A POLYNOMIAL RECURSION FOR PRIME CONSTELLATIONS

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ABSTRACT. An algorithm for recursively generating the sequence of solutions of a prime constellation is described. The algorithm is based on a polynomial equation formed from the first n elements of the constellation. A root of this equation is the next element of the sequence.

1. Introduction

Hypothesis H is one of the few mathematics conjectures that is distinguished by having its own Wikipedia page. The hypothesis, proposed independently by Schinzel-Sierpinski [1] and Bateman-Horn [2], describes a pattern of integers and then hypothesizes that there is an instance of the pattern such that all the integers in the pattern are prime numbers. It is a small step to conjecture that there are an infinite number of such occurrences.

The twin prime pattern, $n, n+2$, is one of the forms characterized Hypothesis H but the hypothesis also subsumes the conjectures of de Polignac [3], Bunyakovskii [4], Hardy-Littlewood [5], Dickson [6], Shanks [7], and many others regarding the infinitude and density of patterns of primes.

Hypothesis H. Let m be a positive integer and let $F = \{f_1(x), f_2(x), \ldots, f_m(x)\}$ be a set of irreducible polynomials with integral coefficients and positive leading coefficients such that there is not a prime p which divides the product

$$f_1(n) \cdot f_2(n) \cdot \ldots \cdot f_i(n) = \prod_{i=1}^{m} f_i(n) \qquad (1)$$

for every integer n. Then there exists an integer q such that $f_1(q), f_2(q), \ldots, f_m(q)$ are all prime numbers.

A sequence of functions F which satisfies Hypothesis H is traditionally called a *prime constellation*. A value q such that $f_1(q), f_2(q), \ldots, f_m(q)$ are all prime numbers is called a *solution of* F while F is said to be *solved by* q. Table 1 lists some familiar examples of prime constellations.

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Familiar Name	**Pattern**
Twin Primes	$\{x, x+2\}$
Sophie Germain Primes	$\{x, 2x+1\}$
Shanks Primes	$\{x^4 + a\}$
Hardy-Littlewood Primes	$\{ax^2 + bx + c\}$
Dickson Chains	$\{a_i x + b_i\}$
Cunningham Chains	$\{2^{i-1}x + (2^{i-1} - 1)\}$

Table 1. Examples of Prime Constellations

Given the first n solutions of a prime constellation we describe a polynomial one of whose roots is the next solution in this sequence. The polynomial can be regarded as a generalization of Rowland [8] which is, in turn, based on the formula for generating the next prime of Gandhi [9]. See also Golomb [10] and [11], Vanden Eynden [12], and Ellis [13]. An interpretation of the recursion is that the first n solutions of an instance of Hypothesis H algebraically encode the $(n+1)^{\text{st}}$ solution.

2. Generation of Prime Constellations

The recursion for prime constellation generation is based on the following primality test:

Lemma 1. *Let*

$$Q_d(x) = \sum_{k=1}^{d-1} \gcd(x, x-k) - 1 = \sum_{i=1}^{d-1} \gcd(i, x-i) - 1. \tag{2}$$

p is prime if and only if $Q_p(p) = 0$.

Let $F = \{f_1(x), f_2(x), \ldots, f_m(x)\}$ be a prime constellation and let p be a solution of F. Set

$$\mathcal{Q}_{F,p}(x) = \sum_{i=1}^{m} Q_{f_i(p)}(f_i(x)). \tag{3}$$

As an example of a $\mathcal{Q}_{F,p}(x)$, take $F = \{x, x+2, x+6\}$. This prime constellation is solved by $n = 5$, viz., $(5, 7, 11)$. In this case,

$$\mathcal{Q}_{F,5}(x) = Q_5(x) + Q_7(x+2) + Q_{11}(x+6).$$

Recursion. *Let p be solution of the prime constellation F so that*

$$\mathcal{Q}_{F,p}(p) = 0.$$

If q is the next integer greater than p such that $\mathcal{Q}_{F,p}(q) = 0$, then q is a solution of the prime constellation F.

