Self-controlled dynamics

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Abstract

A new class of dynamical system described by ODE coupled with their Liouville equation has been introduced and discussed. These systems called self-controlled, or self-supervised since the role of actuators is played by the probability produced by the Liouville equation. Following the Madelung equation that belongs to this class, non-Newtonian properties such as randomness, entanglement, and probability interference typical for quantum systems have been described. Special attention was paid to the capability to violate the second law of thermodynamics, which makes these systems neither Newtonian, nor quantum. It has been shown that self-controlled dynamical systems can be linked to mathematical models of livings as well as to models of AI. The central point of this paper is the application of the self-controlled systems to NP-complete problems known as being unsolvable neither by classical nor by quantum algorithms. The approach is illustrated by solving a search in unsorted database in polynomial time by resonance between external force representing the address of a required item and the response representing location of this item.

1. Introduction.

Self-controlled dynamical systems, i.e. system described by ODE coupled with their Liouville equation, until recently have been represented only by the Madelung equations, [1]. Madelung decomposed the Schrödinger complex variables into real and imaginary parts and obtained his famous hydrodynamic version of quantum mechanics

$$\frac{\partial \rho}{\partial t} + \nabla \bullet \left(\frac{\rho}{m} \nabla S\right) = 0 \qquad (1)$$

$$\frac{\partial S}{\partial t} + (\nabla S)^2 + F - \frac{\hbar^2 \nabla^2 \sqrt{\rho}}{2m\sqrt{\rho}} = 0 \qquad (2)$$

Here ρ and S are the components of the wave function $\psi = \sqrt{\rho} e^{iS/\hbar}$, and $\hbar$ is the Planck constant divided by 2π. The last term in Eq. (2) is known as quantum potential. From the viewpoint of Newtonian mechanics, Eq. (1) expresses continuity of the flow of probability density, and Eq. (2) is the Hamilton-Jacobi equation for the action S of the particle. Actually the quantum potential in Eq. (2), as a feedback from Eq. (1) to Eq. (2), represents the difference between the Newtonian and quantum mechanics, and therefore, it is solely responsible for fundamental quantum properties. The detailed analysis of these equations can be found in [2].

But what happened if the quantum potential is replaced by other feedbacks? Such replacement was introduced and investigated in [3,4,5]. Obviously the modified version of the Madelung equation became independent of quantum mechanics, and new class of dynamic equations has been created. Surprisingly this new dynamics belongs neither to quantum, nor to Newtonian physics: it represents a quantum-classical hybrid. Philosophical implications of that are discussed in [5]. At the same time, these new dynamical systems can be called self-controlled, or self-supervised since the role of actuators there is played by the probability produced by the Liouville equation that, in turn, is produced by the original ODE.

2. Self-controlled dynamical systems.

In order to illuminate specific features of the self-controlled systems under consideration, we will start with control dynamics that described by a system of ODE:

$$\frac{d\mathbf{v}}{dt} = \mathbf{F}[\mathbf{v}, U] \qquad (3)$$

Here

$\mathbf{v} = v_1, v_2, \ldots v_n$ is the vector of state variables to be controlled,

$u = u_1, u_2, \ldots u_m$ is the control vector that represents ***external*** actuators.

Let us compare the control system Eq. (3) with the following system

$$\frac{d\mathbf{v}}{dt} = \mathbf{F}[\rho(\mathbf{v})] \qquad (4)$$

where the probability ρ is introduced via the Liouville equation corresponding to Eq. (1)

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{F}) = 0 \qquad (5)$$

It describes the continuity of the probability density flow originated by the error distribution

$$\rho_0 = \rho(t = 0) \qquad (6)$$

in the initial condition of ODE (4).

Comparison of Eqs.(3) and (4) shows that they have similar structure, and the role of the external actuator U in the control system (3) is played by the term $\rho(\mathbf{v})$ in the system (4). However the origins of these actuators are fundamentally different: the actuator U represents an external force, while the actuator $\rho(\mathbf{v})$ is an ***internal*** one. Indeed it is defined by Eq. (5) that, in turn, uniquely follows from Eq. (4). That is why the system (4),(5) can be called self- controlled, or self-supervised.

From the physical viewpoint, the feedback from the Liouville equation is a fundamental step in our approach: in Newtonian dynamics, the probability never explicitly enters the equation of motion. In addition to that, the Liouville equation generated by Eq. (4) could be nonlinear with respect to the probability density ρ

$$\frac{\partial \rho}{\partial t} + \nabla \bullet \{\rho \mathbf{F}[\rho(\mathbf{V})]\} = 0 \qquad (7)$$

and therefore, the system (4),(5) departs from Newtonian dynamics. However although it has the same topology as quantum mechanics (since now the equation of motion is coupled with the equation of continuity of probability density as it does in the Madelung version of the Schrödinger equation), it belongs neither to quantum, nor to Newtonian mechanics.. Indeed Eq. (4) is more general than the Hamilton-Jacoby equation: it is not necessarily conservative, and **F** is not necessarily the quantum potential although further we will impose some restriction upon it that links **F** to the concept of information, see Fig.1.

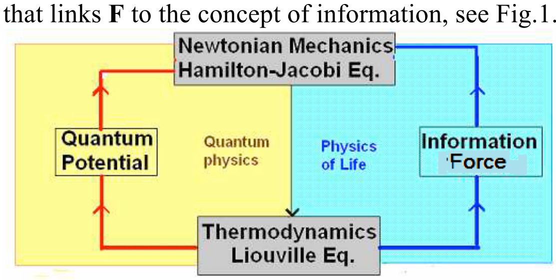


Figure 1. Topology of self-controlled dynamics.

Remark. Here and below we make distinction between the random *variable v(t)* and its *values V* in probability space.

3. Self-controlled dynamics with diffusion feedback.
In this section, following [3,5,6] we will concentrate on a special type of the self-supervised system Eqs. (4),(5). We will start with derivation of an auxiliary result that illuminates departure from Newtonian dynamics. For mathematical clarity, we will consider here a one-dimensional motion of a unit mass under action of a force *f* depending upon the *velocity v* and time *t* and present it in a dimensionless form

$$\dot{v} = f(v,t) \qquad (8)$$

referring all the variables to their representative values $v_0, t_0, etc.$
If initial conditions are not deterministic, and their probability density is given in the form

$$\rho_0 = \rho_0(V), \qquad where \quad \rho \geq 0, \quad and \qquad \int_{-\infty}^{\infty} \rho dV = 1 \qquad (9)$$

while ρ is a *single- valued* function, then the evolution of this density is expressed by the corresponding Liouville equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial V}(\rho f) = 0 \qquad (10)$$

The solution of this equation subject to initial conditions and normalization constraints (9) determines probability density as a function of *V* and *t*:

$$\rho = \rho(V,t) \qquad (11)$$

In order to deal with the constraint (9) let us integrate Eq. (10) over the whole space assuming that $\rho \to 0$ at $|V| \to \infty$ and $|f| < \infty$. Then

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \rho dV = 0, \int_{-\infty}^{\infty} \rho dV = const, \qquad (12)$$

Hence, the constraint (9) is satisfied for $t > 0$ if it is satisfied for $t = 0$.
Let us now specify the force f as a feedback from the Liouville equation

$$f(v,t) = \varphi[\rho(v,t)] \qquad (13)$$

and analyze the motion after substituting the force (13) into Eq.(9)

$$\dot{v} = \varphi[\rho(v,t)], \qquad (14)$$

Although the theory of ODE does not impose any restrictions upon the force as a function of space coordinates, the Newtonian physics does: equations of motion are never coupled with the corresponding Liouville equation. Moreover, it can be shown that such a coupling leads to non-Newtonian properties of the underlying model. Indeed, substituting the force f from Eq. (13) into Eq. (10), one arrives at the *nonlinear* equation of evolution of the probability density

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial V}\{\rho \varphi[\rho(V,t)]\} = 0 \qquad (15)$$

Let us now demonstrate the destabilizing effect of the feedback (13). For that purpose, it should be noticed that the derivative $\partial \rho / \partial v$ must change its sign at least once, within the interval $-\infty < v < \infty$, in order to satisfy the normalization constraint (9).
But since

$$Sign\frac{\partial \dot{v}}{\partial v} = Sign\frac{d\phi}{d\rho}Sign\frac{\partial \rho}{\partial v} \qquad (16)$$

there will be regions of *v* where the motion is unstable, and this instability generates randomness with the probability distribution guided by the Liouville equation (15). It should be noticed that the condition (18) may lead to exponential or polynomial growth of *v* (in the last case the motion is called neutrally stable, however, as will be shown below, it causes the emergence of randomness as well if prior to the polynomial growth, the Lipchitz condition is violated).

3.1. Emergence of self-generated stochasticity. In order to illustrate mathematical aspects of the concepts of Liouville feedback in systems under consideration as well as associated with its instability and randomness, let us take the feedback (13) in the form

$$f = -\sigma^2 \frac{\partial}{\partial v} \ln \rho, \qquad (17)$$

to obtain the following equation of motion

$$\dot{v} = -\sigma^2 \frac{\partial}{\partial v} \ln \rho, \qquad (18)$$

This equation should be complemented by the corresponding Liouville equation (in this particular case, the Liouville equation takes the form of the Fokker-Planck equation)

$$\frac{\partial \rho}{\partial t} = \sigma^2 \frac{\partial^2 \rho}{\partial V^2} \qquad (19)$$

Here v stands for a particle velocity, and σ^2 is the diffusion coefficient.

If

$$\sigma^2 = const., \qquad (20)$$

the solution of Eq. (19) subject to the sharp initial condition

$$\rho = \frac{1}{2\sigma\sqrt{\pi t}} \exp\left(-\frac{V^2}{4\sigma^2 t}\right) \qquad (21)$$

describes diffusion of the probability density, and that is why the feedback (17) will be called a diffusion feedback. Substituting this solution into Eq. (18) at $\boldsymbol{V = v}$**,** one arrives at the differential equation with respect to $v\ (t)$

$$\dot{v} = \frac{v}{2t} \qquad (22)$$

and therefore,

$$v = C\sqrt{t} \qquad (23)$$

where C is an arbitrary constant. Since $v = 0$ at $t = 0$ for any value of C, the solution (23) is consistent with the sharp initial condition for the solution (21) of the corresponding Liouvile equation (19). The solution (23) describes the simplest irreversible motion: it is characterized by the “beginning of time” where all the trajectories intersect (that results from the violation of Lipcsitz condition at $t = 0$, Fig.2), while the backward motion obtained by replacement of t with $(-t)$ leads to imaginary values of velocities.

