The Theory of n-Scales

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Abstract

We provide a theory of n-scales previously called as n dimensional time scales. In previous approaches to the theory of time scales, multi-dimensional scales were taken as product space of two time scales [4, 5]. n-scales make the mathematical structure more flexible and appropriate to real world applications in physics and related fields. Here we define an n-scale as an arbitrary closed subset of $\mathbb{R}^n$. Modified forward and backward jump operators, Δ-derivatives and multiple integrals on n-scales are defined.

1 Introduction

A time-scale is an arbitrary closed subset of $\mathbb{R}$ in the usual topology [1]. The calculus on time-scales has been introduced by B. Aulbach and S. Hilger in order to unify discrete and continuous analysis [1]. For example the set $[0, 1]$ is a time-scale where as $(0, 1)$ is not. The sets $\mathbb{Z}$, $\mathbb{N}$ and $[0, 1] \cup \mathbb{Z}$ are also examples of time-scales.

Previously, Δ-derivatives and Δ-integrals were defined on the time-scale structure. A Δ-derivative is the usual derivative on continuous subsets of a time-scale however it departs from the usual definition on discrete subsets. On discrete subsets it becomes the forward difference operator divided by the distance between two coordinate values. Readers may find more on time-scales in the references [1, 2, 3, 6].

2 n-Scales

Previous studies in the literature considered multi-dimensional time scales as product space of two or many time scales [4, 5, 7]. However this is a rather severe restriction. In this section we define multi-dimensional time scales and name them as n-scales. This is mainly because when multiple dimensions are introduced, other dimensions may denote space rather than time.

Just as a 1-scale (*i.e.* a time scale) is a nonempty arbitrary closed subset of $\mathbb{R}$, we define an n-scale as a nonempty arbitrary closed subset of $\mathbb{R}^n$. Though it may not be problematic for continuous parts of n-scales, defining neighborhood

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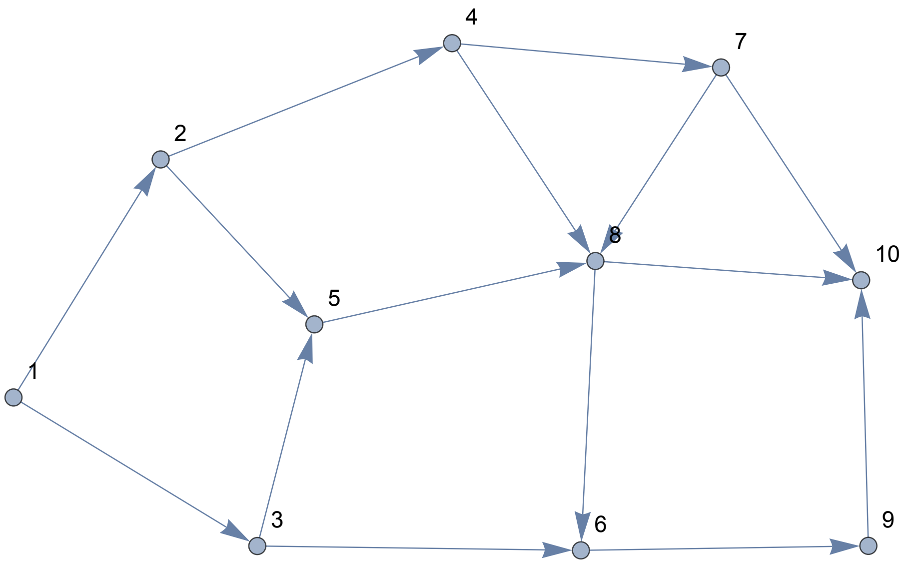


Figure 1: A 2-scale consisting of ten points. The arrows denote the neighborhood structure.

relations for discrete parts is crucial. When an n-scale is taken as a direct product of two time scales [4], the very nature of direct product gives neighborhood relations and the problem does not appear. However when generalizing the time scale structure to n-scales one must specify neighborhood relations in the form of a directed graph. This graph can connect discrete points to continuous parts of an n-scale.

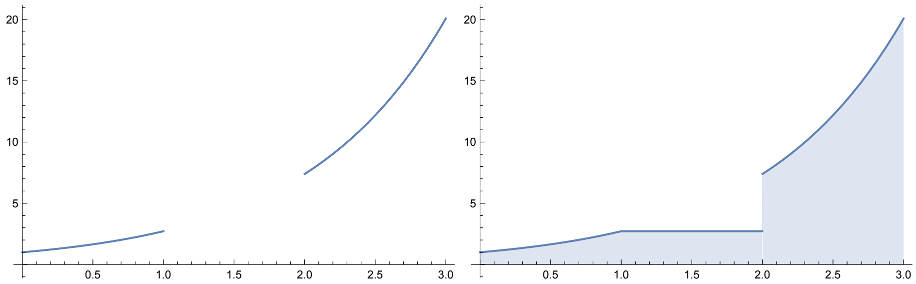
Definition 2.1. *An n-scale $(\mathbb{T}^n, G)$ is a tuple where $\mathbb{T}^n$ is a nonempty arbitrary closed subset of $\mathbb{R}^n$ and G is a directed graph that does not contain cyclic edge appointments. The graph should cover the measure zero subsets of $\mathbb{T}^n$ and connect them with non-zero measure subsets of $\mathbb{T}^n$. G indicates the neighborhood structure of measure zero subsets of $\mathbb{T}^n$.*

