On New Quantum Search Algorithms and Complexity of *NP-Complete* problems

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Abstract

We develop three new quantum algorithms for searching the desired target state in the unstructured database of size *N*. The first algorithm requires *Log N* iterative steps. It constructs two quantum bags of equal size in terms of two quantum states, out of which exactly one quantum state will have nonzero overlap with the target state. This determination of overlap is done by taking the inner product, in *Log N* time [2], of the implicitly known target state with any one of these two quantum states. The second algorithm requires just one single step which uses a new suitable operator and the choice of this operator is problem dependent, i.e. it depends upon the number of qubits required to be used to represent an element in the index set. The third algorithm again requires only a single step and this algorithm makes use of a fixed (same) operator. It is known that algorithm for unstructured database search can be easily adaptable for solving *NP-Complete* problems. However, the computational complexity of *NP-Complete* problems after the adaptations of both the classical as well as quantum [1] search algorithms remains of the exponential order as the exponent for quantum [1] algorithm changes only to one-half times the exponent for classical algorithm. But for our quantum algorithms the exponent falls substantially so that our new quantum algorithms for unstructured search are capable if reducing the computational complexity of *NP-Complete* problems to polynomial order!

1. **Introduction:** In this paper we propose three new quantum algorithms for unstructured database search. If *N* is the size of the unstructured database then we show that we can pick out the desired target in just $LogN$ steps by the first algorithm, and in just single step by the other two algorithms! The innovation in the first algorithm consists of dividing the given database into two equal sized databases in terms of two quantum states and By using the idea of taking inner product in $LogN$ time [2] of any one these quantum states representing the quantum bags with the target state which enables one to find out the quantum state to which the target state belongs!

   We now proceed to propose our first quantum algorithm for unstructured search. This new quantum algorithm proceeds roughly as follows: It begins with the preparation of the implicitly known desired target state. Starting with a quantum bag that contains target, i.e. starting with a quantum state that contains the target state it then carries out the construction of two suitable initial quantum states using state that contains the target state in the superposition. It then evaluates the inner product of the target state with any one of the two initially constructed quantum states mentioned above. The value of this inner product determines to which quantum state the target state belongs. This quantum state to which target state belong is used further to construct two more new suitable quantum states and the same procedure is repeated iteratively. By iterating these steps for $LogN$ times we will see that with these steps one directly arrives at the desired target state and completes the search. As mentioned above the generate-and-test type classical algorithm or quantum [1] algorithm for unstructured database search though can be easily adapted to solve the *NP-Complete* problems still the computational complexity of these algorithms after the adaptations remains that of the exponential order as the exponent for quantum [1] algorithm changes only to one-half times the exponent for classical algorithm. But for our first quantum algorithm the exponent becomes the polynomial of logarithm of the exponent for the classical algorithm. Therefore, our first quantum algorithm reduces the computational complexity of *NP-Complete* problems to polynomial order! If N is the size of the unstructured database then we show that we can attain the desired target in just $LogN$ steps! To attain the desired target the best known generate-and-test type classical algorithm and quantum [1] algorithm for unstructured search requires roughly $\frac{N}{2}$ steps and $\sqrt{N}$ steps respectively. This implies that only quadratic speedup is achievable by quantum [1] algorithm over classical algorithm. Though such speedup is quite good one still it is not good enough as it doesn't tame the problems with exponential complexity. A formal statement of unstructured search problem is as follows: Consider a search problem that requires to find a particular element of the database. Given a set containing N candidates, and suppose these N candidates are labeled by indices, x in the range $0 \leq x \leq (N-1)$, and that the index of the sought after target item is $x = t$. Let there be a computational oracle, or "black-box function", $f_t(x)$, that when presented with an index x can pronounce on whether or not it is the index of the target. Specifically, $f_t(x)$ is defined such that $f_t(x) = 1$ if $x = t$ and $f_t(x) = 0$ if $x \neq t$ where 1 stands for YES and 0 stands for NO. The search problem is unstructured because there is no discernible pattern to the values of $f_t(x)$ to provide any guidance in finding $x = t$. Our job is to find index $x = t$, using fewest calls to the oracle $f_t(x)$. Oracle is nothing but a factitious mathematical device that allows one to estimate the computational cost of some algorithm measured in the units of the "number of calls to this oracle", required to reach the solution. "Oracle" or "black-box function" or "knowledge holder" are synonyms, and if we consider for example the problem of finding name given telephone number what is the oracle? The 'oracle' in this case is the 'telephone directory' itself. We now express the search problem in quantum mechanical language. A quantum analog of the bag of indices can be regarded as an equally weighted superposition of all the indices in the range $0 \le x \le (N-1)$, i.e. the quantum state $|\Psi\rangle = \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}|x\rangle$. Thus, the bag of all indices can be looked upon as a wave function $|\Psi\rangle$ given above. Let us suppose that $N = 2^n$. Therefore, using binary representation for all the indices in the bag we can express the wave function representing bag of indices as