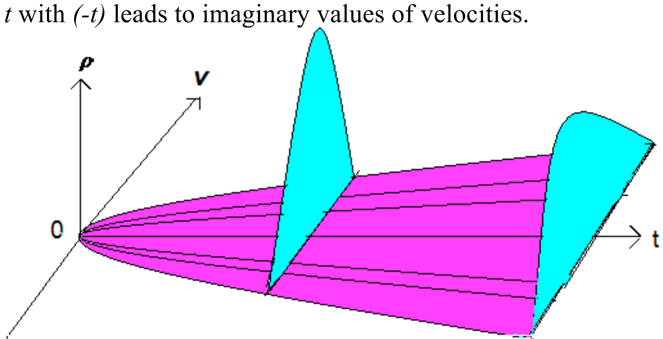


Figure 2. Stochastic process and probability density.

One can notice that the probability density (21) possesses the same properties. As shown in [2] and [4], the solution (23) has the same structure as the solution of the Madelung Eqs.(1) and (2), although the dynamical system (18), (19) is not quantum!

The explanation of such a "coincidence" is very simple: the system (18), (19) has the same dynamical topology as that of the Madelung equation where the equation of conservation of the probability is coupled with the equation of conservation of the momentum, (see Fig.1) As will be shown below, the system (18), (19) neither quantum nor Newtonian, and we will call such systems quantum-inspired, or self-controlled

Further analysis of the solution (23) demonstrates that this solution is *unstable* since

$$\frac{d\dot{v}}{dv} = \frac{1}{2t} > 0 \qquad (24)$$

and therefore, an initial error always grows generating *randomness.* Initially, at *t=0*, this growth is of infinite rate since the Lipchitz condition at this point is violated

$$\frac{\partial \dot{v}}{\partial v} \to \infty \qquad at \qquad t \to 0 \qquad (25)$$

This type of instability has been introduced and analyzed in [7]. The unstable equilibrium point ($v = 0$) has been called a terminal repeller, and the instability triggered by the violation of the Lipchitz condition – non-Lipchitz, or terminal instability. The basic property of the non- Lipchitz instability is the following: if the initial condition is infinitely close to the repeller, the transient solution will escape the repeller during a *bounded* time while for a regular repeller the time would be *unbounded*. Indeed, an escape from the simplest regular repeller can be described by the exponent $v = v_0 e^t$. Obviously $v \to 0$ if $v_0 \to 0$, unless the time period is unbounded. On the contrary, the period of escape from the terminal repeller (23) is bounded (and even infinitesimal) if the initial condition is infinitely small, (see Eq. (25)).

Considering first Eq. (23) at fixed *C* as a sample of the underlying stochastic process (21), and then varying C, one arrives at the whole ensemble characterizing that process, (see Fig. 2). The curves that envelope the cross-sectional blue areas at $t^* = const$ present the probability density distribution at constant times. One can verify that, as follows from Eq. (21), [8], the expectation and the variance of this process are, respectively

$$\bar{v} = 0, \; \tilde{v} = 2\sigma^2 t \qquad (26)$$

The same results follow from the ensemble (23) at $-\infty \le C \le \infty$. Indeed, the first equality in (26) results from symmetry of the ensemble with respect to *v = 0*; the second one follows from the fact that

$$\tilde{v} \propto v^2 \propto t \qquad (27)$$

It is interesting to notice that the stochastic process (21) is an alternative to the following Langevin equation, [8]

$$\dot{v} = \Gamma(t), \qquad \bar{\Gamma} = 0, \; \tilde{\Gamma} = \sigma \qquad (28)$$

that corresponds to the *same* Fokker-Planck equation (19). Here $\Gamma(t)$ is the Langevin (random) force with zero mean and constant variance σ.

Thus, the emergence of self-generated stochasticity is the first basic non-Newtonian property of the self-controlled dynamics with the Liouville feedback.

3.2. Entanglement. In this sub-section we will introduce a fundamental and still mysterious property that was predicted theoretically and corroborated experimentally in quantum systems: entanglement. Quantum entanglement is a phenomenon in which the quantum states of two or more objects have to be described with reference to each other, even though the individual objects may be spatially separated. This leads to correlations between observable physical properties of the systems. As a result, measurements performed on one system seem to be instantaneously influencing other systems entangled with it. Different views of what is actually occurring in the process of quantum entanglement give rise to different interpretations of quantum mechanics. Here we will demonstrate that entanglement is not a prerogative of quantum systems: it occurs in *self-controlled* systems under consideration. That will shed light on the concept of entanglement as a special type of global constraint imposed upon a broad class of dynamical systems that includes quantum as well as self-controlled systems.

In order to introduce entanglement, we will start with Eqs.(18) and (19) and generalize them to the two-dimensional case

$$\dot{v}_1 = -a_{11}\frac{\partial}{\partial v_1}\ln\rho - a_{12}\frac{\partial}{\partial v_2}\ln\rho, \qquad (29)$$

$$\dot{v}_2 = -a_{21}\frac{\partial}{\partial v_1}\ln\rho - a_{22}\frac{\partial}{\partial v_2}\ln\rho, \qquad (30)$$

$$\frac{\partial\rho}{\partial t} = a_{11}\frac{\partial^2\rho}{\partial V_1^2} + (a_{12}+a_{21})\frac{\partial^2\rho}{\partial V_1\partial V_2} + a_{22}\frac{\partial^2\rho}{\partial V_2^2}, \qquad (31)$$

As in the one- dimensional case, this system describes diffusion without a drift
The solution to Eq. (31) has a closed form

$$\rho = \frac{1}{\sqrt{2\pi\det[\hat{a}_{ij}]t}}\exp(-\frac{1}{4t}b'_{ij}V_iV_j), \qquad i = 1,2. \qquad (32)$$

Here

$$[b'_{ij}] = [\hat{a}_{ij}]^{-1}, \hat{a}_{11} = a_{11}, \hat{a}_{22} = a_{22}, \hat{a}_{12} = \hat{a}_{21} = a_{12} + a_{21}, \ \hat{a}_{ij} = \hat{a}_{ji}, b'_{ij} = b'_{ji}, \qquad (33)$$

Substituting the solution (32) into Eqs. (29) and (30), one obtains

$$\dot{v}_1 = \frac{b_{11}v_1 + b_{12}v_2}{2t} \qquad (34)$$

$$\dot{v}_2 = \frac{b_{21}v_1 + b_{22}v_2}{2t}, \qquad b_{ij} = b'_{ij}\hat{a}_{ij} \qquad (35)$$

Eliminating t from these equations, one arrives at an ODE in the configuration space

$$\frac{dv_2}{dv_1} = \frac{b_{21}v_1 + b_{22}v_2}{b_{11}v_1 + b_{12}v_2}, \ v_2 \to 0 \qquad at \qquad v_1 \to 0, \qquad (36)$$

This is a classical singular point treated in textbooks on ODE.
Its solution depends upon the roots of the characteristic equation

$$\lambda^2 - 2b_{12}\lambda + b_{12}^2 - b_{11}b_{22} = 0 \qquad (37)$$

Since both the roots are real in our case, let us assume for concreteness that they are of the same sign, for instance, $\lambda_1 = 1, \quad \lambda_2 = 1$. Then the solution of Eq. (36) is presented by the family of straight lines

$$v_2 = \tilde{C}v_1, \qquad \tilde{C} = const. \qquad (38)$$

Thus, the solutions of Eqs. (29) and (30) are presented by two-parametrical families of random samples, as expected, while the randomness enters through the time-independent parameters C and $\tilde{C}$ that can take any real numbers. Let us now find such a combination of the variables that is deterministic. Obviously, such a combination should not include the random parameters C or $\tilde{C}$. It easily verifiable that

$$\frac{d}{dt}(\ln v_1) = \frac{d}{dt}(\ln v_2) = \frac{b_{11} + \tilde{C}b_{12}}{2t} \qquad (40)$$

and therefore,

$$(\frac{d}{dt}\ln v_1)/(\frac{d}{dt}\ln v_2) \equiv 1 \qquad (41)$$

Thus, the ratio (41)is deterministic although both the numerator and denominator are random,(see Eq.40). This is a fundamental non-classical effect representing a global constraint. Indeed, in theory of stochastic processes, two random functions are considered statistically equal if they have the same statistical invariants, but their point-to-point equalities are not required (although it can happen with a vanishingly small probability). As demonstrated above, the *diversion of determinism into randomness via instability (due to a Liouville feedback), and then conversion of randomness to partial determinism (or coordinated randomness) via entanglement* is the fundamental non-classical paradigm that.

