Mathematical Formulas: Part 6

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Abstract-Resumen

In this paper we give some formulas for the number pi

En esta nota mostramos una colección de fórmulas para la constante pi:

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right) = 3.141592 \ldots$$

Introducción

Descripción y Notación:

En las fórmulas (1)-(2)-(3)-(5)-(10)-(11)-(18)-(19)-(20)-(21)-(24)-(25) , aparecen los números de Bernoulli B_n :

$$B_n = \left\{\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \ldots\right\}$$

En las fórmulas (6)-(7)-(8) , aparece la sucesión:

$$u_n = Im((1+i)^n) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n}{2k+1} \quad , n \in \mathbb{N}$$

$$\{u_n\} = \{1,2,2,0,-4,-8,-8,0,16,32,32,0,-64,-128,-128, \ldots\}$$

El símbolo de Pochhammer es:

$$(a)_n = a(a+1)(a+2)\ldots(a+n-1) \ , \qquad (a)_0 = 1$$

Los números H_n , se definen como:

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

Si $z = x + i\,y$, $x, y \in \mathbb{R}$, $i = \sqrt{-1}$, entonces: $Re(z) = x$, $Im(z) = y$

En las fórmulas (26)-(27)-(28)-(29)-(30)-(31)-(37)-(38)-(48)-(49)-(50) , aparece la función hipergeométrica de Gauss:

$$F(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n\,n!}\,x^n\quad,|x|<1,c\neq 0,-1,-2,\ldots$$

que es solución de la ecuación diferencial

$$x(1-x)y''+(c-(a+b+1)x)y'-aby=0$$

Una representación integral para $F(a,b;c;x)$, es:

$$F(a,b;c;x)=\frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)}\int_0^1 t^{a-1}\,(1-t)^{c-a-1}(1-xt)^{-b}\,dt$$

donde $Re(a)>0, Re(c-a)>0, |x|<1$, y $\Gamma(x)$ es la clásica función Gamma.

En las fórmulas (68),(69),(117),(118),(119),(120), aparecen las funciones de Bessel:

$$J_n(x)=\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta-n\theta)\,d\theta\quad,n\in\mathbb{Z}$$

$$x^2\frac{d^2w}{dx^2}+x\frac{dw}{dx}+(x^2-n^2)w=0\quad,w=J_n(x)$$

En las fórmulas (74)-(75)-(76) , aparece la función Gamma incompleta:

$$\gamma(a,x)=\int_0^x e^{-t}\,t^{a-1}\,dt\quad,a>0$$

$$\gamma(n+1,x)=n!\left(1-e^{-x}\sum_{k=0}^{n}\frac{x^k}{k!}\right)\quad,n=0,1,2,3,\ldots$$

$$\frac{d^2w}{dx^2}+\left(1+\frac{1-a}{x}\right)\frac{dw}{dx}=0\quad,w=\gamma(a,x)$$

En las fórmulas (83),(84),(90),(91), aparecen los números de Euler:

$$E_n=\{1,5,61,1385,50521,\ldots\}$$

Fórmulas

(1) $$\pi = \frac{3}{4} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{6n} B_{2n-1} a^{4n-2}}{(2n-1)(4n-2)!}$$

donde $a = 0.888539$ ..., satisface la ecuación: $\tanh a = \frac{1}{\sqrt{3}} \tan a$

(2) $$\pi = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{6n} B_{2n-1} a^{4n-2}}{(2n-1)(4n-2)!}$$

donde $a = 1.091769$ ..., satisface la ecuación: $\tanh a = \left(\sqrt{2} - 1\right) \tan a$

(3) $$\pi = \frac{3}{8} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{6n} B_{2n-1} a^{4n-2}}{(2n-1)(4n-2)!}$$

donde $a = 1.266367$ ..., satisface la ecuación: $\tanh a = \left(2 - \sqrt{3}\right) \tan a$

(4) $$\pi = 6 \sum_{n=0}^{\infty} \frac{1}{2^n\, n!}\, u_n(a) v_n(b) I_n(a,b)$$

donde

$$u_{n+1}(a) = 2a u_n(a) - 2n u_{n-1}(a)\ , u_0(a) = 1, u_1(a) = 2a\ , a \in \mathbb{R}$$

$$v_{n+1}(b) = 2b v_n(b) - 2n v_{n-1}(b)\ , v_0(b) = 1, v_1(b) = 2b\ , b \in \mathbb{R}$$

$$I_n(a,b) = \int_0^{1/2} x^n\ exp\left(-\frac{2abx - (a^2 + b^2)x^2}{1 - x^2}\right) dx\ , n = 0,1,2,3, \ldots$$

Si $a = b = 0$, entonces (4) se reduce a:

$$\pi = 3 \sum_{n=0}^{\infty} \frac{(1/2)_n^2}{(2n+1)!}$$

(5) $$\pi = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n} (2^{2n} - 1) B_n}{n\, (2n)!}\ Im(z^{2n})$$

donde

$$z = \ln\left(\varphi + \sqrt{\varphi}\right) + i\ \tan^{-1}\left(\sqrt{\varphi}\right)\ , \varphi = \frac{1 + \sqrt{5}}{2}$$

(6) $$\pi = 12 \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{3-\sqrt{3}}{6}\right)^n \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n}{2k+1}$$

(7) $$\pi = 12 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(\frac{\sqrt{3}-1}{2}\right)^n \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n}{2k+1}$$

(8) $$\pi = 4 \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{2}\right)^n \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n}{2k+1}$$

(9) $$\pi = 4 - 4 \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{Im(z_n)}{Re(z_n)}\right) = 4 - 4 \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{b_n}{a_n}\right)$$

donde

$$z_{n+1} = 1 + \frac{i}{n+2}\left(1 - \frac{1}{z_n}\right) , z_1 = \frac{3+i}{4}$$

$$\begin{cases} a_{n+1} = (n+2)(a_n^2 + b_n^2) - (n+1)b_n(a_{n-1}^2 + b_{n-1}^2) \\ b_{n+1} = (a_n^2 + b_n^2) - (n+1)a_n(a_{n-1}^2 + b_{n-1}^2) \\ a_0 = 1, b_0 = 1, a_1 = 3, b_1 = 1 \end{cases}$$

