An integral with one parameter

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abstract

In this note we exhibit an integral with one parameter.the integrand is of the form:

$$A(t)\cos(f(x,t))\,\mathrm{senh}(g(x,t)) + B(t)\,\mathrm{sen}(f(x,t))\cosh(g(x,t))$$

$$0 \le x \le 1\,,\ \ 0 < t \le \pi/2$$

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1. An integral with one parameter

Sea $0 < \theta \le \dfrac{\pi}{2}$, $f(x,\theta)$, $g(x,\theta)$, $h(x,\theta)$, definidas como :

$$f(x,\theta) = \frac{1-(1-\cos\theta)\,x}{1-2\,(1-\cos\theta)\,x\,(1-x)}\,,\ \ 0 \le x \le 1 \tag{1}$$

$$g(x,\theta) = \frac{x\,\mathrm{sen}\,\theta}{1-2\,(1-\cos\theta)\,x\,(1-x)}\,,\ \ 0 \le x \le 1 \tag{2}$$

$$h(x,\theta) = (1-\cos\theta)\cos\left(f(x,\theta)\right)\mathrm{senh}\left(g(x,\theta)\right) + \mathrm{sen}\,\theta\,\mathrm{sen}\left(f(x,\theta)\right)\cosh(g(x,\theta)) \tag{3}$$

se tiene :

$$\int_0^1 h(x,\theta)\,dx = \theta - \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(2\,n)\,(2\,n+1)\,!}\,\mathrm{sen}(2\,n\,\theta) \tag{4}$$

2. Particular formulas

$$\pi = 2\int_0^1 \left(\mathrm{sen}\left(\frac{1-x}{1-2\,x(1-x)}\right)\cosh\left(\frac{x}{1-2\,x(1-x)}\right) + \cos\left(\frac{1-x}{1-2\,x(1-x)}\right)\mathrm{senh}\left(\frac{x}{1-2\,x(1-x)}\right)\right)dx \tag{5}$$

$$\pi = 2\int_0^1 \left(\mathrm{sen}\left(\frac{x}{1-2\,x(1-x)}\right)\cosh\left(\frac{1-x}{1-2\,x(1-x)}\right) + \cos\left(\frac{x}{1-2\,x(1-x)}\right)\mathrm{senh}\left(\frac{1-x}{1-2\,x(1-x)}\right)\right)dx \tag{6}$$

$$\pi = \frac{3\sqrt{3}}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{(6\,n+2)\,(6\,n+3)\,!} + \frac{1}{(6\,n+4)\,(6\,n+5)\,!}\right) + \frac{3}{2}\int_0^1\left(\sqrt{3}\ \mathrm{sen}\left(\frac{2-x}{2-2\,x(1-x)}\right)\cosh\left(\frac{\sqrt{3}\ x}{2-2\,x(1-x)}\right) + \cos\left(\frac{2-x}{2-2\,x(1-x)}\right)\mathrm{senh}\left(\frac{\sqrt{3}\ x}{2-2\,x(1-x)}\right)\right)dx \tag{7}$$

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+2)(4n+3)!} + 2\sqrt{2}\int_0^1\left(\operatorname{sen}(F(x))\cosh(G(x)) + \left(\sqrt{2}-1\right)\cos(F(x))\operatorname{senh}(G(x))\right)dx \tag{8}$$

donde

$$F(x) = \frac{\sqrt{2}-\left(\sqrt{2}-1\right)x}{\sqrt{2}-2\left(\sqrt{2}-1\right)x(1-x)}, \quad G(x) = \frac{x}{\sqrt{2}-2\left(\sqrt{2}-1\right)x(1-x)} \tag{9}$$

$$\pi = 3\sqrt{3}\sum_{n=0}^{\infty}\left(\frac{1}{(6n+2)(6n+3)!} - \frac{1}{(6n+4)(6n+5)!}\right) + 3\int_0^1\left(\operatorname{sen}(F(x))\cosh(G(x)) + \left(2-\sqrt{3}\right)\cos(F(x))\operatorname{senh}(G(x))\right)dx \tag{10}$$