$$|\Psi\rangle = \frac{1}{\sqrt{N}}\sum_{i_1,i_2,\cdots i_n=0}^{1}|i_1 i_2 \cdots i_n\rangle$$

To prepare such state is in fact a very easy task. For this one just need to take as a starting state a tensor product of *n* number of zero kets, $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and then apply Hadamard operator, H, on each zero ket, $|0\rangle$, in the tensor product. Thus,

$$|\Psi\rangle = \frac{1}{\sqrt{N}}\sum_{i_1,i_2,\cdots i_n=0}^{1}|i_1 i_2 \cdots i_n\rangle = H^{\otimes n}|00\cdots 0\rangle$$

The implicitly known target state $|t\rangle = |t_1 t_2 t_3 \cdots t_n\rangle$, where each $t_i \in \{0,1\}$, can be prepared using the oracle, $f_t(x)$, which gives value 1 when $x = t$ and 0 when $x \ne t$, by expressing the target state, $|t\rangle$ corresponding to index $x = t$ by using the relation of the target state, $|t\rangle$, and the oracle function, $f_t(x)$. This relation can be expressed in the following two equivalent forms:

$$|t\rangle = \frac{\sqrt{N}}{2}\left[1-(-1)^{f_t(x)}\right]|\Psi\rangle \qquad \text{(A)}$$

or,

$$|t\rangle = \frac{\sqrt{N}}{2}\left[1-\left[1-2|t\rangle\langle t|\right]\right]|\Psi\rangle \qquad \text{(B)}$$

We now divide the elements in the bag containing indices x, such that $0 \le x \le (N-1)$, into two bags such that the first bag will contain half indices, i.e. all those indices, x, such that $0 \le x \le (\frac{N}{2}-1)$ and the second bag will contain remaining half indices, i.e. all those indices, x, such that $\frac{N}{2} \le x \le (N-1)$. It is easy to achieve this by constructing these bags in terms of two quantum states, $|\Psi_0\rangle$ and $|\Psi_1\rangle$ as follows, where

$$|\Psi_0> = |0> \otimes H^{\otimes(n-1)}|00\cdots 0>$$

and,

$$|\Psi_1> = |1> \otimes H^{\otimes(n-1)}|00\cdots 0>$$

Note that the ket $|00\cdots 0>$ in the above expressions for $|\Psi_0>$ and $|\Psi_1>$ is of length $(n-1)$, i.e. a computational basis state in $(2^{(n-1)})$ dimensional Hilbert space, while $|\Psi_0>$ and $|\Psi_1>$ are obviously states in 2^n dimensional Hilbert space. Also, $|\Psi_0>$ represents the bag that contains all those indices, x, such that $0 \le x \le (\frac{N}{2} - 1)$ and $|\Psi_1>$ represents the bag that contains all those indices, x, such that $\frac{N}{2} \le x \le (N-1)$, as desired.

The idea behind our new quantum algorithm in simple terms is to divide "the bag which contains the target state" at each iterative step into two separate bags of equal size such that now the target state will belong to some one and only one of these two bags which now has become equal to half of the size of the original bag and then to determine by taking inner product of any one state representing these bags with the target state to which the target state belongs. Thus we manage to reduce the size of the bag that contains the target state in each of the iterations to half of its size at that stage. By proceeding along these lines finally the bag that contains the target state will become of size one, i.e. it will contain only the target state itself. Thus, we first begin with the bag represented by the wave function, $|\Psi>$, this original bag contains all numbers from 0 to *N*-1., i.e. it contains the target state.