(10) $$\pi = 2 \sum_{n=1}^{\infty} \frac{2^{2n} B_n}{n\,(2n)!} Im(z^{2n})$$

donde

$$z = 1.93737197 \ldots + i\ 1.04467026 \ldots$$

$$z_{n+1} = z_n - \frac{z_n - (1+i)\sin z_n}{1 - (1+i)\cos z_n} \ , z_1 = 2 + i \ , \lim_{n\to\infty} z_n = z$$

(11) $$\pi = 2 \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} B_n}{n\,(2n)!} Im(z^{2n})$$

donde

$$z = 1.04467026 \ldots - i\ 1.93737197 \ldots$$

$$z_{n+1} = z_n - \frac{z_n - (1+i)\sinh z_n}{1 - (1+i)\cosh z_n} \ , z_1 = 1 - 2i \ , \lim_{n\to\infty} z_n = z$$

(12) $$\pi = 4\ \tan^{-1}\left(\frac{\tanh 1}{\tan 1}\right) - 4\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{Im(z_n)}{Re(z_n)}\right)$$

donde

$$z_{n+1} = 1 - \frac{i}{(n+1)(2n+3)}\left(1 - \frac{1}{z_n}\right)\ , z_1 = \frac{3-i}{3}$$

(13) $$\pi = 4\ \tan^{-1}\left(\frac{\tanh 1}{\tan 1}\right) + 4\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{Im(z_n)}{Re(z_n)}\right)$$

donde

$$z_{n+1} = 1 + \frac{i}{(n+1)(2n+3)}\left(1 - \frac{1}{z_n}\right)\ , z_1 = \frac{3+i}{3}$$

(14) $$\pi = 4\sum_{n=1}^{\infty}(-1)^{n-1}\, H_n\ Im((z-1)^n)$$

donde

$$z = 0.641026\ldots + i\ 0.523628\ldots$$

$$z_{n+1} = (1+i)^{z_n}\quad , z_1 = \frac{1+i}{2}\ , \lim_{n\to\infty} z_n = z$$

(15) $$\pi = 6\sum_{n=1}^{\infty}(-1)^{n-1}\, H_n\ Im((z-1)^n)$$

donde

$$z = 0.800814\ldots + i\ 0.416047\ldots$$

$$z_{n+1} = \left(1 + \frac{i}{\sqrt{3}}\right)^{z_n}\quad , z_1 = \frac{1+i}{2}\ , \lim_{n\to\infty} z_n = z$$

(16) $$\pi = 8\sum_{n=1}^{\infty}(-1)^{n-1}\, H_n\ Im((z-1)^n)$$

donde

$$z = 0.874399\ldots + i\ 0.339395\ldots$$

$$z_{n+1} = \left(1 + i\left(\sqrt{2} - 1\right)\right)^{z_n} \quad , z_1 = \frac{1+i}{2} \; , \lim_{n\to\infty} z_n = z$$

(17) $$\pi = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{y_n}{(1-x_n)^2 + y_n^2} - 4 \sum_{n=2}^{\infty} \tan^{-1}\left(\frac{y_n}{1+x_n}\right)$$
$$= 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{y_n}{1 + (x^2 + (1-x)^2)^n - 2x_n} - 4 \sum_{n=2}^{\infty} \tan^{-1}\left(\frac{y_n}{1+x_n}\right)$$

donde

$$x_{n+1} = -x\,x_n - (1-x)y_n \; , y_{n+1} = (1-x)x_n - x\,y_n \; , x_1 = -x, y_1 = 1-x$$

$$0 < x < 1$$

Ejemplo: $x = 1/3$,

$$\pi = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{y_n}{1 + (5/9)^n - 2x_n} - 4 \sum_{n=2}^{\infty} \tan^{-1}\left(\frac{y_n}{1+x_n}\right)$$

$$x_{n+1} = -\frac{1}{3}x_n - \frac{2}{3}y_n$$

$$y_{n+1} = \frac{2}{3}x_n - \frac{1}{3}y_n$$

$$x_1 = -\frac{1}{3} \; , \qquad y_1 = \frac{2}{3}$$

(18) $$\pi = 4\, \tan^{-1}\left(\frac{\sinh x}{\sin x}\right) - 8 \sum_{n=1}^{\infty} \frac{(2^{2n-1} - 1)B_n\, x^{2n}}{(2n)!} \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} \binom{2n-1}{2k}$$

$$0 < x < \pi/\sqrt{2}$$

(19) $$\pi = 24 \sum_{n=1}^{\infty} \frac{(2^{2n-1} - 1)B_n\, x^{2n}}{(2n)!} \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} \binom{2n-1}{2k}$$

donde $x = 1.280021 \ldots$, es solución de la ecuación no lineal: $\sinh x = \sqrt{3}\ \sin x$

(20) $$\pi = 16 \sum_{n=1}^{\infty} \frac{(2^{2n-1} - 1)B_n\, x^{2n}}{(2n)!} \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} \binom{2n-1}{2k}$$

donde $x = 1.614027 \ldots$, es solución de la ecuación no lineal: $\sinh x = \left(\sqrt{2} + 1\right) \sin x$

(21) $$\pi = 12 \sum_{n=1}^{\infty} \frac{(2^{2n-1}-1)B_n\ x^{2n}}{(2n)!} \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} \binom{2n-1}{2k}$$

donde $x = 1.954759\ldots$, es solución de la ecuación no lineal: $\sinh x = \left(2+\sqrt{3}\right) \sin x$

(22) $$\pi = 2 + \sum_{n=0}^{\infty} (n+1) \left(\frac{1}{2}\right)^{n+1} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \left(\frac{1}{2k+1} + \frac{1}{2k+3}\right)$$

(23) $$\pi = 2 + \frac{1}{3} \sum_{n=0}^{\infty} (n+1) \left(\frac{2}{3}\right)^{n+1} \sum_{k=0}^{[n/2]} \left(-\frac{1}{4}\right)^k \binom{n}{2k} \left(\frac{1}{2k+1} + \frac{1}{2k+3}\right)$$

