A Definite Integral

Abstract

In this paper we give some formulas associated with
an integral hidden in Gradshteyn and Ryzhik.

Sobre La Integral $ln\frac{2}{\pi}=\int_0^1\frac{1-x}{1+x}\frac{dx}{lnx}$, La que aparece en Gradshteyn y Ryzhik

Edgar Valdebenito

Julio 2008

Resumen

Se muestran algunas integrales y fórmulas relacionadas con la integral: $ln\left(\frac{2}{\pi}\right)=\int_0^1\frac{1-x}{1+x}\frac{dx}{ln(x)}$, la que aparece en: I.S. Gradshteyn y I.M. Ryzhik , Table of Integrals , Series , and Products , Seventh Edition (Alan Jeffrey , Daniel Zwillinger – Academic Press , 2007 , Pag. 545 , 4.267 Nº1).

1. Fórmulas - Integrales

$$ln\left(\frac{2}{\pi}\right)=\int_0^1\frac{1-x}{1+x}\frac{dx}{ln(x)}\Leftrightarrow ln\left(\frac{\pi}{2}\right)=-\int_0^1\frac{1-x}{1+x}\frac{dx}{ln(x)} \qquad (1)$$

$$ln\left(\frac{\pi}{2}\right)=\int_{0}^{\infty}\frac{1-e^{-x}}{1+e^{-x}}\frac{e^{-x}}{x}dx \tag{2}$$

$$ln\left(\frac{\pi}{2}\right)=\int_{0}^{\infty}\frac{e^{-x}}{x}th\left(\frac{x}{2}\right)dx \tag{3}$$

$$ln\left(\frac{\pi}{2}\right)=\int_{0}^{\infty}\frac{e^{-2x}}{x}th\left(x\right)dx \tag{4}$$

$$ln\left(\frac{\pi}{2}\right)=-\int_{0}^{1}\frac{x}{\left(2-x\right)ln\left(1-x\right)}dx \tag{5}$$

$$ln\left(\frac{\pi}{2}\right)=-\int_{1}^{2}\frac{2-x}{x}\frac{dx}{ln\left(x-1\right)} \tag{6}$$

$$ln\left(\frac{\pi}{2}\right)=\int_{1}^{\infty}\frac{x-1}{x+1}\frac{dx}{x^{2}\,ln\left(x\right)} \tag{7}$$

$$ln\left(\frac{\pi}{2}\right)=\int_{-\infty}^{\infty}e^{-2e^{x}}th\left(e^{x}\right)dx \tag{8}$$

$$ln\left(\frac{\pi}{2}\right)=\int_{-\infty}^{\infty}\frac{1-e^{-2e^{x}}}{1+e^{2e^{x}}}dx \tag{9}$$

$$ln\left(\frac{\pi}{2}\right)=2\int_{0}^{1}\frac{x}{\left(1+x\right)^{2}ln\left(\frac{1+x}{1-x}\right)}dx \tag{10}$$

$$ln\left(\frac{\pi}{2}\right)=\int_{1}^{\infty}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)\frac{dx}{x\sqrt{x}\,ln\left(x\right)} \tag{11}$$

$$ln\left(\frac{\pi}{2}\right)=-\int_{0}^{1}\left(\frac{1-x^{n}}{1+x^{n}}\right)\frac{x^{n-1}}{ln(x)}dx \quad ,n\in\mathbb{N} \tag{12}$$

$$ln\left(\frac{\pi}{2}\right)=-\frac{1}{2a}\int_{-a}^{a}\left(\frac{a-x}{3a+x}\right)\frac{dx}{ln\left(\frac{x+a}{2a}\right)} \quad ,a>0 \tag{13}$$

$$ln\left(\frac{\pi}{2}\right)=-\int_{0}^{\pi/4}\frac{\left(1-tan(\theta)\right)\left(1+\left(tan(\theta)\right)^{2}\right)}{\left(1+tan(\theta)\right)ln\left(tan(\theta)\right)}d\theta \tag{14}$$

$$ln\left(\frac{\pi}{2}\right)=-\int_{0}^{\pi/2}\left(tan\left(\frac{\theta}{2}\right)\right)^{2}\frac{sen(\theta)}{ln\left(cos(\theta)\right)}d\theta \tag{15}$$

$$ln\left(\frac{\pi}{2}\right)=-2\int_{0}^{\pi/4}\left(tan(\theta)\right)^{2}\frac{sen(2\theta)}{ln\left(cos(2\theta)\right)}d\theta \tag{16}$$

