3} \sum_{n=2}^{\infty} (-1)^{n-1} \tan^{-1}\left(\frac{\sqrt{3}}{3^n}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, 3^n}{(2n-1)\,(1 + 3^{2n-1})} \tag{141}$$

21 Número π , sucesión , integral

La constante pi se puede representar por la siguiente fórmula :

$$\pi = 6 \sum_{n=0}^{\infty} (-1)^n \left(\frac{a_n}{n!}\right) \int_0^{1/\sqrt{3}} x^{2n}\, e^{-x^2/(1+x^2)}\, dx \tag{142}$$

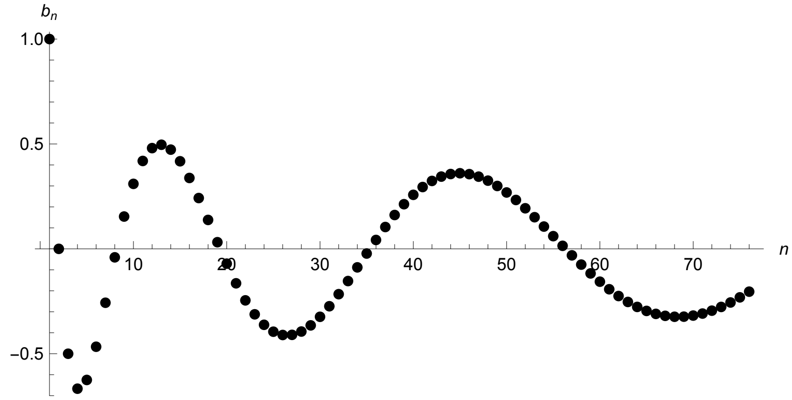
donde

$$a_{n+1} = 2\,n\,a_n - n^2\,a_{n-1}\ ,\ \ n \in \mathbb{N}\ ,\ \ a_0 = 1,\ a_1 = 0 \tag{143}$$

$$a_n = \{1,\ 0,\ -1,\ -4,\ -15,\ -56,\ -185,\ \ldots\} \tag{144}$$

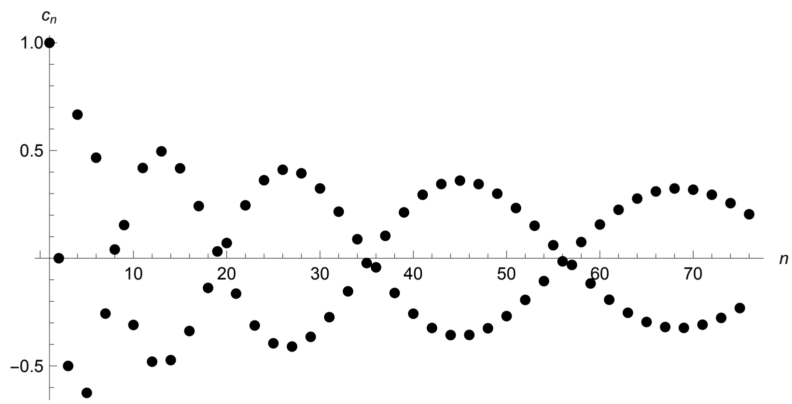
sea $b_n = \dfrac{a_n}{n!}$, $n \in \mathbb{N}_0$, se tiene :

$$b_{n+1} = \frac{2\,n}{n+1}\, b_n - \frac{n}{n+1}\, b_{n-1}\ ,\ \ n \in \mathbb{N},\ \ b_0 = 1,\ b_1 = 0 \tag{145}$$



sea $c_n = (-1)^n \, b_n$, $n \in \mathbb{N}_0$, se tiene :

$$c_{n+1} = -\frac{2\,n}{n+1}\,c_n - \frac{n}{n+1}\,c_{n-1} \ , \ n \in \mathbb{N} \ , \ c_0 = 1, \ c_1 = 0 \tag{146}$$



$$I_n = \int_0^{1/\sqrt{3}} x^{2\,n}\, e^{-x^2/(1+x^2)}\, dx \ , \ n \in \mathbb{N}_0 \tag{147}$$

$$I_n = \frac{1}{2}\int_0^{1/3} x^{n-1/2}\, e^{-x/(1+x)}\, dx \ , \ n \in \mathbb{N}_0 \tag{148}$$

$$I_n = \frac{3^{-n}}{2\,\sqrt{3}}\int_0^{1} x^{n-1/2}\, e^{-x/(3+x)}\, dx \ , \ n \in \mathbb{N}_0 \tag{149}$$

$$0 < I_n < \frac{3^{-n}}{\sqrt{3}\ (2\,n+1)} \ , \ n \in \mathbb{N}_0 \tag{150}$$