3.3. Entanglement as reaction to global constraint. In this sub-section we will establish roots of entanglement in context of classical mechanics, and turn to the concept of global constraint.

a. Criteria for non-local interactions. Based upon analysis of all the known interactions in the Universe and defining them as local, one can formulate the following criteria of *non*-local interactions: they are *not* mediated by another entity, such as a particle or field; their actions are *not* limited by the speed of light; the strength of the interactions does not drop off with distance. All of these criteria lead us to the concept of the global constraint as a starting point.

b. Global constraints in physics. It should be recalled that the concept of a global constraint is one of the main attribute of Newtonian mechanics. It includes such *idealizations* as a rigid body, an incompressible fluid, an inextensible string and a membrane, a non-slip rolling of a rigid ball over a rigid body, etc. All of those idealizations introduce geometrical or kinematical restrictions to positions or velocities of particles and provides "instantaneous" speed of propagation of disturbances. Let us discuss the role of the reactions of these constraints. One should recall that in an incompressible fluid, the reaction of the global constraint $\nabla \cdot v \geq 0$ (expressing non-negative divergence of the velocity *v*) is a non-negative pressure $p \geq 0$; in inextensible flexible (one- or two-dimensional) bodies, the reaction of the global constraint $g_{ij} \leq g^0{}_{ij}, i,j = 1,2$ (expressing that the components of the metric tensor cannot exceed their initial values) is a non-negative stress tensor $\sigma_{ij} \geq 0$, *i,j*=1,2. It should be noticed that all the known forces in physics (the gravitational, the electromagnetic, the strong and the weak nuclear forces) are local. However, the reactions of the global constraints listed above do not belong to any of these local forces, and therefore, they are non-local. Although these reactions are being successfully applied for engineering approximations of theoretical physics, one cannot relate them to the origin of ***entanglement*** since they are result of idealization that ignores the discrete nature of the matter. However, there is another type of the global constraint in physics: the normalization constraint, (see Eq. (9)). This constraint is fundamentally different from those listed above for two reasons. Firstly, it is not an idealization, and therefore, it cannot be removed by taking into account more subtle properties of matter such as elasticity, compressibility, discrete structure, etc. Secondly, it imposes restrictions not upon positions or velocities of particles, but upon the *probabilities* of their positions or velocities, and that is where the ***entanglement*** comes from. Indeed, if the Liouville equation is coupled with equations of motion as in quantum mechanics, the normalization condition imposes a global constraint upon the state variables, and that is the origin of quantum entanglement. In quantum physics, the reactions of the normalization constraints can be associated with the energy eigenvalues that play the role of the Lagrange multipliers in the conditional extremum formulation of the Schrödinger equation, Landay,L., 1997, [17]. In self-controlled systems, the Liouville equation is also coupled with equations of motion (although the feedback is different). And that is why the origin of entanglement in self-controlled systems is the same as in quantum mechanics.

c. Speed of action propagation. Further illumination of the concept of entanglement follows from comparison of quantum and Newtonian systems. Such a comparison is convenient to perform in terms of the Madelung version of the Schrödinger equation: Eq.(1) and (2}

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\frac{\rho}{m} \nabla S) = 0$$

$$\frac{\partial S}{\partial t} + (\nabla S)^2 + V - \frac{\hbar^2 \nabla^2 \sqrt{\rho}}{2m\sqrt{\rho}} = 0$$

As shown in [9], the Newtonian mechanics ($\hbar = 0$), in terms of the S and ρ as state variables, is of a hyperbolic type, and therefore, any discontinuity propagates with the *finite* speed *S/m*, i.e. the Newtonian systems do not have non-localities. But the non-relativistic quantum mechanics ($\hbar \neq 0$) is of a parabolic type. This means that any disturbance of S or ρ in one point of space instantaneously transmitted to the whole space, and this is the mathematical origin of non-locality. But is this property is a prerogative of quantum evolution? Obviously, it is not. Any parabolic equation (such as Navier-Stokes equations or Fokker-Planck equation) has exactly the same non-local properties. However, the difference between the quantum and classical non-localities is in their physical interpretation. Indeed, the Navier-Stokes equations are derived from simple laws of Newtonian mechanics, and that is why a physical interpretation of non-locality is very simple: If a fluid is incompressible, then the pressure plays the role of a reaction to the geometrical constraint $\nabla \cdot v \geq 0$, and it is transmitted instantaneously from one point to the whole space (the Pascal law). One can argue that the incompressible fluid is an idealization, and that is true. However, it does not change our point: Such a model has a lot of engineering applications, and its non-locality is well understood. The situation is different in quantum mechanics since the Schrodinger equation has never been derived from Newtonian mechanics: It has been postulated.

Let us turn now to the self-controlled systems. The formal difference between them and quantum systems is in a feedback from the Liouville equation to equations of motion: the gradient of the quantum potential is replaced by the information forces, while the equations of motion are written in the form of the second Newton's law rather than in the Hamilton-Jacoby form. But as in quantum mechanics, the global constraint is the normalization condition expressed by Eq. (9). That is why both quantum and self-controlled systems possess the same non-locality: instantaneous propagation of changes in the probability density, and this is due to similar topology of their dynamical structure, and in particular, due to a feedback from the Liouville equation.

4. Self-controlled dynamics with shock feedback.

4.1. Shock waves in probability space. In order to introduce more surprising properties of self-controlled dynamics, we replace the diffusion feedback Eq. (17) by the following one

$$f = \xi\rho, \qquad \xi = const., \tag{42}$$

and therefore, the equation of motion and the Liouville equation are

$$\dot{v} = \xi\rho \tag{43}$$

$$\frac{\partial\rho}{\partial t} + \xi\frac{\partial}{\partial V}(\rho^2) = 0, \quad , \tag{44}$$

The solution of Eq. (44) subject to the initial conditions $\rho_0(V)$ and the normalization constraint (9) is given in the following implicit form, [10],

$$\rho(V,t) = \rho_0(V - \xi\rho t), \quad \rho_0 = \rho_{t=0} \tag{45}$$

This solution subject to initial conditions and the normalization constraint, describes propagation of initial distribution of the density $\rho_0(V)$ with the speed *V* that is proportional to the values of this density, i.e. the higher values of ρ propagate faster than lower ones. As a result, any compressive part of the wave, where the propagation velocity is a decreasing function of *V*, ultimately "breaks" to give a *triple-valued* (but still continuous) solution for $\rho(V,t)$. Eventually, this process leads to the formation of strong discontinuities that are related to propagating jumps of the probability density. In the theory of nonlinear waves, this phenomenon is known as the formation of a shock wave. Thus, as follows from the solution (45), a single-valued continuous probability density spontaneously transforms into a triple-valued, and then, into discontinuous distribution. That is why the feedback (42) can be called the shock feedback.

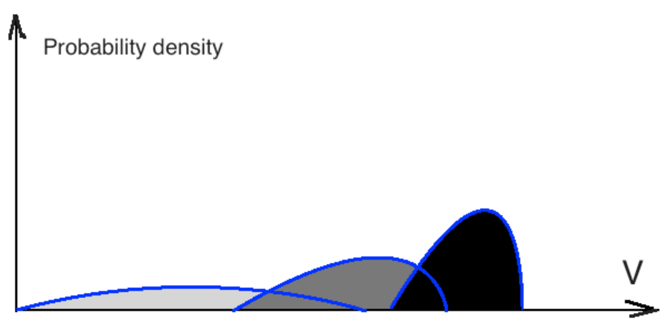


Figure 3. Formation of shock wave in probability space.

In aerodynamical application of Eq. (44), when ρ stands for the gas density, these phenomena are eliminated through the model correction: at the small neighborhood of shocks, the gas viscosity ν cannot be ignored, and the model must include the term describing dissipation of mechanical energy. The corrected model is represented by the Burgers' equation

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial V}(\rho^2)=\nu\frac{\partial^2\rho}{\partial V^2} \qquad (46)$$

As shown in [10], this equation has continuous single-valued solution (no matter how small is the viscosity ν), and that provides a perfect explanation of abnormal behavior of the solution of Eq. (44). Similar correction can be applied to the case when ρ stands for the probability density if one includes Langevin forces $\Gamma(t)$ into Eq. (43)

$$\dot{v}=\rho+\sqrt{\nu}\Gamma(t), \qquad <\Gamma(t)>=0, \; <\Gamma(t)\Gamma(t')>=2\delta(t-t') \qquad (47)$$

Then the corresponding Fokker-Planck equation takes the form (46). It is reasonable to assume that small random forces of strength $\sqrt{\nu}<<1$ are always present, and that protects the mathematical model (43), (44) from singularities and multi-valuedness in the same way as it does in the case of aerodynamics.

It is interesting to notice that Eq. (46) can be obtained from Eq. (43) in which random force is replaced by an additional Liouville feedback

$$f=\xi\rho-\sigma^2\frac{\partial}{\partial v}\ln\rho, \qquad (48)$$

i.e. the shock feedback (42) is combined with the diffusion feedback (17) and therefore

$$\dot{v}=\xi\rho-\sigma^2\frac{\partial}{\partial v}\ln\rho, \qquad (49)$$

It is easily verifiable that the solution of Eq. (46) satisfies the constraint (9) if the corresponding initial condition does, [10].

It should be emphasized that despite the mathematical similarity between Eq.(46) and the Bergers equation, the physical interpretation of Eq.(46) is fundamentally different from that of the Burgers equation: it is a part of the dynamical system (46),(49) in which Eq. (46) plays the role of the Liouville equation generated by Eq. (49). Mathematical details of the short-time behavior of solution to the system (46), (49) can be found in [4]. Here, following [10], we describe the long-term behavior when $t\to\infty$.

The long-term behavior of the solutions to Eq.(46) can be adopted from the theory of the Burger equation by considering an initial step

$$\rho(V)=\begin{Bmatrix}\rho_1 & V>0\\ \rho_2>\rho_1 & V<0\end{Bmatrix} \qquad (50)$$

that diffuses into the steady profile:

$$\rho=\rho_1+\frac{\rho_2-\rho_1}{1+\exp\frac{\rho_2-\rho_1}{2\nu}(V-Ut)} \qquad (51)$$

moving with the constant speed

$$U=\frac{\rho_1+\rho_2}{2} \quad \text{for } \xi=\frac{1}{2} \qquad (52)$$

Special solutions for a moving single hump, triangle wave and *N* wave can also be presented in closed analytical form. For instance, the exact solution for Eq. (46) is, [10],

$$\rho(V,t)=\frac{\int_{-\infty}^{\infty}\frac{V-\eta}{t}e^{-G/2\sigma}d\eta}{\int_{-\infty}^{\infty}e^{-G/2\sigma}d\eta} \qquad (53)$$

where

$$G(\eta,v.t)=\int_{0}^{\eta}\rho(\eta',t=0)d\eta'+\frac{(V-\eta)^2}{2t} \qquad (54)$$

Although this solution is well known, it should be emphasized that in our case it occurs in the ***probability*** space. This means that if one runs Eq. (49) independently many times starting with different initial conditions and compute the statistical characteristics of the family of these solutions, he will arrive at the evolution of the probability density described by Eq. (46).