(24) $$\pi = \sum_{n=1}^{\infty} \frac{2^{4n}(2^{4n-2}-1)B_{2n-1}}{(2n-1)(4n-2)!}\ Im(z^{4n-2})$$

donde

$$z = 0.791035\ldots + i\ 0.501872\ldots$$

z , es solución de la ecuación no lineal:

$$\cosh z - (1+i) \cos z = 0$$

(25) $$\pi = 8 \sum_{n=1}^{\infty} \frac{2^{4n}(2^{4n}-1)B_{2n}}{(2n)(4n)!}\ Im(z^{4n})$$

donde

$$z = 1.159277\ldots + i\ 0.482198\ldots$$

z , es solución de la ecuación no lineal:

$$\cosh z\ \cos z - \left(1 - i\left(\sqrt{2}-1\right)\right) = 0$$

(26) $$a\,\pi = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} (\tan(a\,\pi))^{2k+1} F(n, 2k+1; 2k+2; -\tan(a\,\pi)) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k+1} \binom{n+k-1}{k} (\tan(a\,\pi))^{k+1} F\left(1, \frac{k+1}{2}; \frac{k+3}{2}; -(\tan(a\,\pi))^2\right)$$

$$0 < a < \frac{1}{4}\,, \qquad n \in \mathbb{N}$$

(27) $$a\,\pi = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \frac{(\tan(a\,\pi))^{2k+1}}{(1+\tan(a\,\pi))^n} F\left(n, 1; 2k+2; \frac{\tan(a\,\pi)}{1+\tan(a\,\pi)}\right) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k+1} \binom{n+k-1}{k} \frac{(\tan(a\,\pi))^{k+1}}{1+(\tan(a\,\pi))^2} F\left(1,1; \frac{k+3}{2}; (\sin(a\,\pi))^2\right)$$

$$0 < a < \frac{1}{4}\,, \qquad n \in \mathbb{N}$$

(28) $$a\,\pi = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \left(\frac{\tan(a\,\pi)}{1+\tan(a\,\pi)}\right)^{2k+1} F\left(2k+1, 2k+2-n; 2k+2; \frac{\tan(a\,\pi)}{1+\tan(a\,\pi)}\right) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k+1} \binom{n+k-1}{k} (\sin(a\,\pi))^{k+1} F\left(\frac{k+1}{2}, \frac{k+1}{2}; \frac{k+3}{2}; (\sin(a\,\pi))^2\right)$$

$$0 < a < \frac{1}{4}\,, \qquad n \in \mathbb{N}$$

(29) $$\pi = 2\sqrt{3} \sum_{k=0}^{\infty} \frac{(-1)^k\, 3^{-k}}{2k+1} F\left(n, 2k+1; 2k+2; -\frac{1}{\sqrt{3}}\right) + 6 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k+1} \binom{n+k-1}{k} \left(\frac{1}{\sqrt{3}}\right)^{k+1} F\left(1, \frac{k+1}{2}; \frac{k+3}{2}; -\frac{1}{3}\right)$$

$$n \in \mathbb{N}$$

(30) $$\pi = \frac{2\left(\sqrt{3}\right)^{n+1}}{\left(1+\sqrt{3}\right)^n} \sum_{k=0}^{\infty} \frac{(-1)^k 3^{-k}}{2k+1} F\left(n, 1; 2k+2; \frac{1}{1+\sqrt{3}}\right) + \frac{9}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k+1} \binom{n+k-1}{k} \left(\frac{1}{\sqrt{3}}\right)^{k+1} F\left(1,1; \frac{k+3}{2}; \frac{1}{4}\right)$$

$$n \in \mathbb{N}$$

(31) $$\pi = 6 \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \left(\frac{1}{1+\sqrt{3}}\right)^{2k+1} F\left(2k+1, 2k+2-n; 2k+2; \frac{1}{1+\sqrt{3}}\right) + 6 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k+1} \binom{n+k-1}{k} \left(\frac{1}{2}\right)^{k+1} F\left(\frac{k+1}{2}, \frac{k+1}{2}; \frac{k+3}{2}; \frac{1}{4}\right)$$

$$n \in \mathbb{N}$$

(32) $$a\,\pi = \sum_{n=0}^{\infty} (-1)^n \binom{2n}{n} 2^{-2n} \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{n}{k} \left(1 - (1 - \sin(a\pi))^{k+\frac{1}{2}}\right)$$

$$0 < a < \frac{1}{2}$$

(33) $$a\,\pi = \sum_{n=0}^{\infty} (-1)^n \binom{2n}{n} 2^{-2n} I_n$$

donde

$$I_n = -\frac{2(\sin(a\pi))^n \sqrt{1 - \sin(a\pi)}}{2n+1} + \frac{2n}{2n+1}\, I_{n-1} \;, n \in \mathbb{N}$$

$$I_0 = 2\left(1 - \sqrt{1 - \sin(a\pi)}\right)$$

$$0 < a < \frac{1}{2}$$

(34) $$\pi = 8 \sum_{n=0}^{\infty} (-1)^n \binom{2n}{n} 2^{-2n} \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{n}{k} \left(1 - \left(1 - \frac{1}{\sqrt{2}}\right)^{k+\frac{1}{2}}\right)$$

(35) $$\pi = 12 \sum_{n=0}^{\infty} (-1)^n \binom{2n}{n} 2^{-2n} \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{n}{k} \left(1 - \left(\frac{1}{2}\right)^{k+\frac{1}{2}}\right)$$

(36) $$\pi = 16 \sum_{n=0}^{\infty} (-1)^n \binom{2n}{n} 2^{-2n} \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{n}{k} \left(1 - \left(1 - \frac{\sqrt{2-\sqrt{2}}}{2}\right)^{k+\frac{1}{2}}\right)$$