We now proceed systematically with our first new quantum algorithm through precise steps as follows:

2. The First New Quantum Search Algorithm implying *P* = *NP* :

(i) Construct quantum state, $|\Psi>$ say, representing the bag of all indices x, such that $0 \le x \le (N-1)$. Namely,

$$|\Psi> = \frac{1}{\sqrt{N}} \sum_{i_1, i_2, \cdots i_n = 0}^{1} |i_1 i_2 \cdots i_n> = H^{\otimes n}|00\cdots 0>$$

Let $N = 2^n$.Since, $t \in [0,(2^n - 1)]$, therefore $|t> \in |\Psi>$, i.e. certainly, $<t|\Psi> \neq 0$.

(ii) Since $t \in [0,(2^n - 1)]$, we divide the indices in this bag into two parts of identical size and put them into two new bags. This is done in equivalent terms as follows. Construct two quantum states $|\Psi_0>$ and $|\Psi_1>$ representing these two bags such that $|\Psi_0>$ will represent the bag that contains all those indices, x, such that $0 \le x \le (2^{(n-1)} - 1)$ and $|\Psi_1>$ will represent the bag that contains all those indices, x, such that $2^{(n-1)} \le x \le (2^n - 1)$. Thus, we have

$$|\Psi_0> = |0> \otimes H^{\otimes(n-1)} |00\cdots 0>$$

and,

$$|\Psi_1> = |1> \otimes H^{\otimes(n-1)} |00\cdots 0>$$

This further implies that either $|t> \in |\Psi_0>$ or $|t> \in |\Psi_1>$.

(iii) Take inner product of the implicitly known target state $|t>$, expressed above in two equivalent forms, (A) or (B), with any one of the two quantum states representing two bags of indices, namely, $|\Psi_0>$ and $|\Psi_1>$ given above.

Case (a): Without loss of generality (WLOG), suppose if $<t|\Psi_0> \neq 0$ then clearly we can infer that $|t> \in |\Psi_0>$, i.e. $t \in [0, (2^{(n-1)} - 1)]$, i.e. t belongs to the first bag that contains all those indices, x, such that $0 \le x \le (2^{(n-1)} - 1)$.

Case (b): Without loss of generality (WLOG), suppose if $<t|\Psi_0> = 0$, i.e. $<t|\Psi_1> \neq 0$, then clearly we can infer that $|t> \in |\Psi_1>$, i.e. $t \in [2^{(n-1)}, (2^n - 1)]$, i.e. t belongs to the second bag that contains all those indices, x, such that $2^{(n-1)} \le x \le (2^n - 1)$.

(iv) Case (a): Since $t \in [0, (2^{(n-1)} - 1)]$, we divide this bag of indices into two equal parts and put them into two new bags. This is done in equivalent terms as follows. Construct two quantum states $|\Psi_{00}>$ and $|\Psi_{01}>$ representing these two bags such that $|\Psi_{00}>$ will represent the bag that contains all those indices, x, such that $0 \le x \le (2^{(n-2)} - 1)$ and $|\Psi_{01}>$ will represent the bag that contains all those indices, x, such that $2^{(n-2)} \le x \le (2^{(n-1)} - 1)$. Thus, we have

$$|\Psi_{00}> = |0>|0> \otimes H^{\otimes(n-2)} |00\cdots 0>$$

and,

$$|\Psi_{01}> = |0>|1> \otimes H^{\otimes(n-2)} |00\cdots 0>$$

This further implies that either $|t> \in |\Psi_{00}>$ or $|t> \in |\Psi_{01}>$.

Case (b): Since $t \in [2^{(n-1)}, 2^n - 1)]$, we divide this bag of indices into two equal parts and put them into two new bags. This is done in equivalent terms as follows. Construct two quantum states $|\Psi_{10}>$ and $|\Psi_{11}>$ representing these two bags such that $|\Psi_{10}>$ will represent the bag that contains all those indices, x, such that $2^{(n-1)} \leq x \leq (2^{(n-1)} + (2^{(n-2)} - 1)$ and $|\Psi_{11}>$ will represent the bag that contains all those indices, x, such that $(2^{(n-1)} + 2^{(n-2)}) \leq x \leq (2^n - 1)$. Thus, we have

$$|\Psi_{10}>=|1>|0>\otimes H^{\otimes(n-2)}|00\cdots 0>$$

and,

$$|\Psi_{11}>=|1>|1>\otimes H^{\otimes(n-2)}|00\cdots 0>$$

This further implies that either $|t>\in|\Psi_{10}>$ or $|t>\in|\Psi_{11}>$.