One can verify that the additional (normalization) constraint imposed upon the probability density that is a state variable of the Burgers equation is satisfied

$$\frac{d}{dt}\int_{-\infty}^{\infty}\rho\,dV=[\sigma\frac{\partial\rho}{\partial V}-\frac{1}{2}\rho^2]_{-\infty}^{\infty}=0 \qquad (55)$$

Indeed, as follows from (55), if the normalization constraint is satisfied at the initial condition, it will be satisfies for all times.

Let us now turn to Eq. (51) that describes the long-term behavior of an initial shock of probability and consider a train of such shocks that overtake one another merging to a single shock of increased strength. The solution of the combined shock of probability as a nonlinear function of the original shocks can be written in the following form, [10]

$$\rho=\frac{\rho_1 f_1+\rho_2 f_2}{f_1+f_2}, \qquad \rho_2>\rho_1 \qquad (56)$$

where

$$f_i=\exp(-\frac{\rho_i V}{2\sigma}+\frac{\rho_i^2 t}{4\sigma}), \qquad i=1,2. \qquad (57)$$

Thus, Eqs. (56) and (57) give the rule for combining probabilities in dynamics with the shock feedback.

4.2. Violation of the second law of thermodynamics. In this sub-section we will derive a distinguished property of the system (46),(49) that is associated with violation of the second law of thermodynamics i.e. with the capability of moving from disorder to order without help from outside. That property can be predicted qualitatively even prior to analytical proof: due to the nonlinear term in Eq. (46), the solution form shock waves in probability space, and that can be interpreted as "concentrations" of probability density, i.e. departure from disorder. In order to demonstrate it analytically, let us turn to Eq. (46) and find the change of entropy H

$$\frac{\partial H}{\partial t}=-\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\rho\ln\rho\,dV=-\int_{-\infty}^{\infty}\xi\dot{\rho}(\ln\rho+1)dV=\int_{-\infty}^{\infty}\xi\frac{\partial}{\partial V}(\rho^2)(\ln\rho+1)dV$$
$$=\xi[\,\Big|_{-\infty}^{\infty}\rho^2(\ln\rho+1)-\int_{-\infty}^{\infty}\rho\,dV]=-\xi<0 \qquad (58)$$

Obviously, presence of small diffusion, when $\sigma^2<<1$, does not change the inequality (58) during certain period of time. (However, eventually, for large times, diffusion takes over, and the inequality (58) is reversed).

Thus the system (46), (49) is capable to decrease with no external interactions. Indeed the information force represented by the feedback (48) is generated by the Liouville equation that, in turn, is generated by the equation of motion (49). In addition to that, the particle described by ODE (49) is in equilibrium $\dot{v} = 0$ prior to activation of the feedback (48). Therefore the solution of Eqs. (46), and (49) is capable to violate the second law of thermodynamics, and that means that this class of dynamical systems does not belong to physics as we know it. This conclusion triggers the following question: are there any phenomena in Nature that can be linked to dynamical systems (46), (49)? The answer was given in [3,5]: the self-controlled dynamics provides extension of modern physics to include physics of life and it can provide mathematical model for artificial intelligence.

4.3. Interference of probabilitries. In Newtonian physics, the probability is introduced via the Liouville equation describing the continuity of the probability density flow. This equation is linear with respect to the probability density, and therefore, according to the superposition principle, the probabilities are combined by summation: when an event can occur in several alternative ways, the probability of the event is the sum of the probabilities for each way considered separately, i.e.

$$\rho = \rho_1 + \rho_2 \qquad (59)$$

In quantum physics, the probability is introduced via the Schrödinger equation that is linear with respect to probability amplitude, i.e. with respect to the square root of the probability density. Therefore, when an event can occur in several alternative ways, the probability amplitude of the event is the sum of the probability amplitudes for each way considered separately. In other words, the probability interference in quantum mechanics follows from the *linearity* of the Schrödinger equation with respect to the probability amplitudes ψ_i as state variables. Due to linear superposition of these amplitudes, the following rule can be formulated

$$\psi = \psi_1 + \psi_2, \; \rho_i = |\psi_i|^2, \qquad \rho = |\psi_1 + \psi_2|^2 \neq \rho_1 + \rho_2 \qquad (60)$$

and this phenomenon is known as interference of probabilities: the probabilities are combined as the intensities of waves.

The situation with interference of probabilities in self-controlled dynamics is more complex: it depends upon the type of information forces. Indeed, in the diffusion feedback, the Liouville equation (19) is linear with respect to the probability density, and the probabilities are combined according to Eq. (59). i.e. without interference. But in the shock feedback, the Liouville equation (46) is nonlinear with respect to the probability density, and consequently, the probabilities interfere. However, this interference is different from the quantum one and it is illustrated below.

Indeed, the rule of combining the probabilities is different from both the classical as well as quantum as follows from Eq. (56)

$$\rho = \frac{\rho_1 f_1 + \rho_2 f_2}{f_1 + f_2}, \qquad \rho_2 > \rho_1 \qquad (61)$$

where $$f_i = \exp\left(-\frac{\rho_i V}{2\sigma} + \frac{\rho_i^2 t}{4\sigma}\right), \qquad i = 1,2. \qquad (62)$$

This means that when an event can occur in several alternative ways, the probability of the event is the sum of nonlinear combinations of the probabilities for each way considered separately.

More general case of interference of probability in self-controlled dynamics was discussed in [11].

5. Self-controlled dynamics with soliton feedback.

Let us turn to the feedback (48) and modify it as following

$$f = c_0 + \frac{1}{2}c_1\rho + \frac{b}{\rho}\frac{\partial^2\rho}{\partial v^2} \qquad (63)$$

$b > 0, c_0 > 0, c_1 > 0.$

Then the equation of motion is

$$\dot{v} = c_0 + \frac{1}{2} c_1 \rho + \frac{b}{\rho} \frac{\partial^2 \rho}{\partial v^2} \tag{64}$$

Then the corresponding Liouville equation will turn into the following PDE

$$\frac{\partial \rho}{\partial t} + (c_0 + c_1 \rho) \frac{\partial \rho}{\partial V} + b \frac{\partial^3 \rho}{\partial V^3} = 0 \tag{65}$$

that is a celebrated Korteweg-de Vries (KdV) equation.

The Korteweg–de Vries (KdV) equation was discovered in 1895 by Korteweg and de Vries, but this equation was forgotten during a long time. It was recently rediscovered by M. Kruskal who obtained it from the Fermi-Pasta-Ulam model [12]. Since then, the mathematical theory behind the KdV equation became rich and interesting, and, in the broad sense, it is a topic of active mathematical research. A homogeneous version of this equation that illustrates its distinguished properties is nonlinear PDE of parabolic type (65). However a fundamental difference between the standard KdV equation and Eq. (65) is that Eq. (65) dwells in the probability space, and therefore, it must satisfy the normalization constraint

$$\int_{-\infty}^{\infty} \rho \, dV = 1 \tag{66}$$

But since the KdV equation has the conservation invariants, [10]

$$\int_{-\infty}^{\infty} \rho \, dV = Const., \tag{67}$$

$$\int_{-\infty}^{\infty} \rho^2 \, dV = Const., \text{ etc.} \tag{68}$$

the constraint (66) becomes a particular case of the invariant (67); consequently, if the normalization constraint is satisfied at $t = 0$, it is satisfied all the time. That allows one to apply all the known result directly to Eq. (65). However it should be noticed that the conservation invariants (67) and (68) have different physical meaning: they are not related to conservation of momentum and energy, but rather impose constraints upon the Shannon information.

We will start the analysis of the equation (65) with consideration of its linear version when $c_1 = 0$

$$\frac{\partial \rho}{\partial t} + c_0 \frac{\partial \rho}{\partial V} + b \frac{\partial^3 \rho}{\partial V^3} = 0 \tag{69}$$

The first applications of linear (parabolic) version of KdV equation (1.2) appear in models of shallow water waves [10]. The equation is also conservative, and its solution is represented by a train of traveling waves

$$\rho(v, t) = Ae^{ikv - \omega t} \tag{70}$$

where ω is the frequency, and k is the wave number. For KdV equation, these two constants are connected by the following dispersion relation

$$\omega = c_0 k - bk^3 \tag{71}$$

If the initial profile $\rho = u(v,0)$ is presented as a sum of the Fourier harmonics, then each of this harmonic will propagate with the phase speed

$$C = \omega / k. \qquad (72)$$

Comparing equations (71) and (72), one can see that each Fourier harmonics will propagate with different phase speed that depends upon its wave number *k*. Therefore any initial profile eventually is dispersed over the whole positive subspace, Fig.4.

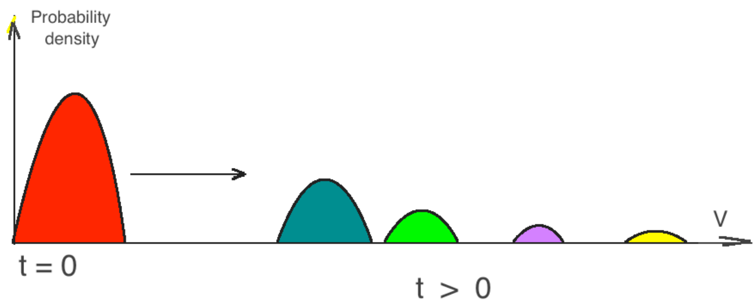


Figure 4. Linear dispersion of initial profile.

An important property of the linear version of the KdV equation is the dependence of its solution on the initial conditions for all times.