$$ln\left(\frac{\pi}{2}\right)=-\int_{0}^{ln\left(1+\sqrt{2}\right)}\left(\frac{1-sh(x)}{1+sh(x)}\right)\frac{ch(x)}{ln\left(sh(x)\right)}dx \tag{17}$$

$$ln\left(\frac{\pi}{2}\right)=\int_{0}^{a}\frac{e^{-x}}{x}th\left(\frac{x}{2}\right)dx+Ie(a)-2\sum_{n=1}^{\infty}(-1)^{n-1}\,Ie\left((n+1)a\right) \tag{18}$$

$$a>0$$

$$ln\left(\frac{\pi}{2}\right)=\int_{0}^{a}\frac{e^{-2x}}{x}th(x)dx+Ie(2a)-2\sum_{n=1}^{\infty}(-1)^{n-1}\,Ie\left(2(n+1)a\right) \tag{19}$$

$$a>0$$

En (18) y (19) :

$$Ie(y) = \int_{y}^{\infty} \frac{e^{-x}}{x} dx \quad , y > 0 \tag{20}$$

$$\pi = \lim_{m \to \infty} 2\left(\frac{m+1}{m+2}\right)^{m+1} \prod_{n=0}^{2m} \left(\frac{n+3}{n+1}\right)^{(-1)^n (n+1)} \tag{21}$$

$$ln\left(\frac{\pi}{2}\right) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{e^{-2k} th(k)}{k} - 4\int_{0}^{\infty} \frac{\left(sen(x)\right)^2}{e^{2\pi x} - 1} dx \tag{22}$$

$$ln\left(\frac{\pi}{2}\right) = \sum_{n=0}^{\infty} e^{-2n} \int_{0}^{1} e^{-2x} \frac{th(x+n)}{x+n} dx \tag{23}$$

$$ln\left(\frac{\pi}{2}\right) = 2\sum_{n=0}^{\infty} (-1)^n \, th^{-1}\left(\frac{1}{2n+3}\right) \tag{24}$$

$$\frac{\pi}{2} = \prod_{n=0}^{\infty} \left(\frac{n+2}{n+1}\right)^{(-1)^n} \tag{25}$$

$$\pi = 2\prod_{n=0}^{\infty} \left(\prod_{k=0}^{n} \left(\frac{k+2}{k+1}\right)^{(-1)^k a^{n-k} \binom{n}{k}} \right)^{\frac{1}{(1+a)^{n+1}}} \quad , a > 0 \tag{26}$$

En (26) con $a = 1$, se tiene:

$$\pi = 2\prod_{n=0}^{\infty} \left(\prod_{k=0}^{n} \left(\frac{k+2}{k+1}\right)^{(-1)^k \binom{n}{k}} \right)^{\frac{1}{2^{n+1}}} \tag{27}$$

$$ln\left(\frac{\pi}{2}\right) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)^{2n+1}} \tag{28}$$

$$ln\left(\frac{\pi}{2}\right) = 2\sum_{n=0}^{\infty} \frac{(-1)^n \left(1 - \beta(2n+1)\right)}{2n+1} \tag{29}$$

$$ln\left(\frac{\pi}{2}\right)=2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\beta(2n+1)}{2n+1} \tag{30}$$

En (29) y (30) $\beta(s)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^s}$ $,s>0$ es la función beta de Dirichlet.

$$ln\left(\frac{\pi}{2}\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}E_n\,\pi^{2n+1}}{2^{2n+1}(2n+1)!} \tag{31}$$

En (31) E_n son los números de Euler: $E_n=\{1,5,61,1385,50521,...\}$.

$$\begin{aligned} ln\left(\frac{\pi}{2}\right)&=\sum_{n=1}^{\infty}\int_0^1\left(\frac{x+n-1}{x+n+1}\right)\frac{dx}{(x+n)^2\,ln(x+n)}= \\ &=\int_0^1\sum_{n=1}^{\infty}\left(\frac{x+n-1}{x+n+1}\right)\frac{1}{(x+n)^2\,ln(x+n)}dx \end{aligned} \tag{32}$$

$$ln\left(\frac{\pi}{2}\right)=\lim_{n\to\infty}\left(\frac{1}{2n}+\frac{1}{n}\sum_{k=1}^{n-1}\frac{n-k}{(n+k)ln(n/k)}\right) \tag{33}$$

Referencias

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