(v) As is done in **(iii),** by taking inner product of the target state $|t>$ now with $|\Psi_{00}>$ or $|\Psi_{01}>$ when case (a) is true, or with $|\Psi_{10}>$ or $|\Psi_{11}>$ when case (b) is true we determine to which quantum bag represented by these quantum states the target state is part of, i.e. the target state has a nonzero overlap with. We continue on these lines with dividing, each time the correct quantum bag (the one containing the target state), into two separate new quantum bags till (assuredly) the size of the correct quantum bag (that has nonzero overlap with target state) will reduce to the bag containing just one entry, i.e. the target state itself!!

□

3. The Second New Quantum Search Algorithm implying *P* = *NP* :

Let x, $0 \leq x \leq (N-1)$, be an element in the unstructured database of size N. Let $N = 2^n$ hence $0 \leq x \leq (2^n - 1)$. Our aim in the unstructured database search problem is to locate and pick out the target index, $t \in [0,(2^n - 1)]$. Note that with each index x we can associate a computational basis state, $|x>$ made up of n qubits, i.e. $|x>=|x_1 x_2 \cdots x_n>$, where, $x_i \in \{0,1\}, 1 \leq i \leq n$. So, our aim in the unstructured database search problem is to substantially amplify the amplitude of the target state, $|t>$. Suppose we have a 1-YES quantum oracle defined in terms of operator, O, which performs the operation $O|x>=(-1)^{f_t(x)}|x>$, where as mentioned previously $f_t(x)$ is defined such that $f_t(x) = 1$ if $x = t$ and $f_t(x) = 0$ if $x \neq t$. where $f_t(x) = 1$ stands for YES and $f_t(x) = 0$ stands for NO. It is clear to check that the operator O is unitary. We can see that the real operator O is an inversion operator which only changes the sign of the target state $|t>$ and keeps all other states $|x>$ unchanged. If we take a wave function, $|\Psi>$

say, made up of some superposition of computational basis states and operate the operator O on it then by its definition it will leave all the computational basis states as they are and will change the sign only that of the computational basis state which is the target state. Now if we will operate O one more time then again it will leave all the computational basis states as they are and will restore the sign of the target state. Thus, $O^2 = OO^+ = O^+O = I$ .We define $M_k = \left[(2^k |\Psi><\Psi| - (2^k - 1)I)\right]$, a new operator. We now check the following:

Claim: $M_k^+ M_k |\Psi> = |\Psi>$.

Proof: Note that $<\Psi|\Psi> = 1$. Consider the case $k = 1$ as follows:
We have $M_1 = [2|\Psi><\Psi| - I]$, therefore,
$M_1^+ M_1 |\Psi> = [2|\Psi><\Psi| - I][2|\Psi><\Psi| - I]|\Psi>$
$= [4|\Psi><\Psi|\Psi><\Psi| - 2|\Psi><\Psi| - 2|\Psi><\Psi| + I]|\Psi>$
$= [4|\Psi><\Psi|\Psi><\Psi| - 4|\Psi><\Psi| + I]|\Psi>$
$= [4|\Psi><\Psi| - 4|\Psi><\Psi| + I]|\Psi>$
$= |\Psi>$. Let us now consider the case $k = 2$ as follows:
$M_2^+ M_2 |\Psi> = [4|\Psi><\Psi| - 3I][4|\Psi><\Psi| - 3I]|\Psi>$
$= [16|\Psi><\Psi|\Psi><\Psi| - 12|\Psi><\Psi| - 12|\Psi><\Psi| + 9I]|\Psi>$
$= [16|\Psi><\Psi|\Psi><\Psi| - 24|\Psi><\Psi| + 9I]|\Psi>$
$= [16|\Psi><\Psi| - 24|\Psi><\Psi| + 9I]|\Psi>$
$= [16|\Psi> - 24|\Psi> + 9|\Psi>]|$
$= |\Psi>$. On similar lines the general case also follows:
$M_k^+ M_k |\Psi> = [2^k |\Psi><\Psi| - (2^k - 1)I][2^k |\Psi><\Psi| - (2^k - 1)I]|\Psi>$
$= [2^{2k} |\Psi><\Psi| - 2(2^k(2^k - 1))|\Psi><\Psi| + (2^k - 1)(2^k - 1)I]|\Psi>$
$= [2^{2k} - 2^{2k} - 2^{2k} + 2^k + 2^k + 2^{2k} - 2^k - 2^k + 1]|\Psi>$
$= [2^{2k} - 2(2^{2k}) + 2(2^k) + 2^{2k} - 2(2^k) + 1]|\Psi>$
$= |\Psi>$.