Let us assume now that $b = 0, \quad c_0 = 0$. We get the equation

$$\frac{\partial \rho}{\partial t} + c_1 \rho \frac{\partial \rho}{\partial V} = 0 \qquad (73)$$

Unlike the previous versions of the KdV equation, this is a nonlinear KdV of hyperbolic type. It appears in models of free particles flow, traffic jam, etc. This is the simplest equation that describes formation of shock waves. Its closed analytical solution can be written only in an implicit form, and here we will analyze it only qualitatively. We will start our analysis with studying a propagation of an initial profile $\rho = \rho(v,0)$. This equation has been considered in the Section 4. As follows from Eq. (73), the higher values of ρ propagate faster than lower ones. As a result, the moving front becomes steeper and steeper, and finally a strong discontinuity representing a shock emerges, see Fig.3. Since closed form solution of Eq. (65) is not available, we will continue with the solution for large time. The rationale for that is the assumption that eventually the solution tends to a stationary shape as a result of a balance between dispersion and shock wave formation. Therefore we will seek the solution in the form of a stationary motion

$$\rho(v,t) = f(v - Ut) = u(\zeta) \quad at \quad t \to \infty \qquad (74)$$

Substituting Eq.(74) into Eq. (65) one obtains

$$-U\frac{\partial \rho}{\partial \zeta} + (c_0 + c_1\rho)\frac{\partial \rho}{\partial \zeta} + b\frac{\partial^3 \rho}{\partial \zeta^3} = 0 \qquad (75)$$

Integrating this equation with respect to ζ and setting the arbitrary constant to zero, one arrives at the ODE in its final form

$$b\frac{\partial^2 \rho}{\partial \zeta^2} + (c_0 - U)\rho + \frac{c_1}{2}\rho^2 = 0 \qquad (76)$$

The solution of this equation is a soliton moving with the speed *U*

$$\rho = aSech^2[\frac{\sqrt{c_1 a}}{\sqrt{12b}}(v - Ut)] \quad (77)$$

where

$$U = c_0 + \frac{1}{3}c_1 a \quad (78)$$

see Fig. 5. It should be emphasized that the soliton (77) does not depend upon initial conditions, and consequently it can be considered as a static attractor in probability space. This means that in physical space, a solution of Eq. (77) eventually approach a stochastic attractor.

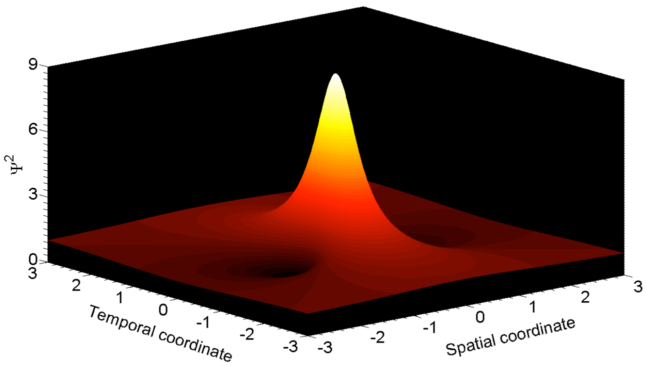


Figure 5. Soliton as an attractor of KdV solution.

6. Self-controlled dynamics with integral feedback

Let us specify the feedback as

$$f = \frac{\xi}{\rho(v,t)}\int_{-\infty}^{v}[\rho(\eta,t) - \rho^*(\eta)]d\eta \quad (79)$$

that we call integral feedback.

Here $\rho^*(v)$ is a preset probability density satisfying the constraints (9), and ξ is a positive constant with dimensionality [1/sec]. With the feedback (79), the equation of motion and the corresponding Liouville equation take the form, respectively

$$\dot{v} = \frac{\xi}{\rho(v,t)}\int_{-\infty}^{v}[\rho(\eta,t) - \rho^*(\eta)]d\eta \quad (80)$$

$$\frac{\partial\rho}{\partial t} + \xi[\rho(t) - \rho^*] = 0 \quad (81)$$

The last equation has the analytical solution

$$\rho = [(\rho_0 - \rho^*)e^{-\xi t} + \rho^*] \quad (82)$$

Subject to the initial condition

$$\rho(t = 0) = \rho_0 \quad (83)$$

that satisfies the constraints (9).

This solution converges to a preset stationary distribution $\rho^*(V)$. Obviously the normalization condition for ρ is satisfied if it is satisfied for ρ_0 and ρ^*. Indeed,

$$\int_{-\infty}^{\infty} \rho V dV = [\int (\rho_0 - \rho^*) V dV] e^{-\xi t} + \int_{-\infty}^{\infty} \rho^* V dV = 1 \qquad (84)$$

Rewriting Eq. (82) in the form

$$\rho = \rho_0 e^{-\xi t} + \rho^* (1 - e^{-\xi t}) \qquad (85)$$

one observes that $\rho \geq 0$ at all $t \geq 0$ and $-\infty > V > \infty$.

As follows from Eq. (82), the solution of Eq. (81) has an attractor that is represented by the preset probability density $\rho^*(V)$. Substituting the solution (82) into Eq. (80), one arrives at the ODE that simulates the stochastic process with the probability distribution (82)

$$\dot{v} = \frac{\zeta e^{-\xi t}}{[\rho_0(v) - \rho^*(v)] e^{-\xi t} + \rho^*(v)} \int_{-\infty}^{v} [\rho_0(\eta) - \rho^*(\eta)] \, d\eta \qquad (86)$$

It is reasonable to choose the solution (82) as starting with a sharp initial condition

$$\rho_0(V) = \delta(V) \qquad (87)$$

As a result of that assumption, all the randomness is supposed to be generated *only* by the controlled instability of Eq. (86). Substitution of Eq. (87) into Eq. (86) leads to two different domains of *v*: $v \neq 0$ and v=0 *where* the solution has two different forms, respectively

$$\int_{-\infty}^{v} \rho^*(\xi) d\xi = \left(\frac{C}{e^{-\xi t} - 1}\right)^{1/\xi}, \quad v \neq 0 \qquad (88)$$

$$v \equiv 0 \qquad (89)$$

Indeed, $\dot{v} = \dfrac{\zeta e^{-\xi t}}{\rho^*(v)(e^{-\xi t} - 1)} \displaystyle\int_{-\infty}^{v} \rho^*(\eta)] d\eta$

whence $\dfrac{\rho^*(v)}{\displaystyle\int_{-\infty}^{v} \rho^*(\eta) d\eta} dv = \dfrac{\zeta e^{-\xi t}}{e^{-\xi t} - 1} dt$. Therefore, $\ln \displaystyle\int_{-\infty}^{v} \rho^*(\eta) d\eta = \ln\left(\dfrac{C}{e^{-\xi t} - 1}\right)^{1/\xi}$ and that leads to Eq. (4.10) that presents an implicit expression for *v* as a function of time since ρ^* is the known function. Eq. (4.11) represents a singular solution, while Eq. (4.10) is a regular solution that includes arbitrary constant C. The regular solutions is discontinuous:

$$v \to \infty \quad at \quad t \to 0, \quad v = 0 \quad at \quad t = 0 \qquad (90)$$

the Lipschitz condition is violated

$$\left| \frac{\partial \dot{v}}{\partial v} \right| \to \infty \qquad at \qquad t \to 0 \;, \quad |v| \to 0 \qquad (91)$$

and therefore, the uniqueness of the solution is lost thereby generating *randomness*.

As follows from Eq. (88), all the particular solutions for different values of *C* possess the same property (90), and that leads to non-uniqueness of the solution due to violation of the Lipchitz condition. Therefore, the same initial condition at $t \to 0$ yields infinite number of different solutions forming a family (88); each solution of this family appears with a certain probability guided by the corresponding Liouville equation (81). For instance, in cases plotted in Fig.6, a) and Fig.6, b), the "winner" solution is, respectively,

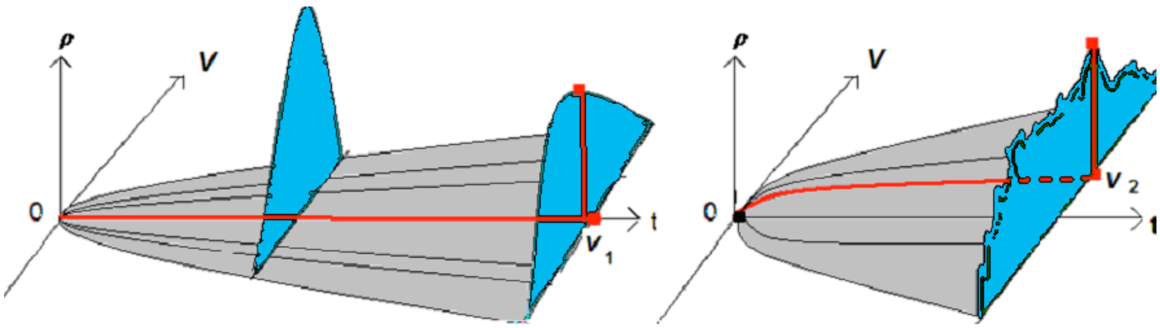


Figure 6. Stochastic processes and their attractors.

$$v_1 = \varepsilon \to 0, \quad \rho(v_1) = \rho_{\max}, \; and \quad v = v_2, \qquad \rho(v_2) = \sup\{\rho\}$$

since it passes through the maximum of the probability density . However, with lower probabilities, other solutions of the same family can appear as well. Obviously, this is a non-classical effect. Qualitatively, this property is similar to those of quantum mechanics: the system keeps all the solutions simultaneously and displays each of them "by a chance", while that chance is controlled by the evolution of probability density (82). It should be recalled that, as in quantum mechanics,[2], the choice of displaying a certain solution is made only once, at *t=0*, i.e. when the solution departs from the deterministic to a random state; since than, it stays with this solution as long as the Liouville feedback is present.

As shown in [16], the self-control dynamics with integral feedback can finds the global maximum of an integrable, but not necessarily differentiable function. The idea of the algorithm is very simple: introduce a positive function to be maximized as the probability density to which the solution is attracted. Then the larger value of this function will have the higher probability to appear.