(37) $$a\,\pi = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{(\sin(a\pi))^{2k+1}}{2k+1}\, F\left(-n+\frac{1}{2}, k+\frac{1}{2}; k+\frac{3}{2}; (\sin(a\pi))^2\right)$$

$$0 < a < \frac{1}{2} \;, n \in \mathbb{N}$$

(38) $$\pi = 6 \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{(1/2)^{2k+1}}{2k+1}\, F\left(-n+\frac{1}{2}, k+\frac{1}{2}; k+\frac{3}{2}; \frac{1}{4}\right)$$

$$n \in \mathbb{N}$$

(39) $$\pi = 4\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k a^{-k-1}}{k+1}\sin\left(\frac{(k+1)\pi}{4}\right)$$

$$a > 1/\sqrt{2}$$

(40) $$\pi = 3\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k a^{-k-1}}{k+1}\sin\left(\frac{(k+1)\pi}{3}\right)$$

$$a > 1$$

(41) $$\pi = 6\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k a^{-k-1}}{k+1}\sin\left(\frac{(k+1)\pi}{6}\right)$$

$$a > 1/\sqrt{3}$$

(42) $$\pi = 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}a^{2n}}{n}\sin\left(\frac{n\pi}{2}\right) + 4\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\frac{(-1)^k a^{k+2m+1}}{k+2m+1}\sin\left(\frac{(k+2m+1)\pi}{4}\right)$$

$$0 < a < 0.66338\ldots$$

(43) $$\pi = \frac{3}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}a^{2n}}{n}\sin\left(\frac{2n\pi}{3}\right) + 3\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\frac{(-1)^k a^{k+2m+1}}{k+2m+1}\sin\left(\frac{(k+2m+1)\pi}{3}\right)$$

$$0 < a < 0.53568\ldots$$

(44) $$\frac{4}{(4n+1)}\left(\frac{2^{2n}}{\binom{2n}{n}}\right)^2\frac{1}{\pi} = 1 + \cfrac{2(8n+2)^{-2}}{1 + \cfrac{9(8n+2)^{-2}}{1 + \cfrac{25(8n+2)^{-2}}{1 + \cfrac{49(8n+2)^{-2}}{1 + \cdots}}}}$$

$$n = 0,1,2,3,\ldots$$

(45) $$\pi = 12\sum_{n=0}^{\infty}\left(-\frac{1}{64}\right)^n c_n\, I_n$$

donde

$$I_n = \int_0^{(\sqrt{6}-\sqrt{2})/4} \frac{x^{2n}(8+x^2)^{3n}}{(1-x^2)^{3n}}\, dx \quad , n = 0,1,2,3,\ldots$$

$$c_n = \sum_{k=0}^{n} \frac{(1/4)_{n-k}(1/4)_k(-1/12)_{n-k}(-1/12)_k}{(2/3)_{n-k}(2/3)_k(n-k)!\,k!}$$

(46) $$\pi = 12 \sum_{n=0}^{\infty} \frac{(-1)^n(1/2)_n(-1/6)_n}{(2/3)_n\, n!}\, I_n$$

donde

$$I_n = \int_0^{(\sqrt{6}-\sqrt{2})/4} \left(\frac{x}{4}\right)^{2n} \frac{(4-x^2)^{3n}}{(1-x^2)^{3n}} dx \quad , n = 0,1,2,3,\ldots$$

(47) $$\pi = 8 \sum_{n=0}^{\infty} \frac{(-1)^n(1/2)_n(-1/6)_n}{(2/3)_n\, n!}\, I_n$$

donde

$$I_n = \int_0^{\sqrt{2-\sqrt{2}}/2} \left(\frac{x}{4}\right)^{2n} \frac{(4-x^2)^{3n}}{(1-x^2)^{3n}} dx \quad , n = 0,1,2,3,\ldots$$

(48) $$\pi = 3 \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-6n}}{4n+1}\, F\left(-\frac{1}{2}, 2n+\frac{1}{2}; 2n+\frac{3}{2}; -\frac{1}{4}\right)$$

(49) $$\pi = 2\sqrt{2} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-4n}}{4n+1}\, F\left(-\frac{1}{2}, 2n+\frac{1}{2}; 2n+\frac{3}{2}; -\frac{1}{2}\right)$$

(50) $$\pi = \frac{3\sqrt{3}}{2} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(3/8)^{2n}}{4n+1}\, F\left(-\frac{1}{2}, 2n+\frac{1}{2}; 2n+\frac{3}{2}; -\frac{3}{4}\right)$$

(51) $$\pi = -12i \sum_{n=0}^{\infty} \int_{e^{i\pi/4}}^{e^{i\pi/3}} e^{-x}(1 - x\, e^{-x})^n \, dx$$

$$= -12i \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \int_{e^{i\pi/4}}^{e^{i\pi/3}} x^k e^{-(k+1)x} \, dx$$

(52) $$\pi = 24 \sum_{n=0}^{\infty} \frac{(1/3)_n}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{3k+1} Im(z^{3k+1})$$

donde

$$z = 1 + i\left(2\sqrt{2+\sqrt{3}} - 2 - \sqrt{3}\right)$$

(53) $$\pi = 24 \sum_{n=0}^{\infty} \frac{(1/3)_n}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{3k+1} \sin\left(\frac{(3k+1)\pi}{24}\right)$$

(54) $$\pi = 4 \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} (1-a)^{n-k} \sin\left(\frac{k\pi}{4}\right)$$

$$- 4 \sum_{n=1}^{\infty} \frac{a^n}{n} Im\left(\left(\frac{a\sqrt{2} + 1 - i}{(1+a^2)\sqrt{2} + 2a}\right)^n\right)$$

$$0 < a < 1$$

(55) $$\pi = \frac{4}{\alpha} \sqrt{\frac{1}{25} + \alpha^2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{5\alpha}\right)^{2n} I_n$$

donde $\alpha = 0.76903994 \ldots$, es solución de la ecuación no lineal: $\alpha - \sqrt{\frac{1}{25} + \alpha^2} \tan\alpha = 0$,

$$I_n = \int_0^{\alpha} (\sin x)^{2n} \, dx \ , n = 0,1,2,3, \ldots$$

$$I_n = -\frac{1}{2\alpha n} \sqrt{\frac{1}{25} + \alpha^2} \left(\frac{25\alpha^2}{1 + 50\alpha^2}\right)^n + \frac{2n-1}{2n} I_{n-1} \ , n \in \mathbb{N}$$

$$I_0 = \alpha$$

(56) $$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(a^{2n+1}-1)} - 4\sum_{n=3}^{\infty}\tan^{-1}(a^{-n})$$

donde

$$a = \frac{1}{3}\left(1+\left(19-3\sqrt{33}\right)^{1/3}+\left(19+3\sqrt{33}\right)^{1/3}\right) = 1.83928675\ldots$$