We now define the operator called the "total operator", $T_k = M_k O$. We are now ready to discuss our second algorithm which requires **only a single step** to find the target state! Before we discuss the algorithm we state one important result which is used in this algorithm.

Claim: Let the initial wave function, $|\Psi>$, representing the quantum bag of indices be an equally weighted superposition of computational basis states, $|x>$, of length k , i.e. the quantum bag contains 2^k indices. Also, let there be only one target state,

$|t>$, then the target state, $|t>$, can be found, or reached, or attained, or achieved by just operating once the operator $T_{(k-1)} = M_{(k-1)}O$ on this wave function, $|\Psi>$.

Proof: It is clear to see that $|\Psi> = \frac{1}{\sqrt{2^k}} \sum_{x=0}^{(2^k-1)} |x>$, where $|x> = |i_1 i_2 \cdots i_k>$

We now operate the operator $T_{(k-1)} = M_{(k-1)}O$ on the wave function $|\Psi>$. Thus we have

$$T_{(k-1)}|\Psi> = M_{(k-1)}O|\Psi> = M_{(k-1)}\left[|\Psi> - \frac{2}{\sqrt{2^k}}|t>\right]$$

$$= \left[(2^{(k-1)}|\Psi><\Psi| - (2^{(k-1)}-1)I)\right]\left[|\Psi> - \frac{2}{\sqrt{2^k}}|t>\right]$$

$$= 2^{(k-1)}|\Psi> - 2^{(k-1)}|\Psi> + |\Psi> - \frac{2^k}{\sqrt{2^k}}|\Psi><\Psi|t> + \frac{2(2^{(k-1)}-1)}{\sqrt{2^k}}|t>$$

$$= 2^{(k-1)}|\Psi> - 2^{(k-1)}|\Psi> + |\Psi> - \left(\frac{2^k}{\sqrt{2^k}}\right).\left(\frac{1}{\sqrt{2^k}}\right)|\Psi> + \frac{2(2^{(k-1)}}{\sqrt{2^k}}|t>$$

$$= 2^{(k-1)}|\Psi> - 2^{(k-1)}|\Psi> + |\Psi> - |\Psi> + \frac{2(2^{(k-1)}}{\sqrt{2^k}}|t>$$

$$= \frac{2(2^{(k-1)}-1)}{\sqrt{2^k}}|t>$$

Thus, we have got only the target state, $|t>$, with nonzero amplitude and all other basis states in $|\Psi>$ vanish, i.e. their amplitude becomes zero! Note that the amplitude of target state becomes large (in fact bigger than unity). This implies that the total "operator" is not unitary since the action of unitary operator on a vector preserves the length of the vector and we have chosen the wave function (vector) $|\Psi>$ such that $\||\Psi>\|^2 = 1$.

We now proceed to formally discuss the steps of the algorithm which consists of just applying the appropriate "total operator" on the wave function, $|\Psi>$, representing the given quantum bag of indices containing a single target index.

Thus, let the given bag of indices contains $N = 2^n$ elements. We will prepare the quantum bag in terms of the wave function, $|\Psi>$, as follows in the following

Steps of the algorithm:

(i) We consider a quantum state containing n qubits, all initialized to zero, i.e. the state $|00\cdots 0>=|0>^{\otimes n}$.

(ii) We apply Hadamard transform to all the n qubits to get

$|\Psi>=H^{\otimes n}|0>^{\otimes n}=\frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x>$. Clearly, $|t>\in|\Psi>$.

(iii) Since the size of the data is $N=2^n$ so we choose $T_{(n-1)}=M_{(n-1)}O$ as our "total operator" to operate on the wave function $|\Psi>$.

(iv) We carry out the action of the chosen operator on the wave function only in terms of the target state itself!

$$T_{(n-1)}|\Psi>=M_{(n-1)}O|\Psi>=M_{(n-1)}\left[|\Psi>-\frac{2}{\sqrt{2^n}}|t>\right]$$

$$=\left[(2^{(n-1)}|\Psi><\Psi|-(2^{(n-1)}-1)I)\right]\left[|\Psi>-\frac{2}{\sqrt{2^n}}|t>\right]$$

$$=\frac{2(2^{(k-1)}-1)}{\sqrt{2^k}}|t>.$$

□

Thus, it is clear that if we carry out measurement then we will get the target state, $|t>$, with probability one!! Thus, this algorithm assures us to obtain the target state with 100% guarantee!!!