donde

$$F(x) = \frac{2-\left(2-\sqrt{3}\right)x}{2-2\left(2-\sqrt{3}\right)x(1-x)}, \quad G(x) = \frac{x}{2-2\left(2-\sqrt{3}\right)x(1-x)} \tag{11}$$

$$\begin{aligned}\pi = 8\sum_{n=0}^{\infty}(-1)^n&\left(\frac{1/\sqrt{2}}{(8n+2)(8n+3)!} - \frac{1}{(8n+4)(8n+5)!} + \frac{1/\sqrt{2}}{(8n+6)(8n+7)!}\right) + \\ 4\int_0^1&\left(\sqrt{2-\sqrt{2}}\ \operatorname{sen}(F(x))\cosh(G(x)) + \left(2-\sqrt{2+\sqrt{2}}\right)\cos(F(x))\operatorname{senh}(G(x))\right)dx\end{aligned} \tag{12}$$

donde

$$F(x) = \frac{2-\left(2-\sqrt{2+\sqrt{2}}\right)x}{2-2\left(2-\sqrt{2+\sqrt{2}}\right)x(1-x)}, \quad G(x) = \frac{x\sqrt{2-\sqrt{2}}}{2-2\left(2-\sqrt{2+\sqrt{2}}\right)x(1-x)} \tag{13}$$

$$\frac{4\pi}{3} + \sqrt{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{(6n+2)(6n+3)!} + \frac{1}{(6n+4)(6n+5)!}\right) = \int_0^1\left(\sqrt{3}\ \operatorname{sen}(F(x))\cosh(G(x)) + 3\cos(F(x))\operatorname{senh}(G(x))\right)dx \tag{14}$$

donde

$$F(x) = \frac{2-3x}{2-6x(1-x)}, \quad G(x) = \frac{x\sqrt{3}}{2-6x(1-x)} \tag{15}$$

$$\frac{3\pi}{2\sqrt{2}} + \sqrt{2}\sum_{n=0}^{\infty}\left(\frac{1}{(8n+2)(8n+3)!} - \frac{1}{(8n+6)(8n+7)!}\right) = \int_0^1\left(\operatorname{sen}(F(x))\cosh(G(x)) + \left(\sqrt{2}+1\right)\cos(F(x))\operatorname{senh}(G(x))\right)dx \tag{16}$$

donde

$$F(x) = \frac{\sqrt{2}-\left(\sqrt{2}+1\right)x}{\sqrt{2}-2\left(\sqrt{2}+1\right)x(1-x)}, \quad G(x) = \frac{x}{\sqrt{2}-2\left(\sqrt{2}+1\right)x(1-x)} \tag{17}$$

3. Complementary formulas

$$\pi - e + e^{-1} + 2\,(\mathrm{Chi}(1) - \mathrm{Ci}(1) + \mathrm{sen}\,1) = 4\int_0^1 \mathrm{sen}\left(\frac{1-x}{1-2\,x(1-x)}\right)\cosh\left(\frac{x}{1-2\,x(1-x)}\right)dx \tag{18}$$

$$\pi + e - e^{-1} - 2\,(\mathrm{Chi}(1) - \mathrm{Ci}(1) + \mathrm{sen}\,1) = 4\int_0^1 \cos\left(\frac{1-x}{1-2\,x(1-x)}\right)\sinh\left(\frac{x}{1-2\,x(1-x)}\right)dx \tag{19}$$

donde

$$\mathrm{Chi}(z) = \gamma + \ln z + \int_0^z \frac{\cosh t - 1}{t}\,dt \tag{20}$$

$$\mathrm{Ci}(z) = -\int_z^\infty \frac{\cos t}{t}\,dt \tag{21}$$

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n\right), \text{ Euler's constant} \tag{22}$$

References

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