(57) $$\pi = 6\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(a^{2n+1}-1)} - 6\sum_{n=3}^{\infty}\tan^{-1}(a^{-n})$$

donde

$$a = \frac{1}{\sqrt{3}}\left(1+\left(1-3\sqrt{2}+3\sqrt{3}\right)^{1/3}+\left(1+3\sqrt{2}+3\sqrt{3}\right)^{1/3}\right) = 2.56088267\ldots$$

(58) $$\pi = 8\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(a^{2n+1}-1)} - 8\sum_{n=3}^{\infty}\tan^{-1}(a^{-n})$$

donde

$$a = \frac{1}{3\left(\sqrt{2}-1\right)}\left(1-\left(2-3\sqrt{2}\right)b^{-1/3}+b^{1/3}\right) = 3.25133913\ldots$$

$$b = -98+72\sqrt{2}+3\sqrt{3\left(744-526\sqrt{2}\right)}$$

(59) $$\pi = 12\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(a^{2n+1}-1)} - 12\sum_{n=3}^{\infty}\tan^{-1}(a^{-n})$$

donde

$$a = \frac{1}{3\left(2-\sqrt{3}\right)}\left(1-\left(3\sqrt{3}-7\right)b^{-1/3}+b^{1/3}\right) = 4.59216986\ldots$$

$$b = 361-207\sqrt{3}+3\sqrt{3\left(9554-5516\sqrt{3}\right)}$$

(60) $$\pi = 6\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sinh\left(\frac{3^{-n}}{\sqrt{3}}\right) - 6\sum_{n=1}^{\infty}\frac{1}{(2n+1)!}\tan^{-1}\left(\frac{3^{-n}}{\sqrt{3}}\right)$$

(61) $$\pi = 6\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sin\left(\frac{3^{-n}}{\sqrt{3}}\right) - 6\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\tan^{-1}\left(\frac{3^{-n}}{\sqrt{3}}\right)$$

(62) $$\pi = 6\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\tan\left(\frac{3^{-n}}{\sqrt{3}}\right) - 6\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)B_n}{(2n)!}\tan^{-1}\left(3^{-n}\sqrt{3}\right)$$

(63) $$\pi = 6\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sin^{-1}\left(\frac{3^{-n}}{\sqrt{3}}\right) - 6\sum_{n=1}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\tan^{-1}\left(\frac{3^{-n}}{\sqrt{3}}\right)$$

(64) $$\pi = 6\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\tan^{-1}(2^{-2n-1}) - 6\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}\sin^{-1}(2^{-2n-1})$$

(65) $$\pi = 8\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\left(\left(\sqrt{2}+1\right)^{2n+1}+\left(\sqrt{2}-1\right)^{2n+1}-2\right)} - 8\sum_{n=2}^{\infty} n\tan^{-1}\left(\left(\sqrt{2}-1\right)^n\right)$$

(66) $$\pi = 8\sum_{n=0}^{\infty}\frac{(-1)^n\left(\sqrt{2}-1\right)^{2n+1}\left(1+\left(\sqrt{2}-1\right)^{2n+1}\right)}{(2n+1)\left(1-\left(\sqrt{2}-1\right)^{2n+1}\right)^3} - 8\sum_{n=2}^{\infty} n^2\tan^{-1}\left(\left(\sqrt{2}-1\right)^n\right)$$

(67) $$\pi = 8\sum_{n=0}^{\infty}\frac{(-1)^n\left(\sqrt{2}-1\right)^{2n+1}\left(1+4\left(\sqrt{2}-1\right)^{2n+1}+\left(\sqrt{2}-1\right)^{4n+2}\right)}{(2n+1)\left(1-\left(\sqrt{2}-1\right)^{2n+1}\right)^4} - 8\sum_{n=2}^{\infty} n^3\tan^{-1}\left(\left(\sqrt{2}-1\right)^n\right)$$

(68) $$\pi = 12\sum_{k=0}^{\infty}(-1)^k\sum_{n=0}^{k}\frac{(-1)^n(2n+1)}{2k-2n+1}J_{2n+1}\left(\frac{3^{n-k}}{\sqrt{3}}\right)$$

(69) $$\pi = 8\sum_{k=0}^{\infty}(-1)^k\sum_{n=0}^{k}\frac{(-1)^n(2n+1)}{2k-2n+1}\left(J_{2n+1}(2^{2n-2k-1})+J_{2n+1}(3^{2n-2k-1})\right)$$

(70) $$\pi = 8\sum_{n=0}^{\infty}(-1)^n\binom{2n}{n}\sum_{k=[n/2]}^{n}\binom{n}{2k-n}\frac{(-1/4)^k\left(\sqrt{2}-1\right)^{2k+1}}{2k+1}$$

(71) $$\pi = 12\sum_{n=0}^{\infty}(-1)^n\binom{2n}{n}\sum_{k=[n/2]}^{n}\binom{n}{2k-n}\frac{(-1/4)^k\left(2-\sqrt{3}\right)^{2k+1}}{2k+1}$$

(72) $$\pi = 16\sum_{n=0}^{\infty}(-1)^n\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\sqrt{2}-1}{2}\right)^{n+k+1}\frac{Im(i^{n+k+1})}{n+k+1}$$

(73) $$\pi = 24\sum_{n=0}^{\infty}(-1)^n\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{2-\sqrt{3}}{2}\right)^{n+k+1}\frac{Im(i^{n+k+1})}{n+k+1}$$

(74) π

$$= 6\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(2n-2k+1)^{-(2k+1)}}{(2k+1)!}\, Im\left(\gamma\left(2k+1,(2n-2k+1)\left(\frac{1+i\sqrt{3}}{2}\right)\right)\right)$$