6.1. Examples.

Example1. Let us start with the following normal distribution

$$\rho^*(V) = \frac{1}{\sqrt{2\pi}} e^{-\frac{V^2}{2}} \qquad (92)$$

Substituting the expression (92) and (87) into Eq. (88) at $V=v$, and $\xi = 1$ one obtains

$$v = erf^{-1}\left(\frac{C_1}{e^{-t}-1}\right), \qquad v \neq 0 \qquad (93)$$

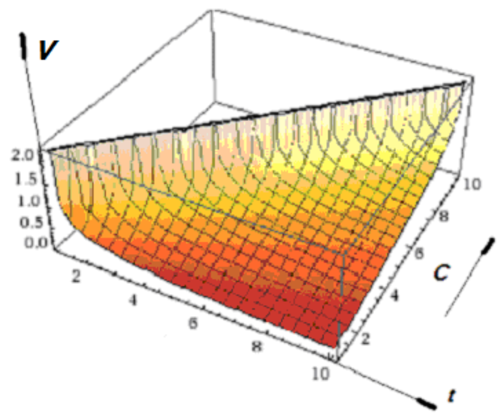
Example 2. Let us choose the target density ρ^* as the Student's distribution, or so-called power law distribution

$$\rho^*(V) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})}\left(1+\frac{V^2}{\nu}\right)^{-(\nu+1)/2} \qquad (94)$$

Substituting the expression (94) and (87) into Eq. (88) at $V=v$, ν=1, and $\xi = 1$ one obtains

$$v = \cot\left(\frac{C}{e^{-t}-1}\right) \; for \quad v \neq 0 \qquad (95)$$

The 3D plot of the solutions of Eqs.(93) and (95), are presented in Figures 7a, and 7b, respectively.



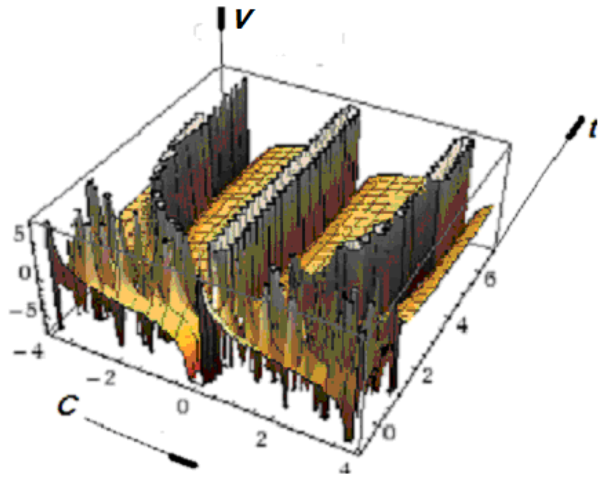


Figure 7a. Dynamics driving random events to normal distribution. **Figure 7b. Dynamics driving random events to power law.**

6.2. A mystery of power-law statistics.

This sub-section was inspired by a mysterious power-law statistics that predicts social catastrophes: wars, terrorist attacks, market crushes est. Resent interest in literature is concentrated on half-a-century finding that the severity of interstate wars is power-law distributed, and that belongs to the most striking empirical regularities in world politics. Surprisingly, similar catastrophes were identified in physics (Ising systems, avalanches, earthquakes), and even in geometry (percolation). Although all these catastrophes have different origins, their similarity is based upon the power law statistics, and as a consequence, on scale invariance, self-similarity and fractal dimensionality, [12]. According to the theory of self-organized criticality, that explains the origin of this kind of catastrophes, each underlying dynamical system is attracted to a critical point separating two qualitatively different states (phases). This attraction is represented by a relaxation process of slowly driven system. Transitions from one phase to another are accompanied by sudden release of energy that can be associated with a catastrophe, and the severity of the catastrophe is power law distributed. However, in order to overcome the critical point and enter a new phase, a slow input of *external* energy is required. The origin of this energy is well understood in physical systems, but not in social ones, since there are no well-established models of social dynamics. For that reason, we turn to the previous section and start with comparison the underlying dynamics of normal and power law distribution, (see Figs. 7a, and 7b). Let us recall that the normal distribution is commonly encountered in practice, and is used throughout statistics, natural sciences, and social sciences as a simple model for complex phenomena. For example, the observational error in an experiment is usually assumed to follow a normal distribution, and the propagation of uncertainty is computed using this assumption. But statistical inference using a normal distribution is not robust to the presence of outliers (data that is unexpectedly far from the mean, due to exceptional circumstances, observational error, etc.). When outliers are expected, data may be better described using a heavy-tailed distribution such as the power-law distribution. As demonstrated in Fig. 8, normal and power law distributions have very close configurations excluding the tails. However despite of that, the types of the random events described by these statistics are of fundamental difference. Indeed, processes described by normal distributions are usually coming from physics, chemistry, biology, etc., and they are characterized by a smooth evolution of underlying dynamical events. On the contrary, the processes described by power laws are originated from events driven by human decisions (wars, terrorist acts, market crushes), and therefore, they are associated with catastrophes. Surprisingly, the 3D plots of Eqs.(93) and (95) (see Figs.7a and 7b) describing dynamics that drives random events to the normal and the power law distributions, respectively, demonstrate the same striking difference between these distributions, that is: a smooth evolution to normal distribution, and “violent”, full of densely distributed discontinuities transition to power law distribution.

Is this a coincidence? Indeed, the proposed self-controlled dynamics is based upon global assumptions, and it does not bear any specific information about a particular statistics as an attractor. However the last statement should be slightly modified:

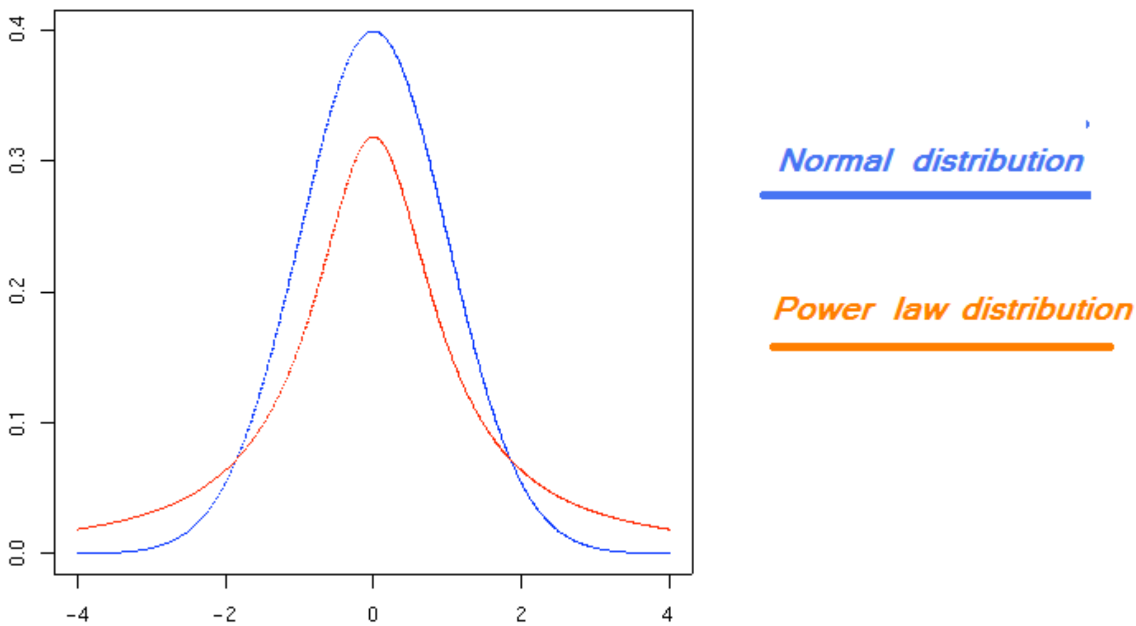


Figure 8. Normal and power law distributions.

actually the model of self-controlled dynamics is tailored to describe Livings' behavior, and in particular, decision making process. Is that why the self-controlled dynamics captures "violent" properties of power law statistics that is associated with human touch? This is an open problem, and the self-control dynamics could have a solution.

7. Self-controlled dynamics with functional feedback

One of the limitations of classical dynamics, is inability to change the structure without an external input. As will be shown below, the self-controlled dynamics can change the locations and even the type of the attractors being triggered only by a feedback from the Liouville equation, i.e. by an internal effort. We will start with a simple dynamical system

$$\dot{v} = 0, \quad v = 0 \quad at\ t = 0 \qquad (96)$$

and than apply the following control

$$f = -k\overline{v} + a\overline{\overline{v}} - \sigma \frac{\partial}{\partial v} \ln \rho, \qquad (97)$$

where $\overline{\overline{V}} = \int_{-\infty}^{\infty} \rho (V - \overline{V})^2 dV, \qquad \overline{V} = \int_{-\infty}^{\infty} \rho V dV,$ (98)

are time-dependent functional, and k, a, σ are constant coefficients.

Then the controlled version of the dynamics of motion (96) is changed to

$$\dot{v} = -k\overline{v} + a\overline{\overline{v}} - \sigma \frac{\partial}{\partial v} \ln \rho \qquad (99)$$

while *f* represents the information forces that play the role of ***internal*** actuator.

Let us notice that the internal actuator (97) is a particular case of the information force (63) at

$$c_0 = -k\bar{v} + a\bar{\bar{v}}, \quad c_1 = 0, \quad c_2 = \sigma, \quad c_3 = 0 \tag{100}$$

while C_0 is a functional.