Example. The sequence of prime numbers

If $F = \{x\}$, then

$$\mathcal{Q}_{F,p}(x) = Q_p(x).$$

According to the above recursion, if p is the i^{th} prime and q is the next larger root of $Q_p(x)$ beyond p, then q is the $i + 1^{st}$ prime.

It is straight-forward to show that this recursion yields the sequence of primes using Bertrand's Postulate ([14], [15], [16], [17]) that guarantees there is always a prime between n and $2n$.

Example. The sequence of twin primes

If $F = \{x, x + 2\}$, then

$$\mathcal{Q}_{F,p}(x) = Q_p(x) + Q_{p+2}(x + 2).$$

According to the above conjecture, if $(p, p + 2)$ is a twin prime and q is the next larger root of $\mathcal{Q}_{F,p}(x)$ beyond p, then $(q, q + 2)$ is a twin prime.

3. Continuations

A continuous rendering of $\mathcal{Q}_{F,p}(x)$ permits existing equation-solving methods to be used in finding its roots.

As one possibility, take

$$P_d(x) = \prod_{\substack{1\le n<d \\ n\nmid d}} \sin^2\left(\frac{\pi(x-n)}{d}\right).$$

Then, $P_d(x)$ is zero if and only if $\gcd(x, d) = 1$. If we set

$$\widetilde{Q}_d(x) = \sum_{k=1}^{d-1} P_k(x),$$

then $\widetilde{Q}_d(x)$ is zero if and only if $Q_d(x)$ is zero so $\widetilde{Q}_d(x)$ can be used in Equation 3 as well as $Q_d(x)$. Since $\widetilde{Q}_d(x)$ is continuous and periodic a next larger is guaranteed to exist.

As a second possibility, Slavin [18] has shown that for odd n

$$\gcd(n, m) = \log_2 \prod_{k=0}^{n-1} (1 + e^{-2i\pi km/n}) = n + \log_2\left(\left(\prod_{k=1}^{(n-1)/2} \cos\frac{km\pi}{n}\right)^2\right).$$

When both arguments of gcd in Equation 3 are even, Slavin's formula produces a negative infinity so it can also be used to find roots of $\mathcal{Q}_{F,p}(x)$.

4. The Dual

The recursion states that given solution p for a prime constellation F, the next element in the sequence of solutions is obtained by finding the next larger root of $\mathcal{Q}_{F,p}$. One can also formulate this recursion using the divisors of the integers between 1 and p rather than the non-divisors. Since the number of divisors grows slightly more quickly than the number of non-divisors, this may yield computational efficiency by reducing the complexity of $\mathcal{Q}_{F,p}$.

To take this dual approach, we set

$$P_d(x) = \prod_{\substack{1 \le n < d \\ n|d}} (x \bmod d - n)^2$$

and

$$Q_d(x) = \prod_{k=1}^{d-1} P_k(x).$$

To generate a sequence of prime constellation solutions using this formulation, we seek non-zero values of a product over the constellation functions rather than a zero value over a sum. The difficulty of seeking a non-zero value as compared to seeking a root may, of course, offset the reduction in complexity of the function being analyzed.

5. The Computation

The next larger root of $\mathcal{Q}_{F,p}(x)$ is readily computed and easily checked as the next solution F after p. The following Mathematica routine computes the next n sequence elements satisfying *constellation* after the solution *start*:

```
Sieve[constellation_, start_, n_] := Module[{f, i, j, q = start, l},
  For[i = 1, i <= n, i++,
    f[x_] :=
     Sum[Q[constellation[[i]][x], constellation[[i]][q]],
                                    {i, 1, Length[constellation]}];
    q = NextZero[f, q];
    l = {};
    For[j = 1, j <= Length[constellation], j++,
     p = pattern[[j]][q];
     AppendTo[l, {p, PrimeQ[p]}];
     ];
    Print[{q, l}];
    ];
  ]
```

Table 2 below lists some prime constellations for which sequences of solutions have been generated using this routine. The starting value Table 2 is a value which when substituted into the pattern yields a prime sequence satisfying the pattern. Thus, for example, when looking for Shank's primes of the form $n^2 + 1$ a starting value could be 4. Tables 3 and 5 lists some other types of prime sequences to which the routine has been applied.