(75) $$\pi = 8\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(2n-2k+1)^{-(2k+1)}}{(2k+1)!}\, Im\left(\gamma\left(2k+1,(2n-2k+1)\left(\frac{1+i}{\sqrt{2}}\right)\right)\right)$$

(76) π

$$= 12\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(2n-2k+1)^{-(2k+1)}}{(2k+1)!}\, Im\left(\gamma\left(2k+1,(2n-2k+1)\left(\frac{\sqrt{3}+i}{2}\right)\right)\right)$$

(77) π

$$= 12\sum_{n=0}^{\infty}(-1)^n\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(n+k+1)}\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)^{n+k+1}\sin\left(\frac{3\pi(n+k+1)}{4}\right)$$

(78) π

$$= 16\sum_{n=0}^{\infty}(-1)^n\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(n+k+1)}\left(\frac{\sqrt{2}-1}{2}\right)^{n+k+1}\sin\left(\frac{3\pi(n+k+1)}{4}\right)$$

(79) π

$$= 24 \sum_{n=0}^{\infty} (-1)^n \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(n+k+1)} \left(\frac{3\sqrt{2}-\sqrt{6}}{12}\right)^{n+k+1} \sin\left(\frac{3\pi(n+k+1)}{4}\right)$$

(80) $$\pi = A \sum_{n=0}^{\infty} \frac{\left(\sqrt{2}-1\right)^{2n+1}}{2n+1} \left(\sin\left((2n+1)x\right) + (-1)^n \cos\left((2n+1)x\right)\right)$$

$$= A \sum_{n=0}^{\infty} \frac{\left(\sqrt{2}-1\right)^{2n+1}}{2n+1} \sum_{k=0}^{n} (-1)^k (\sin x)^{2k} (\cos x)^{2n-2k} \left(\binom{2n+1}{2k+1} \sin x + (-1)^n \binom{2n+1}{2k} \cos x\right)$$

Ejemplo1:

Para $A = 8, x = \pi/2$, y $A = 8, x = 0$, se tiene:

$$\pi = 8 \sum_{n=0}^{\infty} \frac{(-1)^n \left(\sqrt{2}-1\right)^{2n+1}}{2n+1}$$

Ejemplo2:

$$A = 12$$

$$\sin x = \sqrt{\frac{3}{2} - \frac{\sqrt{3}}{2} + \frac{1}{2}\sqrt{6\sqrt{2}-7}}$$

$$\cos x = \sqrt{\frac{3}{2} - \frac{\sqrt{3}}{2} - \frac{1}{2}\sqrt{6\sqrt{2}-7}}$$

$$\frac{\pi}{2} < x < \pi$$

Ejemplo3:

$$A = 16$$

$$\sin x = \frac{1}{2}\left(1+\sqrt{2}\right)\sqrt{5+4\sqrt{2}}\left(2\sqrt{\sqrt{2}-1}-\sqrt{2\left(\sqrt{2}-1\right)}-\sqrt{\frac{4\sqrt{2}-5}{7}}\right.$$
$$\left.+\sqrt{\frac{17293-12228\sqrt{2}-15084\sqrt{3+\sqrt{2}}+10666\sqrt{2\left(3+\sqrt{2}\right)}}{1451-1026\sqrt{2}}}\right)$$

$$\cos x = -\frac{1}{2}\left(1+\sqrt{2}\right)\sqrt{5+4\sqrt{2}}\left(-2\sqrt{\sqrt{2}-1}+\sqrt{2\left(\sqrt{2}-1\right)}+\sqrt{\frac{4\sqrt{2}-5}{7}}\right.$$
$$\left.+\sqrt{\frac{17293-12228\sqrt{2}-15084\sqrt{3+\sqrt{2}}+10666\sqrt{2\left(3+\sqrt{2}\right)}}{1451-1026\sqrt{2}}}\right)$$

$$\frac{\pi}{2} < x < \pi$$

Ejemplo4:

$$A = 24$$

$$\sin x = \frac{1}{26}\sqrt{11-6\sqrt{3}}\left(7+4\sqrt{3}\right)\left(\sqrt{39\left(12+6\sqrt{3}\right)}-2\sqrt{143+78\sqrt{3}}\right.$$
$$\left.+13\left(2\sqrt{2}-\sqrt{6}+r\right)\right)$$

$$\cos x = -\sqrt{11-6\sqrt{3}}\left(\frac{7}{2}+2\sqrt{3}\right)\left(-2\sqrt{2}+\sqrt{6}+2\sqrt{\frac{11+6\sqrt{3}}{13}}-\sqrt{\frac{3\left(11+6\sqrt{3}\right)}{13}}+r\right)$$

donde

$$r = \sqrt{\frac{6411723-3701810\sqrt{3}+464812\sqrt{66-36\sqrt{3}}-805078\sqrt{22-12\sqrt{3}}}{356952\sqrt{3}-618259}}$$

$$\frac{\pi}{2} < x < \pi$$

Ejemplo5:

$$A = 6$$

$$\sin x = \frac{-1 + \sqrt{10} + \sqrt{2\sqrt{10} - 5}}{2\sqrt{3}}$$

$$\cos x = \frac{-1 + \sqrt{10} - \sqrt{2\sqrt{10} - 5}}{2\sqrt{3}}$$

$$0 < x < \frac{\pi}{2}$$

(81) $$\pi = 6 \ln 2 - 12 \sum_{n=1}^{\infty} \frac{1}{n!} \int_{1}^{\sqrt{3}} \frac{(\ln x)^n}{1 + x^2} dx$$

(82) $$\pi = 6 \ln \frac{3}{2} + 12 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \int_{1}^{\sqrt{3}} \frac{(\ln x)^n}{1 + x^2} dx$$

(83) $$\pi = 4 - 9 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} E_n}{(2n)!} \int_{\sqrt{3}}^{(4+\sqrt{7})/3} \frac{(x^4 - 1)(\ln x)^{2n}}{x^2 \sqrt{46x^2 - 9(1 + x^4)}} dx$$