For a closure, Eq. (99) is complemented by the corresponding Liouville equation

$$\frac{\partial \rho}{\partial t} = k\bar{V}\frac{\partial \rho}{\partial V} - a\bar{\bar{V}}\frac{\partial \rho}{\partial V} + \sigma\frac{\partial^2 \rho}{\partial V^2}, \tag{101}$$

to be solved subject to sharp initial condition

$$\rho_0(V) = \delta(V)\, at \qquad t = 0, \tag{102}$$

As shown above, the solution of Eq. (99) is random, while this randomness is controlled by Eq. (101). Therefore in order to describe it, we have to transfer to the mean values $\bar{v}$ and $\bar{\bar{v}}$. For that purpose, let us multiply Eq.(101) by V .Then integrating it with respect to V over the whole space, one arrives at ODE for the expectation $\bar{v}(t)$

$$\dot{\bar{v}} = -k\bar{v} + a\bar{\bar{v}} \tag{103}$$

Multiplying Eq.(101) by V^2, then integrating it with respect to V over the whole space, one arrives at ODE for the variance $\bar{\bar{v}}(t)$

$$\dot{\bar{\bar{v}}} = -2k\bar{\bar{v}} + 2a\bar{v}\,\bar{\bar{v}} + 2\sigma \tag{104}$$

Let us find fixed points of the system (93) and (94) by solving the system of algebraic equations:

$$0 = -k\bar{v} + a\bar{\bar{v}} \tag{105}$$

$$0 = -2k\bar{\bar{v}} + 2a\bar{v}\,\bar{\bar{v}} + 2\sigma \tag{106}$$

By selecting

$$\sigma = \frac{k^3}{2a^2} \tag{107}$$

we arrive at the following single fixed point

$$\bar{v}^* = \frac{k}{2a}, \quad \bar{\bar{v}}^* = \frac{k^2}{2a^2} \tag{108}$$

In order to establish whether this fixed point is an attractor or a repeller, we have to analyze stability of the homogeneous version of the system (103), (104) linearized with respect to the fixed point (108)

$$\dot{\bar{v}} = -k\bar{v} + a\bar{\bar{v}} \tag{109}$$

$$\dot{\bar{\bar{v}}} = -k\bar{\bar{v}} + \frac{k^2}{a}\bar{v} \tag{110}$$

Analysis of its characteristic equation shows that it has non-positive roots:

$$\lambda_1 = 0, \quad \lambda_2 = -2k < 0$$

and therefore, the fixed point (108) is a stochastic attractor with stationary mean and variance. However the higher moments of the probability density are not necessarily stationary: they can be found from the original PDE (101).

Thus as a result of a ***feedback*** control, an ***isolated*** dynamical system (96) that prior to control was at rest, moves to the stochastic attractor (108) having the expectation $\bar{v}^*$ and the variance $\bar{\bar{v}}^*$.

8. Intelligence of self-controlled systems.

The distinguished property of the self-controlled system introduced above definitely fits into the concept of intelligence. Indeed, the evolution of intelligent living systems is directed toward the highest levels of complexity if the complexity is measured by an irreducible number of different parts that interact in a well-regulated fashion. At the same time, the solutions to the models based upon dissipative Newtonian dynamics eventually approach attractors where the evolution stops while these attractors dwell on the subspaces of lower dimensionality, and therefore, of the lower complexity (until a "master" reprograms the model). Therefore, such models fail to provide an autonomous progressive evolution of intelligent systems (i.e. evolution leading to increase of complexity). At the same time, self-controlled systems can create their own complexity based only upon an *internal* effort.

Thus the actual source of intelligent behavior of the systems introduced above is a new type of force - the information force - that contributes its work into the Law of conservation of energy. However this force is internal: it is generated by the system itself with help of the Liouvile equation. The machinery of the intelligence is similar to that of control system with the only difference that control systems are driven by external actuators while the intelligent particle is driven by a feedback from the Liouvile equation without any external resources. New modification of intelligent systems that lead to modeling decisions based upon intuition and utilizing interference of probabilities are introduced in [5].

Remark. It should be noticed that the work of the information forces (17), (42), (48), (63), and (79) could be attributed to internal energy of specific processes in livings in the same way in which the work of the quantum potential is attributed to energy of spin, [18]. In case of livings, the substructure that similar to spin in quantum mechanics could be presented in the form of a Boolean model of mind, introduced in [19]. As shown there, this model is coupled with the original self-control dynamics in the master-slave manner, and it can accumulate or release energy. Actually this property "saves" the law of conservation of energy thereby making the self-controlled ODE presentable in the form of dynamical systems.

9. Self-controlled dynamics for advanced computing.

9.1. Combinatorial optimization. Combinatorial problems are among the hardest in the theory of computations. They include a special class of so called NP-complete problems, which are considered to be intractable by most theoretical computer scientists. A typical representative of this class is a famous traveling-salesman problem (TSP) of determining the shortest closed tour that connects a given set of *n* points on the plane. As for any of NP-complete problem, here the algorithm for solution is very simple: enumerate all the tours, compute their lengths, and select the shortest one. However, the number of tours is proportional to *n!* and that leads to exponential growth of computational time as a function of the dimensionality *n* of the problem, and therefore, to computational intractability. It should be noticed that, in contradistinction to continuous optimization problems where the knowledge about the length of a trajectory is transferred to the neighboring trajectories through the gradient, here the gradient does not exist, and there is no alternative to a simple enumeration of tours.

The class of NP-complete problems has a very interesting property: if any single problem (including its worse case) can be solved in polynomial time, then every NP-complete problem can be solved in polynomial time as well. But despite that, there is no progress so far in removing a curse of combinatorial explosion: it turns out that if one manages to achieve a polynomial time of computation, then the space or energy grow exponentially, i.e., the effect of combinatorial explosion stubbornly reappears. That is why the intractability of NP-complete problems is being observed as a fundamental principle of theory of computations, which plays the same role as the second law of thermodynamics in physics.

At the same time, one has to recognize that the theory of computational complexity is an attribute of a digital approach to computations, which means that the monster of NP-completeness is a creature of the Turing machine. As an alternative, one can turn to an analog device, which replaces digital computations by physical simulations. Indeed, assume that one found such a physical phenomenon whose mathematical description is equivalent to that of a particular NP-complete problem. Then, incorporating this phenomenon into an appropriate analog device one can simulate the corresponding NP-complete problem. In this connection it is interesting to note that, at first sight, NP-complete problems are fundamentally different from natural phenomena: they look like man-made puzzles and their formal mathematical framework is mapped into decision problems with yes/no solutions. However, one should recall that physical laws could also be stated in a "man-made" form: the least time (Fermat), the least action (in modifications of Hamilton, Lagrange, or Jacobi), and the least constraints (Gauss). Moreover self-controlled systems under consideration have the direct link to model of livings. So may be this is the answer to the problem?

Finally let us quote a recent statement posed in [13]: Can **NP-complete problems** be solved efficiently in the ***physical universe?*** The answer given by the author, Scott Aaronson, is negative. To our opinion, it could be positive if we complement the "physical world" with self-controlled systems capable to violate the second law of thermodynamics in order to find short cuts to solutions of combinatorial problems.

9.2. Search in unsorted database.

In this section we will apply the self-controlled dynamics to solve the problem of search in unsorted database by improving the Grover's algorithm. Grover's algorithm is a quantum algorithm that finds with high probability the unique input to a black box function that produces a particular output value, using just O($N^{1/2}$) evaluations of the function, where N is the size of the function's domain. It was originated by Lov Grover in 1996. The problem can be illuminated by a trivial example: find the address of a telephone subscriber in a telephone book based upon his telephone number. The analogous problem in classical computation cannot be solved in fewer than O(N) evaluations (because, in the worst case, the Nth member of the domain might be the correct member). Unlike other quantum algorithms, which may provide exponential speedup over their classical counterparts, Grover's algorithm provides only a quadratic speedup. However, even quadratic speedup is considerable when N is large. At roughly the same time that Grover published his algorithm, it was proved that no quantum solution to the problem can evaluate the function fewer than O($N^{1/2}$) times, so Grover's algorithm is asymptotically optimal [14]. However that proof does not contradict our claim to improve the Grover algorithm since the self-controlled dynamics is neither quantum nor Newtonian: it represents a quantum-classical hybrid capable to violate the second law of thermodynamics. and therefore, it does not belong to the world of physics as we know it, [5].

9.3. Non-homogeneous self-controlled system with diffusion feedback for solving NP-complete problem.

Following [15], let us introduce an inhomogeneous version of Eq. (18)

$$\dot{v} = -\frac{1}{\rho}[a^2 \frac{\partial \rho}{\partial v} - e^{-\omega t} \sum_{k=1}^{k=m} \frac{l}{2\pi k} \sin \frac{2\pi k}{l} v], 0 \leq v \leq l, t > 0, \quad (111)$$

Then the corresponding Liouville equation takes the form of an inhomogeneous parabolic equation subject to an aperiodic force

$$\frac{\partial \rho}{\partial t} - a^2 \frac{\partial^2 \rho}{\partial V^2} = e^{-\omega t} \sum_{k=1}^{k=m} \cos \frac{2\pi k}{l} v \quad (112)$$

It should be noticed that the sums in Eqs. (111) and (112) are finite, and they do not represent even truncated Fourier expansions, while all the harmonic terms are equally powerful. Obviously this system is still self-supervising, but ***not*** isolated any more.

We will solve this equation subject to the following initial and boundary conditions

$$\rho(v,0) = \delta(v = 0.5l), \qquad \frac{\partial \rho}{\partial V}(0,t) = 0, \qquad \frac{\partial \rho}{\partial V}(l,t) = 0 \quad (113)$$

and the normalization constraint

$$\int_0^l \rho(\zeta,t) d\zeta = 1 \quad (114)$$

Before writing down the solution, we will verify satisfaction of the constraint (114). For that purpose, let us integrate Eq. (112) with respect to *v*

$$\frac{\partial}{\partial t}\int_0^l \rho d\zeta - a^2\int_0^l \frac{\partial^2\rho}{\partial V^2} = e^{-\omega t}\sum_{k=1}^{k=m}\int_0^l \cos\frac{2\pi k}{l}\zeta d\zeta \qquad (115)$$

As follows from the boundary conditions in (113),

$\frac{\partial\rho}{\partial V}|_{V=0} = \frac{\partial\rho}{\partial V}|_{V=l} = 0$, and therefore, $\int_0^l \frac{\partial^2\rho}{\partial V^2}d\zeta = 0$; obviously, $\int_0^l \cos\frac{2\pi k}{l}\zeta d\zeta = 0$ as well.