4. The Third New Quantum Search Algorithm implying $P = NP$:

Again, Let x, $0\le x\le(N-1)$, be an element in the unstructured database of size N. Let $N=2^n$ hence $0\le x\le(2^n-1)$. Our aim in the unstructured database search problem is to locate and pick out the target index, $t\in[0,(2^n-1)]$. Also, Suppose we have a 1-YES quantum oracle defined in terms of operator, O, which performs the operation $O|x>=(-1)^{f_t(x)}|x>$, where as mentioned previously $f_t(x)$ is defined such that $f_t(x)=1$ if $x=t$ and $f_t(x)=0$ if $x\neq t$. where $f_t(x)=1$ stands for YES and $f_t(x)=0$ stands for NO. As seen previously, the operator O is unitary. Thus, everything is same as it was in previous algorithms. In this new quantum algorithm we will be doing non-unitary quantum computation, i.e. the operator we will be using to achieve the task of enhancing the amplitude of the target state, $|t>$, as is done in the previous two algorithms is non-unitary. This algorithm also works in just a single step, i.e. it enhances the amplitude of the target state to its full in just one operation of the non-unitary operator chosen for this algorithm.

Steps of the algorithm:

(i) We consider a quantum state containing n qubits, all initialized to zero, i.e. the state $|00\cdots0>=|0>^{\otimes n}$.

(ii) We apply Hadamard transform to all the n qubits to get

$$|\Psi>=H^{\otimes n}|0>^{\otimes n}=\frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x>$$. Clearly, $|t>\in|\Psi>$.

(iii) We apply non-unitary operator, $A=\frac{\sqrt{N}}{2}[I-O]$, on the wave function $|\Psi>$. We get

$$A|\Psi>=\frac{\sqrt{N}}{2}[I-O]|\Psi>=\frac{\sqrt{N}}{2}\left[|\Psi>-|\Psi>+\frac{2}{\sqrt{N}}|t>\right]=|t>$$

.

□

Thus, we have seen that by the action of non-unitary operator, $A=\frac{\sqrt{N}}{2}[I-O]$, and carry out the measurement then we will get the target state, $|t>$, with probability one! The thing to be seen is whether it is possible to build quantum circuit which will perform the action of the non-unitary operator, $A=\frac{\sqrt{N}}{2}[I-O]$.

5. Remarks:

Remark 1: It is clear to see that as the algorithm proceeds we get at each iteration the bag containing proper range of indices to which target index belongs, i.e. we get during each of the iterations a proper quantum bag reduced to half in size, in terms of a quantum state which has nonzero overlap with target state. Thus, as we proceed at an intermediate stage we reach at a wave function,

$$|\Psi_{i_1 i_2 \cdots i_k}>=|i_1 i_2 \cdots i_k > \otimes H^{\otimes(n-k)}|00\cdots 0>$$

which has nonzero overlap with the target state, $|t>$. We then divide the quantum bag into two new quantum bags, i.e. construct two new states out of which only one will have nonzero overlap with the target state, $|t>$, to be determined by taking inner product with any one of these two newly prepared quantum states. Thus, the new quantum states constructed from consideration of the earlier reached above mentioned quantum state will be

$$|\Psi_{i_1 i_2 \cdots i_k 0}>=|i_1 i_2 \cdots i_k > \otimes |0> \otimes H^{\otimes(n-k-1)}|00\cdots 0>$$

and,

$$|\Psi_{i_1 i_2 \cdots i_k 1}>=|i_1 i_2 \cdots i_k > \otimes |1> \otimes H^{\otimes(n-k-1)}|00\cdots 0>.$$

Remark 2: It is interesting to see that the amplitude of each state in the equally weighted superposition of states (including target state) is initially equal to $\frac{1}{\sqrt{N}}$. This state represents the initial quantum bag. After first iteration of the size of the quantum bag reduces to half and this size reaches finally to unity, i.e. finally (at the n^{th} iteration) the quantum bag will contain only the target state itself! Therefore, after first iteration the amplitude of each state in the equally weighted superposition of states becomes $\sqrt{\frac{2}{N}}$. The amplitude of each state including target state in the superposition changes in the successive iterations as follows:

$$\frac{1}{\sqrt{N}} \to \sqrt{\frac{2}{N}} \to \sqrt{\frac{2^2}{N}} \to \cdots \to \sqrt{\frac{2^j}{N}} \to \cdots \to 1$$

Remark 3: It is clear to see that in $n = LogN$ iterations we will attain the target state, i.e. in the final quantum bag, after carrying out $n = LogN$ iterations, will contain only the target state $|t>$ itself which will lead to the value of inner product equal to unity.