(84) $$\pi = 6 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} E_n}{(2n-1)!} \int_{1}^{\sqrt{3}} \frac{(\ln x)^{2n-1}}{x} \left(\frac{x^2 + 1}{x^2 - 1} \right) dx$$

(85) $$\pi = 12 \sum_{n=0}^{\infty} (-1)^n \int_{0}^{u} \frac{x^n}{1 + (\ln(1 + x))^2} dx \ , u = e^{2-\sqrt{3}} - 1$$

(86) $$\pi = 12 \sum_{n=0}^{\infty} \int_{0}^{u} \frac{x^n}{1 + (\ln(1 - x))^2} dx \ , u = 1 - e^{-2+\sqrt{3}}$$

(87) $$\pi = 4 \sum_{n=0}^{\infty} \int_{0}^{u} \frac{x^n}{1 + (\ln(1 - x))^2} dx \ , u = 1 - e^{-1}$$

(88) $$\pi = 12\sum_{n=0}^{\infty}(n+1)\int_0^u \frac{x^n(1-x)^2}{2-2x+x^2}dx \quad ,u = 1-\frac{1}{\sqrt{3}}$$

(89) $$\pi = 6-2\sqrt{3}+6\sum_{n=1}^{\infty}(n+1)\int_0^u \frac{x^n(1-x)^2}{2-2x+x^2}dx \quad ,u = 1-\frac{1}{\sqrt{3}}$$

(90) $$\pi = 6\sqrt{2}\ln\left(1+\sqrt{2}\right)-3\ln 2-6+3\sqrt{3} + 12\sum_{n=2}^{\infty}\frac{(-1)^{n-1}E_n}{(2n-2)!}\int_1^{\sqrt{3}}\frac{x(1+x^2)(\ln x)^{2n-2}}{6x^2-1-x^4}dx$$

(91) $$\pi = 6\sqrt{2}\ln\left(1+\sqrt{2}\right)-3\ln 3+6\sum_{n=2}^{\infty}\frac{(-1)^{n-1}E_n}{(2n-2)!}\int_1^{\sqrt{3}}\frac{(1+x^2)^2(\ln x)^{2n-2}}{x(6x^2-1-x^4)}dx$$

(92) $$\pi = 3\ln\left(\frac{5+4u}{5-4u}\right)+12\sum_{n=1}^{\infty}(-1)^n\binom{2n}{n}2^{-2n}\int_{-u}^{u}\frac{x^n}{5+4x}dx$$

$$u = \frac{1}{4}\sqrt{-263-160\sqrt{3}+8\sqrt{2321+1340\sqrt{3}}}$$

(93) $$\pi = 3\ln\left(\frac{5+4u}{5-4u}\right)+3\sum_{n=1}^{\infty}\left(\frac{5}{16}\right)^n\binom{2n}{n}c_n$$

$$c_n = \ln\left(\frac{5+4u}{5-4u}\right)+\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k}{k}\left(\left(1+\frac{4u}{5}\right)^k-\left(1-\frac{4u}{5}\right)^k\right)$$

$$u = \frac{1}{4}\sqrt{-263-160\sqrt{3}+8\sqrt{2321+1340\sqrt{3}}}$$

(94) $$\pi = 3\ln 2+12\sum_{n=1}^{\infty}\binom{2n}{n}2^{-2n}\int_{1/4}^{3/4}\frac{x^n}{5-4x}dx$$

(95) $$\pi = 3\sum_{n=0}^{\infty}\binom{2n}{n}\left(\frac{5}{16}\right)^n\left(\ln 2+\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k}{k}\left(\left(\frac{4}{5}\right)^k-\left(\frac{2}{5}\right)^k\right)\right)$$

(96) $$\pi = 16\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^{\sqrt{2}}\frac{x^{n+1}e^{-x}}{4+x^4}dx$$

(97) $$\pi = 24\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^{\sqrt{2/\sqrt{3}}}\frac{x^{n+1}e^{-x}}{4+x^4}dx$$

(98) $$\pi = 32\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^{\sqrt{2\sqrt{2}-2}}\frac{x^{n+1}e^{-x}}{4+x^4}dx$$

(99) $$\pi = 48\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^{\sqrt{3}-1}\frac{x^{n+1}e^{-x}}{4+x^4}dx$$

(100) $$\pi = 32\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_0^{\sqrt{2}}\frac{(\tanh(x/2))^{2n+1}}{4+x^4}dx$$

(101) $$\pi = 96\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_0^{\sqrt{3}-1}\frac{(\tanh(x/2))^{2n+1}}{4+x^4}dx$$

(102) $$\pi = 48\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^{\sqrt{3}-1}\frac{(\tan x)^{2n+1}}{4+x^4}dx$$

(103) $$\pi = 48\sum_{n=0}^{\infty}\frac{2^{-2n}}{2n+1}\binom{2n}{n}\int_0^{\sqrt{3}-1}\frac{(\sin x)^{2n+1}}{4+x^4}dx$$

(104) $$\pi = 48\sum_{n=0}^{\infty}\frac{(-1)^n 2^{-2n}}{2n+1}\binom{2n}{n}\int_0^{\sqrt{3}-1}\frac{(\sinh x)^{2n+1}}{4+x^4}dx$$

(105) $$\pi = 48\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\int_0^{u}\frac{x^{2n+1}\cosh x}{4+(\sinh x)^4}dx$$

$$u = \ln\left(\sqrt{3}-1+\sqrt{5-2\sqrt{3}}\right)$$

(106) $$\pi = 48\sum_{n=1}^{\infty}\frac{1}{n}\int_{0}^{\sqrt{3}-1}\frac{(1-e^{-x})^n}{4+x^4}\,dx$$

(107) $$\pi = 2+4\sum_{n=1}^{\infty}\frac{1}{(2n)!}\int_{1}^{u}\frac{(x^2-1)(\ln x)^{2n}}{\sqrt{4x-x^2-1}\,(x^2+1)^{3/2}}\,dx \ ,u=2+\sqrt{3}$$