22 La constante γ , fórmulas

Algunas representaciones para la constante γ :

$$\gamma = \int_{-\infty}^{\infty} x\, e^{-x-e^{-x}}\, dx \tag{151}$$

$$\gamma = \int_0^\infty x\, e^{-x-e^{-x}}\, dx - \int_0^\infty x\, e^{x-e^{x}}\, dx \tag{152}$$

$$\gamma = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!\, n} - \int_1^\infty e^{-x} \ln x\, dx \tag{153}$$

$$\gamma = -\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!\,(2n)} - \int_1^\infty \left(\cos x - \frac{1}{1+x}\right) \frac{1}{x}\, dx \tag{154}$$

$$\gamma = 1 - \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n+1)!\,(2n)} - \int_1^\infty \left(\frac{\operatorname{sen} x}{x} - \frac{1}{1+x}\right) \frac{1}{x}\, dx \tag{155}$$

$$\gamma = -\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!\, n} - \int_1^\infty \left(e^{-x} - \frac{1}{1+x}\right) \frac{1}{x}\, dx \tag{156}$$

$$\gamma = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!\, n} - 2^k \int_1^\infty \frac{e^{-x^{2^k}}}{x}\, dx\ ,\ \ k \in \mathbb{Z} \tag{157}$$

$$\gamma = H_n - \ln(n+1) + \int_0^1 \int_0^1 \frac{x^n (1-x^y)}{1-x}\, dy\, dx \quad n \in \mathbb{N} \tag{158}$$

$$(p-q)\,\gamma = \sum_{n=1}^{\infty} \frac{(-1)^n\,(q\, a^{p\,n} - p\, a^{q\,n})}{n!\, n} + p\, q \int_a^\infty \frac{e^{-x^p} - e^{-x^q}}{x}\, dx \tag{159}$$

$$p > 0\,,\ \ q > 0\,,\ \ a > 0$$

$$\gamma = \int_0^1 \int_0^1 \frac{x^y - (1 - y \ln x)^{-2}}{x}\, dy\, dx \tag{160}$$

$$\gamma = \int_0^1 \int_0^1 \frac{1 + \ln x}{1 - y\,(1 + \ln x)}\, dy\, dx \tag{161}$$

$$\gamma = \int_0^\infty \int_0^1 \frac{(1-x)\, e^{-x}}{1 - y + x\, y}\, dy\, dx \tag{162}$$

$$\gamma = \int_0^\infty \int_0^\infty \frac{(1-x)\, e^{-x-y}}{1 - e^{-y}(1-x)}\, dy\, dx \tag{163}$$

$$\gamma = 2 \int_0^\infty \int_0^1 \left(e^{-x\,y} - 2\, x\, y\, e^{-(x\,y)^2}\right) dy\, dx \tag{164}$$

$$\gamma = \int_0^\infty \int_0^1 \left(e^{-x\,y} - (1 + x\,y)^{-2}\right) dy\, dx \tag{165}$$

$$\gamma = \int_0^\infty \int_0^\infty \left(e^{-x\, e^{-y}} - (1 + x\, e^{-y})^{-2}\right) e^{-y}\, dy\, dx \tag{166}$$

$$\gamma = 2 \int_0^1 \int_0^1 \frac{x^y + 2\, y\, (\ln x)\, e^{-y^2 (\ln x)^2}}{x}\, dy\, dx \tag{167}$$

$$\gamma = -\int_0^\infty \int_0^\infty e^{-x\,y-x}\, x \ln x\, dy\, dx \tag{168}$$

$$\gamma = -\int_0^\infty \int_0^1 \frac{y + \ln(1-y)}{y}\, (1-y)^x\, dy\, dx \tag{169}$$

$$\gamma + 1 = -\int_0^\infty \int_0^\infty \frac{e^{-x\,y-x} \ln x}{1+y}\, d\,y\, d\,x \tag{170}$$

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