General integration theory defined from extended cohomology.

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Abstract

We engage in an approach towards integration theory divorced from measure theory concentrating on the differentiable functions instead of the measurable ones. In a sense, we do for "measure theory" what differential geometry does for topology; the final goal of this paper being the rigorous definition of a generalization of the Feynman path integral. The approach taken is an axiomatic one in which it is more important to understand relationships between certain quantities rather than to calculate them exactly. In a sense, this is how the field of algebraic geometry is developed in opposition to the study of partial differential equations where in the latter case, the stress is unfortunately still too much on the construction of explicit solutions rather than on structural properties of and between solutions.

1 Introduction to the idea.

Upon reflection, the construction of an integral is a rather elaborate task: first, one has to define the sigma algebra according to which measurable functions can be derived. Next, one constructs the Lebesgue integral for positive functions and hence for general real functions and ultimately for complex valued functions. So, at first, one would suspect differentiable functions having not much to do with integration theory just like one would perhaps be inclined to think that differential geometry would not have much to say about topology. The latter is well known to be false and John Milnor has written the most lovely book about topology from the differentiable viewpoint. Likewise will we treat integrals here from the differentiable viewpoint, or at least something proportional to an integral. The ultimate motivation for this paper comes from the Feynman path integral, where one has to calculate the normalized partition function

$$Z\left[J\right] = \frac{\int e^{iJ(\phi)} d\mu(\phi)}{\int d\mu(\phi)}$$

where neither the nominator, nor the denominator are well defined but expressions which blow up towards infinity. The question is whether one can endow

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$Z[J]$ directly with meaning; for a one dimensional integral this boils down to

$$F(f(t)dt) = \frac{\int_a^b f(t)G(t)dt}{\int_a^b G(t)dt}$$

a functional on the one forms. One notices that for one forms of the kind $(dg)G^{-1}$ with $g(a) = g(b)$ that

$$F(dg\, G^{-1}) = 0.$$

Together with the linearity of F and the normalization $F(dx) = 1$ does this determine F completely. Indeed, any continuous function f, assuming that G is continuous too, may be written as

$$f(x)G(x) = (\partial g)(x) + \frac{1}{b-a}\int_a^b f(t)G(t)dt$$

where $g(a) = g(b) = 0$ leading to

$$F(f(x)dx) = \frac{F(G^{-1}(x)dx)}{b-a}\int_a^b f(t)G(t)dt$$

by linearity and the cohomology condition. Normalization then implies that

$$F(G^{-1}(x)dx) = \frac{b-a}{\int_a^b G(t)dt}$$

leading to the original definition. We now generalize this idea to higher dimensions.

2 Higher dimensional integrals.

Consider again the functional

$$F(\mu, f) = \frac{\int_M f\mu}{\int_M \mu}$$

where M is an n-dimensional compact[1], oriented manifold, possibly with boundary, μ a measure on M and f a function (which one might choose to vanish on ∂M). Then, one obtains that F is constant on the cohomology class of $f\mu$ meaning if $f\mu = g\nu + d\lambda$ and μ, ν determine the same volume, where λ is an $n-1$ form which vanishes on the boundary, then

$$F(\mu, f) = F(\nu, g).$$

F satisfies the following properties: (a) $F(\mu, f+g) = F(\mu, f) + F(\mu, g)$ (linearity in f) (b) $F(\mu, 1) = 1$ (normalization) (c) $F(\mu, f) = F(\nu, g)$ in case $f\mu = g\nu + d\kappa$ (cohomology condition) where κ is an $n-1$ form vanishing on the boundary and $F(\mu, G) = 1$ where $\nu = G\mu$ (such a positive function G can always be found)

[1] A topological space is compact if and only if every covering by means of open sets has a finite subcovering.

(d) $F(G\nu, f) = \frac{F(\nu, Gf)}{F(\nu, G)}$ (quotient rule) and $F(\tilde{\mu}, \chi_i)$ is given for some reference $\tilde{\mu}$ and characteristic function χ_i on a component $\mathcal{M}_i$ of $\mathcal{M}$. Axiom (d) is mandatory in case the manifold consists out of different connected components. We now prove that those four conditions and the determination of F on the n'th cohomology module[2] determine F completely. Let me give a reason why this should work; in our first example, the cohomology of the interval implies that $f\mu = dg + c\,dx$ for any f and any measure μ where c is a fully specified constant and the function g vanishes on the boundary, which is a disconnected space with two components. Note that an equation of the type $f\mu = dg + c\mu$ is possible but with a different c of course. Therefore, in the second case, F is fully determined by c by means of the normalization condition *in spite of* the two disconnected components of the boundary. This suggests that it suffices indeed to restrict to $n-1$ forms λ which vanish on the boundary. The reader should notice that c depends upon the interval which implies that for multiple disjoint intervals the volume ratios of the separate components with regard to the total space are needed (cfr.(d)). To be precise, the type of cohomology equivalence considered here is defined by means of the space Z_n of n-forms and the space V_n of exterior derivatives of the $n-1$ forms λ satisfying $\lambda = 0$ at ∂M. We do *not* simply take the quotient since one needs the supplementary condition on the relative volumes between two measures. The proof of our assertion now reduces to showing that for any component $\mathcal{M}_i$ of $\mathcal{M}$ and any pair (μ, f) one has that