(108) $$\pi = 4\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\int_{u}^{v}\frac{(x^2+1)(\ln x)^{2n+1}}{\sqrt{1+2x-x^2}\,(x^2-1)^{3/2}}\,dx$$

$$u=\frac{1+\sqrt{5}}{2}\ \ ,v=1+\sqrt{2}$$

(109) $$\pi = 6\left(\sqrt{6}-\sqrt{2}\right)+12\ln\left(\frac{2+\sqrt{2}}{2+\sqrt{6}}\right)$$
$$+12\sum_{n=1}^{\infty}\binom{4n}{2n}2^{-4n}\sum_{k=0}^{2n-1}\frac{(-1)^k}{2k+3}\binom{2n-1}{k}\left(\left(\frac{1}{2}\right)^{k+\frac{3}{2}}-\left(\frac{3}{2}\right)^{k+\frac{3}{2}}\right)$$

(110) $$\pi = 4\sqrt{4+2\sqrt{2}}-4\sqrt{4-2\sqrt{2}}-8\ln\left(\frac{1+\sqrt{1+\frac{1}{\sqrt{2}}}}{1+\sqrt{1-\frac{1}{\sqrt{2}}}}\right)$$
$$+8\sum_{n=1}^{\infty}\binom{4n}{2n}2^{-4n}\sum_{k=0}^{2n-1}\frac{(-1)^k}{2k+3}\binom{2n-1}{k}\left(\left(1-\frac{1}{\sqrt{2}}\right)^{k+\frac{3}{2}}-\left(1+\frac{1}{\sqrt{2}}\right)^{k+\frac{3}{2}}\right)$$

(111) $$\pi = 12\sqrt{4-\sqrt{2}+\sqrt{6}}-12\sqrt{4+\sqrt{2}-\sqrt{6}}+24\ln\left(\frac{2+\sqrt{4+\sqrt{2}-\sqrt{6}}}{2+\sqrt{4-\sqrt{2}+\sqrt{6}}}\right)$$
$$+24\sum_{n=1}^{\infty}\binom{4n}{2n}2^{-4n}\sum_{k=0}^{2n-1}\frac{(-1)^k}{2k+3}\binom{2n-1}{k}\left(\left(1-\frac{\sqrt{6}-\sqrt{2}}{4}\right)^{k+\frac{3}{2}}\right.$$
$$\left.-\left(1+\frac{\sqrt{6}-\sqrt{2}}{4}\right)^{k+\frac{3}{2}}\right)$$

(112) $$\pi = 3\sum_{n=0}^{\infty}\binom{4n+2}{2n+1}2^{-4n}\sum_{k=0}^{2n}\frac{(-1)^k}{2k+3}\binom{2n}{k}\left(\left(\frac{3}{2}\right)^{k+\frac{3}{2}}-\left(\frac{1}{2}\right)^{k+\frac{3}{2}}\right)$$

(113) $$\pi = 2\sum_{n=0}^{\infty}\binom{4n+2}{2n+1}2^{-4n}\sum_{k=0}^{2n}\frac{(-1)^k}{2k+3}\binom{2n}{k}\left(\left(1+\frac{1}{\sqrt{2}}\right)^{k+\frac{3}{2}}-\left(1-\frac{1}{\sqrt{2}}\right)^{k+\frac{3}{2}}\right)$$

(114) $$\pi = 6\sum_{n=0}^{\infty}\binom{4n+2}{2n+1}2^{-4n}\sum_{k=0}^{2n}\frac{(-1)^k}{2k+3}\binom{2n}{k}\left(\left(1+\frac{\sqrt{6}-\sqrt{2}}{4}\right)^{k+\frac{3}{2}}-\left(1-\frac{\sqrt{6}-\sqrt{2}}{4}\right)^{k+\frac{3}{2}}\right)$$

(115) $$\pi = 4\sum_{n=0}^{\infty}\frac{2^{-2n}}{(2n+1)!}\int_{1}^{u}\frac{(\ln x)^{2n+1}}{x^2-1}dx = 8\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\int_{0}^{v}\frac{x^{2n+1}}{\sinh(2x)}dx$$

$$u = 3+2\sqrt{2}\ , v = \ln\left(1+\sqrt{2}\right)$$

(116) $$\pi = 24\sum_{n=0}^{\infty}\frac{2^{-2n}}{(2n+1)!}\int_{u^2}^{v^2}\frac{(\ln x)^{2n+1}}{x^2-1}dx = 48\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\int_{\ln u}^{\ln v}\frac{x^{2n+1}}{\sinh(2x)}dx$$

$$u = 2-\sqrt{3}+2\sqrt{2-\sqrt{3}}\ , v = \sqrt{2}-1+\sqrt{4-2\sqrt{2}}$$

(117) $$\pi = 16\sum_{n=0}^{\infty}(2n+1)\int_{0}^{1}\frac{J_{2n+1}(x)}{1+x^4}dx$$

(118) $$\pi = 96\sum_{n=1}^{\infty}n^2\int_{0}^{1}\frac{J_{2n}(x)}{1+x^6}dx$$

(119) $$\pi = 16\sum_{n=0}^{\infty}(2n+1)\int_{0}^{1}\sqrt{1-x^4}\ J_{2n+1}(x)\,dx$$

(120) $$\pi = 96\sum_{n=1}^{\infty}n^2\int_{0}^{1}\sqrt{1-x^6}\ J_{2n}(x)\,dx$$

(121) $$\pi = 12m^2 \sum_{n=0}^{\infty} \binom{n+m}{n} \int_u^v \frac{x^n(1-x)^{2m}}{1+m^2(1-x)^{2m}} dx$$

$$m \in \mathbb{N}, u = 1-\left(\frac{1}{m}\right)^{1/m} \quad , v = 1-\left(\frac{1}{m\sqrt{3}}\right)^{1/m}$$

(122) $$\pi = 6\ln\frac{3}{2} + 12\sum_{n=1}^{\infty}\frac{1}{n!}\int_1^{\sqrt{3}} \frac{(\ln x)^n}{x(1+x^2)} dx$$

(123) $$\pi = 6\ln 2 + 12\sum_{n=1}^{\infty}\frac{1}{n!}\int_{1/\sqrt{3}}^{1} \frac{(\ln x)^n}{x(1+x^2)} dx$$

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