Remark 4: It is important to note that actually in each iteration of the algorithm we are getting one bit of the target state. i.e. if the target state is $|t>=|t_1 t_2 \cdots t_j \cdots t_n >=|t_1> \otimes |t_2> \otimes \cdots \otimes |t_j> \otimes \cdots \otimes |t_n>$ then in first iteration we determine the first bit namely, $|t_1>$, in the successive iterations we determine $|t_2>, \cdots, |t_j>, \cdots, |t_n>$. Thus in $n = LogN$ iterations we will be able to determine the target state, $|t>$, completely.

Remark 5: Alternatively, instead of one oracle we may define implicitly *n* number of oracles, $f_t^i(x)$, which gives rise to *n* number of target states $|t^i>=|t_i t_{(i+1)}\cdots t_n>$. Clearly, $f_t^1(x)=f_t(x)$ and it gives rise to target state $|t^1>=|t>$. Further, by finding the nonzero inner product between the inner products taken that of $|t^i>$ with any one of the wave functions, $|\Psi_{i0}>$ and $|\Psi_{i1}>$ that we build, namely, $|\Psi_{i0}>=|0>\otimes H^{(n-i-1)}|00\cdots 0>$ and $|\Psi_{i1}>=|1>\otimes H^{(n-i-1)}|00\cdots 0>$, we can determine separately each bit $|t_i>$ of the target state $|t>$ and then build it as $|t>=|t_1 t_2\cdots t_j\cdots t_n>$.

Remark 6: As far as the value of inner product is concerned we are only interested to know whether it is zero or nonzero, and we are not interested in its exact value. Therefore we can use the existing quantum algorithm [2] to evaluate the inner product with complexity $\sim O(LogN)$. Since our new quantum algorithm requires $LogN$ steps to reach the desired target state and each iterative step requires to find out one inner product which again takes time $\sim O(LogN)$ therefore, our new quantum search algorithm is of the order $\sim O((LogN)^2)$.

Remark 7: For a typical ***NP-Complete* problem** in which one has to find an assignment of one of the b values to each of the C variables, the number of candidate solutions, $N=b^C$, grows exponentially with C. Hence, the classical algorithm for unstructured search would therefore take time of the order, $\sim O(b^C)$, to find desired solution (as the target state) e.g. minimum weight Hamiltonian circuit among the all possible Hamiltonian circuits as a solution for the traveling salesman problem, whereas the Grover's quantum algorithm [1] would take a time of the order, $\sim O(b^{\frac{C}{2}})$. But from the complexity of the order $\sim O((LogN)^2)$ that we get for our quantum search algorithm it is easy to check that our quantum search algorithm will takes time of the order, $\sim O(b^{(LogC)^2})$, thus an impressive (exponential) speedup over existing classical or quantum algorithm. We thus have managed $P=NP$ using our new quantum search algorithm.

Example 1: Let the bag of indices contains numbers $\{0,1,2,\cdots,15\}$ and let the target element, $t=11$. We begin with the wave function, $|\Psi>$, namely, $|\Psi>=H^{\otimes 4}|0000>$ which contains the target state, $|t>=|11>=|1011>$.
We now follow the steps of the algorithm:

Cleary, $<t|\Psi>\neq 0$, therefore, we divide quantum bag represented by $|\Psi>$ into two bags, represented by $|\Psi_0>$ and $|\Psi_1>$, where

$|\Psi_0>=|0>\otimes H^{\otimes 3}|000>$, and $|\Psi_1>=|1>\otimes H^{\otimes 3}|000>$.