In Figure 1 a simplistic 2-scale is drawn. This definition is much more flexible. For example the discrete meshes used to numerically solve partial differential equations are seen to be n-scales. The difference in n-scales is that it unifies discrete and continuous structures.

Definition 2.2. *The graph structure lets one know the neighbors of a point p that can be reached by following directions indicated by the graph. The neighbors can be labelled with numbers such as $1, 2, \ldots n_p$. The directed forward jump operator σ_i yields the i^{th} neighbor of p. In a similar manner, ρ_i denotes the i^{th} neighbor of p in the reverse direction that is found in the graph.*

Theorem 2.1. *A graph G can be chosen for an n-scale such that one can find one point in every cell of G where all of its neighbors can be reached via σ operator.*

Proof. The proof is algorithmic. We begin with an undirected graph and will make it directed with the desired property.



(a) The plot of e^x on $\mathbb{T}$. (b) The integral of e^x on $\mathbb{T}$.

Figure 2

1. Choose a random point (p) and connect it with one of its neighbors (say, q) via a forward directed edge.
2. Choose a cell that includes the point p.
3. Connect p with its other neighbor in the cell with forward directed edge and repeat this same procedure until q is reached.
4. Choose one edge in the previous cell label the point where edge emerges as p' and reaches to q'.
5. Repeat commands 2–5 until all the graph becomes directed.

This guarantees the existence of at least one G, however there may be other graphs that may be suitable for a specific application.

□

Now, let us move on to definition of directed Δ-derivatives. Suppose that we want to calculate a derivative of a function f at a point p. There are two cases to consider: 1) p lies in the interior of a continuous region, 2) p is an element of a measure-zero set.

Definition 2.3. *In the first case, the definition directed Δ-derivative is usual partial derivative. However in the second case, we need the graph structure of the n-scale. It allows one to navigate through the neighboring points. The derivative directed to the i'th neighbor is just the difference equation:*

$$f_i^{\Delta}(p) = \frac{f^{\sigma_i}(p) - f(p)}{\Delta x_i(p)}, \tag{1}$$

where $f^{\sigma_i}(p)$ is the function f evaluated at the i^{th} neighbor of the point p and $\Delta x_i(p)$ is the distance between the point p and its i^{th} neighbor.

We move on to define integrals on n-scales. It would be useful to begin with an example from 1-scales. Let $\mathbb{T} = [0, 1] \cup [2, 3]$. We would like to integrate e^x on $\mathbb{T}$. The result of the integral is the sum of areas that lie below the function e^x on $\mathbb{T}$ plus the area of the rectangle where time scale makes a jump. See Figure 2 for an illustration.

What one observes in Figure 2 is two fold. First, the region between 1 and 2 that is excluded by the time-scale gives a contribution to the integral.

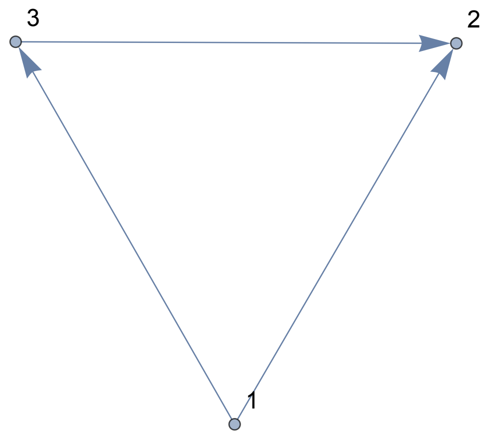


Figure 3: A simple 2-scale consisting of three points is depicted. The integral of a function f equals its value at the point $1 \times$ the area of the triangle.

Hence we understand that the *connection structure* of a time scale is important in calculating integrals as well as its domain. Second, the contribution to the integral from the separation site is proportional to the value of the function on right scattered point. For example if we integrated e^{3-x} on $\mathbb{T}$ the result of the integral would be higher because the contribution to the integral from the separation site would be e^2 instead of just e. Hence we conclude time-scales induce *a direction* in space. This directionality can be traced back to definition of Δ-derivatives in that whether one uses right scattered points or left scattered points in the definition of difference equations.