$$f\mu = \alpha_i + d\lambda_i$$

on $\mathcal{M}_i$ where $\alpha_i \in \frac{Z_n}{V_n}$ is of the form $c_i\tilde{\mu}$ almost everywhere with c_i some constant which follows from Betti duality (which implies that the dimension of the zero'th cohomology class over the complex numbers equals the dimension of the n'th cohomology class), $\lambda_i \in V_n$ and $\tilde{\mu}$ some *fixed* reference volume (which may be chosen to have the same volume than or to be equal to μ). This can be shown to be true by reconstructing every manifold by means of surgery theory applied to n-dimensional cubes with boundary conditions imposed on two chosen opposite $n-1$ dimensional faces and gluing conditions imposed on the remaining $2(n-1)$ faces[3]. The reader may then easily show by means of Fourier analysis that one has $n-1$ local degrees of freedom supplemented with the boundary condition and one global constraint per mode which *enforces* the condition

$$c_i = \frac{\int_{\mathcal{M}_i} f\mu}{\int_{\mathcal{M}_i} \tilde{\mu}}.$$

Hence, if $\mathcal{M}$ were to consist out of one component: linearity (a), normalization (b) and the cohomology condition (c) with $\nu = \mu$ would fix F; multiple components are taken care of due to (d). Indeed, $f\mu$ is equivalent to $(\sum_i c_i\chi_i)(c\tilde{\mu})$ by the cohomology condition assuming that $F(\mu, G) = c^{-1}$ where $\tilde{\mu} = G\mu$ and therefore, by (a) and (c) $F(\mu, f) = \sum_i c_i F(c\tilde{\mu}, \chi_i)$ which is fixed given $F(\tilde{\mu}, \chi_i)$ and

$$F(\mu = H\tilde{\mu}, \chi_i) = \frac{F(\tilde{\mu}, H\chi_i)}{F(\tilde{\mu}, H(\sum_j \chi_j))} = \frac{d_i F(\tilde{\mu}, \chi_i)}{\sum_j d_j F(\tilde{\mu}, \chi_j)}$$

[2]Each cohomology class has a representative of the type $f\mu$.

[3]A one dimensional cube has two faces (points), a square has four lines, a cube has six faces ....

where we have used formula (d) and the cohomology condition on each separate component. Mind, that the $F(\tilde{\mu}, \chi_i)$ have to be "well chosen" in order for F to be the quotient of two well defined integrals over $\mathcal{M}$; one cannot do any better than this however.

3 Generalization towards infinite dimensions.

As mentioned in the introduction, neither $\int d\mu(\phi)$ nor $\int e^{iJ\phi} d\mu(\phi)$ are well defined in quantum field theory but their quotients are given some regularization scheme. Also, the measures here are complex which forces us to impose that $\int \mu \neq 0$ in order for our scheme to be consistent. By a measure μ, we mean an object which locally corresponds to an expression of the type

$$\mu = \alpha(x_j) dx_1 \wedge dx_2 \wedge \ldots$$

where the x_j form an oriented complete coordinate system of the infinite dimensional manifold $\mathcal{M}$ and α can be any complex valued function one likes such as

$$\alpha(x_j) = e^{iS(x_j)}$$

where S is some classical real valued action. Hence, we arrive at an expression of the type

$$F(\mu(\phi), e^{iJ(\phi)})$$

satisfying our previous four axioms. In general, the manifold $\mathcal{M}$ has an infinite number of components in case the first homotopy group of configuration space has an infinite number of elements. In that case, the definition of the relative volume of one component comes into danger supposing that one would have a "translation symmetry" which ought to be measure preserving up to a constant unitary number. Here, only complex measures can survive given that one would be in a situation of the type

$$1 + z + z^2 + \ldots = \frac{1}{1 - z}$$

with $|z| = 1 - \epsilon$ as opposed to

$$1 + 1 + 1 \ldots = \infty$$

in the case of positive measures. It might be that this problem can indeed be resolved in this way but it does not need to. I conjecture, moreover, that given suitable convergence criteria, our proof that F is fully determined from those four axioms still goes trough. Effectively, the great virtue of our reasoning was that we reduced the quotient

$$\frac{\int_{\mathcal{M}_i} f\mu}{\int_{\mathcal{M}_i}}$$

to a quotient of two one dimensional integrals of the Fourier modes of zero momentum which depend on one (transversal to the boundary) coordinate only. It remains to be seen how well this scheme works in practise, but there is at least some hope that we have concentrated on the relevant finite numbers and neglected infinite multiplication factors.

4 Conclusions.

The idea in this paper has been clearly introduced and Fourier analysis played a rather crucial role in the construction; much remains to be examined before any conclusive results can be given, but at least our definition has a non-trivial chance to survive given that the situation effectively reduces to a one dimensional one[4].

[4]Of course, the determination of the zero'th Fourier mode is a global problem and requires an infinite number of integrations.