Cleary, $<t|\Psi_1>\neq 0$, therefore, we further divide quantum bag represented by $|\Psi_1>$ into two bags, represented by $|\Psi_{10}>$ and $|\Psi_{11}>$, where

$$|\Psi_{10}>=|1>\otimes|0>\otimes H^{\otimes 2}|00>\text{ , and}$$

$$|\Psi_{11}>=|1>\otimes|1>\otimes H^{\otimes 2}|00>.$$

Cleary, $<t|\Psi_{10}>\neq 0$, therefore, we further divide quantum bag represented by $|\Psi_{10}>$ into two bags, represented by $|\Psi_{100}>$ and $|\Psi_{101}>$, where

$$|\Psi_{100}>=|1>\otimes|0>\otimes|0>\otimes H|0>\text{, and}$$

$$|\Psi_{101}>=|1>\otimes|0>\otimes|1>\otimes H|0>.$$

Cleary, $<t|\Psi_{101}>\neq 0$, therefore, we further divide quantum bag represented by $|\Psi_{101}>$ into two bags, represented by $|\Psi_{1010}>$ and $|\Psi_{1011}>$, where

$$|\Psi_{1010}>=|1>\otimes|0>\otimes|1>\otimes|0>\text{, and}$$

$$|\Psi_{1011}>=|1>\otimes|0>\otimes|1>\otimes|1>.$$

Clearly, $<t|\Psi_{1011}>\neq 0$, and in fact $<t|\Psi_{1011}>=1$, therefore,
We have located (reached to) the desired target state, $|t>=|11>=|1011>$, present in the given database (initial quantum bag containing target) in terms of the superposition state, $|\Psi>=H^{\otimes 4}|0000>$.

Example 2: Let the bag of indices contains numbers $\{0,1,2,\cdots,7\}$ and let the target element, $t=3$. We begin with the wave function, $|\Psi>$, namely, $|\Psi>=H^{\otimes 3}|000>$ which contains the target state, $|t>=|3>=|011>$.
Carrying out **step (iv) of the second algorithm** we have

$$\left[(2^2|\Psi><\Psi|-(2^2-1)I)\right]\left[O\right]\left[|\Psi>\right]$$

$$=\left[(2^2|\Psi><\Psi|-(2^2-1)I)\right]\left[|\Psi>-\frac{2}{\sqrt{2^3}}|t>\right]$$

$$=\left[4|\Psi><\Psi|-3I\right]\left[|\Psi>-\frac{1}{\sqrt{2}}|t>\right]$$

$$=4|\Psi>-3|\Psi>-\frac{4}{\sqrt{2}}|\Psi><\Psi|t>+\frac{3}{\sqrt{2}}|t>$$

$$= 4|\Psi> - 3|\Psi> - \frac{4}{\sqrt{2}}|\Psi>.\frac{1}{2\sqrt{2}} + \frac{3}{\sqrt{2}}|t> = \frac{3}{\sqrt{2}}|t>.$$

Example 3: Consider example same as Example 1 above. We solve it now using second algorithm: Let the bag of indices contains numbers $\{0,1,2,\cdots,15\}$ and let the target element, $t = 11$. We begin with the wave function, $|\Psi>$, namely, $|\Psi> = H^{\otimes 4}|0000>$ which contains the target state, $|t> = |11> = |1011>$. Carrying out **step (iv) of the second algorithm** we have

$$\left[(2^3|\Psi><\Psi| - (2^3-1)I)\right]\left[O\right]\left[|\Psi>\right]$$

$$= \left[(2^3|\Psi><\Psi| - (2^3-1)I)\right]\left[|\Psi> - \frac{2}{\sqrt{2^4}}|t>\right]$$

$$= \left[8|\Psi><\Psi| - 7I\right]\left[|\Psi> - \frac{1}{2}|t>\right]$$

$$= 8|\Psi> - 7|\Psi> - 4|\Psi><\Psi|t> + \frac{7}{2}|t>$$

$$= 8|\Psi> - 7|\Psi> - 4|\Psi>.\frac{1}{4} + \frac{7}{2}|t> = \frac{7}{2}|t>.$$

Example 4: Consider same example above. We now solve it using third algorithm: Let the bag of indices contains numbers $\{0,1,2,\cdots,15\}$ and let the target element, $t = 11$. We begin with the wave function, $|\Psi>$, namely, $|\Psi> = H^{\otimes 4}|0000>$ which contains the target state, $|t> = |11> = |1011>$. We carry out **step (iii) of the third algorithm, i.e.** we apply non-unitary operator, $A = \frac{\sqrt{N}}{2}[I - O]$, on the wave function $|\Psi>$. This gives $A = \frac{\sqrt{N}}{2}[I - O] = \frac{4}{2}[I - O]H^{\otimes 4}|0000> = |t>$.

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