Integration on an n-dimensional subset of an n-scale is the usual Riemann integral. Hence, the integration procedure is standard in this part. What is important, however, is integration in measure zero sets. These sets will give contribution to the integral as we have seen in the example of integration of e^x on a 1-scale. For that purpose let us illustrate the integration on discrete subsets with a simple example. The general rule can be inferred via induction.

Suppose we want to integrate a function f on a discrete subset of an n-scale. For that purpose we focus on a small subset where each vertex is connected to one another. See Figure 3. It could be of rectangular shape instead of a triangle, it is not important. In the figure we see three points. 1 is connected to 2 and 3, 3 is connected to 2. Observe that all of the neighbors of 1 can be reached via σ. The integral of f is simple to calculate: value of f at $1 \times$ the area of the triangle. The value of the function should be calculated in a cell at the point where its other neighbors can be reached using forward jump operators. This makes the definition of integral on n-scales consistent with the one on the direct product of time-scales.

Definition 2.4 (Source points)**.** *A source point in $\mathbb{T}$ is a point where in the cell it is included, all of the neighbors can be reached via forward jump operators.*

Definition 2.5 (Integration on n-scales)**.** *Let $\mathbb{T}$ be an n-scale, and f be a piecewise continuous function on each subset of $\mathbb{T}$ with respect to dimension of each subset. The integral of f is the sum of Riemann integrals on n-dimensional*

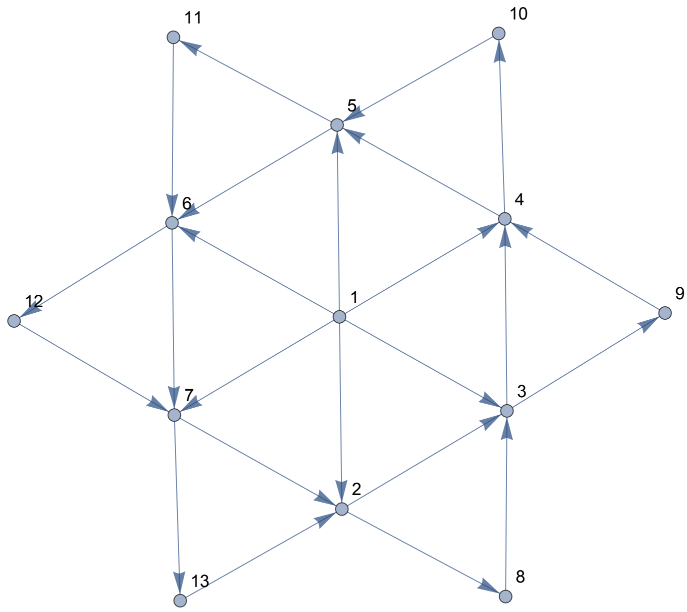


Figure 4: A highly symmetrical 2-scale consisting of thirteen points. Elements of 2-scale are located at vertices of equilateral triangles. Each edge has the length a.

subsets of $\mathbb{T}$ *and integrals on measure zero subsets whose values are calculated using the source points.*

We illustrate how to take Δ-integrals on n-scales using a simple example of the 2-scale drawn in Figure 4. The edge length is a where as the points are positioned at vertices of equilateral triangles.

Suppose we want to integrate $f(x, y)$. As one looks carefully to the connection structure of the 2-scale, one sees that the points that contributes to the integral are $1, 2, 3, 4, 5, 6, 7$. However note that point 1 contributes with multiplicity six. (This may be the case with highly symmetric n-scales as well.) The area of each equilateral triangle with side length of a is $a^2\sqrt{3}/4$. Let $p(n)$ be the coordinates of the point n. Then the result of the integral is as follows:

$$\int_{\mathbb{T}} f(x, y)\Delta x \Delta y = \frac{a^2\sqrt{3}}{4}\sum_{i=1}^{7} m_i f(p(i)) \tag{2}$$

where m_i is the number of triangles considered with $p(i)$ as the source point. As it is seen in Figure 4 $m_1 = 6$ where as $m_i = 1$ for $i \neq 1$. The point 1 seems to have a special role but it is illusory. Its multiplicity being six is because of the special way that the graph is chosen. It is also important to show that by choosing a suitable graph, one can favor some points over the others.

3 Conclusion

Previous studies [4, 7] considered direct products of time-scales to generalize the concept of time-scale to dimensions higher than one. In this article, we proposed a new concept called n-scale to evade the restrictions posed by the product space structure and still do derivation and integration. We provided definitions for directed Δ-derivatives and multiple Δ-integrals on n-scales for that purpose.

As for applications, lattice Yang-Mills theories in physics uses directed edges as connections between the lattice sites. We clearly see that this is a specific form of an n-scale. On the other hand, suppose the approximate value of a multidimensional scalar function is known in space except a spherical region. Then n-scales can be used to numerically connect the values of the function across the excluded region where it is hard to calculate the original function.

4 Acknowledgements

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