Familiar Name	**Pattern**	**Start**
Primes	`{#&}`	5
Twin Primes	`{#&,#+2&}`	3
Cousin Primes	`{#&,#+4&}`	3
Prime Constellation	`{#&,#+2&,#+6&}`	5
Sophie Germain Primes	`{#&,2#+1&}`	5
Gaussian Primes	`{#&,4#+3&}`	5
Cunningham Chain	`{#&,2#+1&,4#+3&}`	5
Dickson Chain	`{#&,2#+1&,3#+4&}`	5
Star Primes	`{6#(#-1)+1&}`	2
Shanks Primes	`{#^2+1&}`	4
Shanks Twins	`{(#-1)^2+1 &,(#+1)^2+1 &}`	3
Shanks Quads	`{(#-1)^2+1 &,(#+1)^2+1 &}`	4
Hardy-Littlewood Primes	`{#^4+#+1&}`	3
Safe Primes	`{#&,(#-1)/2&}`	11
Centered Heptagonal Primes	`{(7#^2-7#+2)/2&}`	4
Centered Square Primes	`{#^2+(#+1)^2&},3`	4
Centered Triangular Primes	`{(3#^2+3#+2)/2&}`	3
Centered Decagonal Primes	`{5(#^2-#)+1&}`	2
Pythagorean Primes	`{4#+1&}`	0
Prime Quadruplets	`{#&,#+2&,#+6&,#+8&}`	3
Sexy Primes	`{#&, #+6&}`	5

Table 2. Hypothesis H Constellations

Familiar Name	**Pattern**	**Start**
Thabit Primes	{3*2^#-1&}	3
Wagstaff Primes	{(2^#+1)/3&}	5
Proth Primes	{2^#+1&},3	4
Kynea Primes	{(2^#+1)^2-2&}	2
Mersenne Primes	{2^#-1&}	1
Double Mersenne Primes	{2^(2^#-1)-1&}	2
Mersenne Prime Exponents	{#&,2^#-1&}	2
Carol Primes	{(2^#-1)^2-2&}	2
Cullen Primes	{#(2^#)+1&}	1
Fermat Primes	{2^(2^#)+1&}	0
Generalized Fermat Primes Base 10	{10^#+1&}	0
Factorial Primes	{ # + 1 &} or { # - 1 &}	0

Table 3. Other Single-Variable Prime Sequences

Familiar Name	**Pattern**	**Start**
Leyland Primes	$\{ \#1^{\#2} + \#2^{\#1}\ \&\ \}$	0
Pierpont Primes	$\{ 2^{\#1}\ 3^{\#2}\ \&\ \}$	0
Solinas Primes	$\{ 2^{\#1} \pm 2^{\#2} \pm 1\ \&\}$	0
Primes of Binary Quadratic Form	$\{ \#1^2 + \#1\ \#2 + 2\ \#2^2\ \&\}$	0
Quartan Primes	$\{ \#1^4 + \#2^4\ \&\}$	0

Table 4. Two-Variable Prime Sequences

The following Mathematica routine implements the dual.

```
Sieve[constellation_, start_, n_] := Module[{f, i, j, q = start, l},
  For[i = 1, i <= n, i++,
    f[x_] :=
     Sum[Q[constellation[[i]][x], constellation[[i]][q]],
                                  {i, 1, Length[constellation]}];
    q = NextNonZero[f, q];
    l = {};
    For[j = 1, j <= Length[constellation], j++,
     p = pattern[[j]][q];
     AppendTo[l, {p, PrimeQ[p]}];
     ];
    Print[{q, l}];
    ];
  ]
```

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