Hence, $\frac{\partial}{\partial t}\int_0^l \rho d\zeta = 0$. But, according to the initial condition in (113) $\int_o^l \rho d\zeta = \int_o^l \delta(\zeta = 0.5l)d\zeta = 1$ at *t=0.*

Therefore, the normalization constraint will be satisfied for all $t \geq 0$
Exploiting the superposition principle for the linear equation (112), we will represent the solution as a sum of free and forced components. These components are, respectfully

$$\rho_1 = \frac{2}{l}\sum_{j=1}^{\infty} e^{-(\frac{\pi j}{l})^2 a^2 t}\cos\frac{\pi j}{l}V\cdot\cos\frac{\pi j}{2l} + \frac{1}{l} \qquad (116)$$

$$\rho_2 = \sum_{k=1}^{m}[\int_0^t e^{-(\frac{\pi k}{l})^2 a^2 (t-\tau)-\omega t}d\tau]\cos\frac{2\pi k}{l}V \quad if \quad \omega \neq (\frac{\pi k}{l})^2 a^2 \qquad (117)$$

$$\rho_2{*} = te^{-\omega t}\cos\frac{2\pi k}{l}V \quad if \quad \omega = (\frac{\pi k}{l})^2 a^2 \qquad (118)$$

Here we will be interested only in the case (118) that represents a resonance between two aperiodic terms, namely: exponentially decaying force and exponentially decaying free motion. Indeed, the solution (118) has a well-pronounced maximum at

$$t{*} = 1/\omega \quad if \quad \omega = (\frac{\pi k}{i})^2 a^2 \qquad (119)$$

while the solutions (116) and (117) are monotonously decay.

Let us now reaffirm the scenario of transition from deterministic to random state described by Eqs. (21), (22), and(23). For that purpose, rewrite Eq. (116) in a different, but an equivalent form (based upon reflections from the boundaries)

$$\rho_1 = \frac{1}{2a\sqrt{\pi t}}\sum_{n=-\infty}^{\infty}\{e^{-\frac{(V-\zeta+2nl)^2}{4a^2t}} + e^{-\frac{(V+\zeta-2nl)^2}{4a^2t}}\} \qquad (120)$$

It can be verified that for vanishingly small times

$$\rho_1 \to \frac{1}{2a\sqrt{\pi t}}\exp(-\frac{V^2}{4a^2t}), \quad and \quad \rho_2{*} \to 0 \quad at \quad t \to 0 \qquad (121)$$

and therefore, the transition scenario remains the same.

It should be noticed that prior to running Eq. (111), the analytical solution of Eq. (112) in the form of the sum of Eqs. (116), (117), and (120) is to be substituted for *ρ.*

Turning to *n*-dimensional case we have (122)

$$\frac{\partial\rho}{\partial t} - a^2 \sum_{i=1}^{n} \frac{\partial^2 \rho}{\partial V_i^2} = e^{-\omega t} \sum_{i=1}^{n} \sum_{k=1}^{m} \cos\frac{2\pi k}{l_i} V_i, \tag{123}$$

$$i = 1, 2, \ldots n, \quad k = 1, 2 \ldots m.$$

Eq. (123) has the following eigen-values of decay

$$\omega_{j_{k_1},\ldots j_{k_n}} = (a^2\pi^2 \sum_{i=1}^{i=n} \frac{j_{k_i}{}^2}{l_i^2}),\ j = 1,2,\ldots n, i = 1,2,\ldots n, k = 1,2\ldots m \tag{124}$$

If the excitation ω in Eqs. (122) and (123) are selected as following

$$\omega = \omega_{j_{k_1},\ldots j_{k_n}} = (a^2\pi^2 \sum_{i=1}^{i=n} \frac{j_{k_i}^2}{l_i^2}),\ j = 1,2,\ldots n, i = 1,2,\ldots n, k = 1,2\ldots m \tag{125}$$

it will generate resonance with the eigen-value (124), and the corresponding "decay" will dominate over the rest of decays; in terms of Eq. 123) this means that the probability density ρ will tend to its maximum at

$$t^* = 1/\omega_{j_{k_1},\ldots j_{k_n}} = (a^2\pi^2 \sum_{i=1}^{i=n} \frac{j_{k_i}{}^2}{l_i^2})^{-1}, \tag{126}$$

along that trajectory which is the "winning" solution of the system (122), Fig. 6. The value of this maximum is irrelevant, but its location is important: it is given by the following values of the coordinates

$$v_i^* = v_i(t^*), i = 1,2,\ldots n. \tag{127}$$

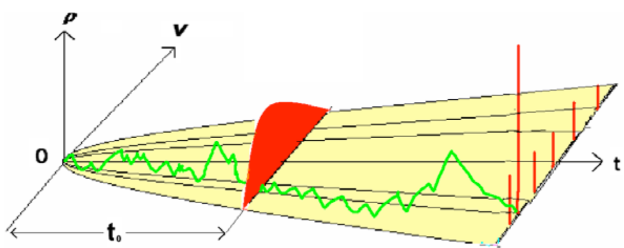


Figure 6. Resonance in the probability space

The algorithm is formulated as the following. Consider an unsorted data-base consisting of n^n items labeled with a string of numbers $j_1, j_2, \ldots j_n$ as shown in Eq.(125) for *n=m*. Obviously a label includes permutations of these numbers. Turning to Eqs. (122), notice that each solution to this system can be labeled similarly if the winning solution in Eqs. (123) has its maximum at a point with the coordinates $v_1^*, v_2^*, \ldots v^*_n$ defined by Eqs (126). Then one can introduce the forced excitation defined by Eq. (125) that provides the resonant solution of Eq. (123), and as a result, the coordinates $v_1^*, v_2^*, \ldots v^*_n$ of this maximum will represent the address of the item in question. According to Eq. (125), the number of possible values of forced excitations ω providing required resonances is equal to n^n, and that is exactly the number of the items to be retrieved. Therefore, each item can be retrieved by the corresponding resonance with the forced excitation (having the values from the set (55)) with the probability that dominates over the probabilities for wrong addresses to occur. Strictly speaking, a non-resonance solution has a smaller, but non-zero probability to occur; then by a few number of Bernoulli trials, the most probable solution can be found. Indeed, the probability of success ρ_s and failure ρ_f after the first trial is, respectively

$$\rho_s = \overline{\psi}_1, \qquad \rho_f = 1 - \overline{\psi}_1 \tag{128}$$

Then the probability of success after M trials is

$$\rho_{sM} = 1 - (1 - \overline{\psi})^M \to 1 \qquad at \qquad M \to \infty \tag{129}$$

Therefore, after *polynomial* number of trials, one arrived at the solution to the problem (unless the function ψ is flat).

Let us now briefly review the procedure of the retrieval. Assume that the label of the item to be found is $\omega_{j_{k_1}} \ldots j_{k_n}$.

The first step is to write down the analytical solution to Eq. (123) that consists of free and forced motions as in the one-dimensional case:

$$\rho = \rho_1 + \rho_2 * \tag{130}$$

Here

$$\rho_1 = \sum_{j=1}^{\infty} C_j e^{-(\frac{\pi j}{l})^2 a^2 t}, \tag{131}$$

$$\rho_2 * = te^{-\omega^* t} \sum_{i=1}^{n} \sum_{k=1}^{m} \cos \frac{2\pi k}{l_i} V_i \tag{132}$$

where C_j are constants to be found from the initial conditions, and

$$\omega^* = \omega_{j_{k_1},\ldots j_{k_n}} = (a^2 \pi^2 \sum_{i=1}^{i=n} \frac{j_{k_i}^2}{l_i^2}), j = 1,2,\ldots n, i = 1,2,\ldots n, k = 1,2\ldots m \tag{133}$$

The second step is to substitute the solution (130) into Eq. (122). The third step is to run the system (122), measure the values of v_i at $t = 1/\omega^*$ and obtain the address of the item in the form of a string of coordinates $v_1^*, v_2^*, \ldots v^*_n$, Fig. 7.

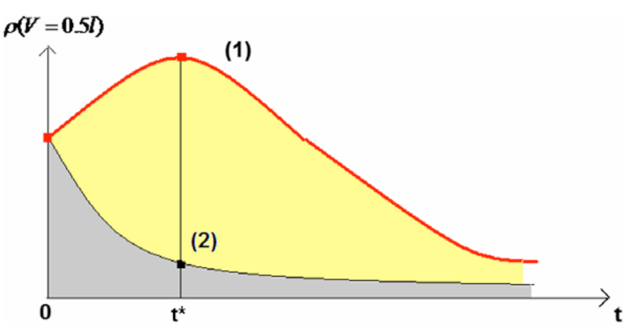


Figure 7. Maximum probability, selected (1) and not selected (2) items.

It should be noticed that the capacity of the unsorted database is of order $O(n^n)$ i.e. exponential with respect to its dimensionality *n*, while all the resources providing its implementation are of order *O(n)*, i.e. polynomial since the number of equations in the system (122) is *n*, and the number of terms in the analytical solution to Eq.(123) (to be substituted into Eqs. (122)) are of the order *O(n)* as well. Indeed, the infinite sum in Eq. (131) converges very fast to equal distribution of the probability density, and practically, only the forced component of the solution represented by Eq. (132) is important, and this component contains $O(n^2)$ number of terms.

10. Discussion and conclusion

A new class of dynamical system described by ODE coupled with their Liouville equation has been introduced, discussed and illustrated. These systems called self-controlled, or self-supervised since the role of actuators is played by the probability produced by the Liouville equation. Following the Madelung equation that belongs to this class, non-Newtonian properties such as randomness, entanglement, and probability interference typical for quantum systems have been described. Special attention was paid to the capability to violate the second law of thermodynamics, which makes these systems neither Newtonian, nor quantum. It has been shown that self-controlled dynamical systems can be linked to mathematical models of livings as well as to models of AI. The central point of this paper is the application of the self-controlled systems to NP-complete problems known as being unsolvable neither by classical nor by quantum algorithms. The approach is illustrated by solving a search in unsorted database in polynomial time by resonance between external force representing the address of a required item and the response representing location of this item. The basic idea of this approach is to create a new kind of dynamical systems that would preserve superposition of random solutions, while allowing one to measure its state variables using classical methods. In other words, such a hybrid system would reinforce the advantages and minimize limitations of both quantum and classical aspects. These systems have been analyzed in [5] and [16]. It has been shown there that along with preservation of superposition, such an important property of quantum systems as direct-product-decomposability in hybrids is lost. Let us recall that the main advantage of this property in terms of quantum information is in blowing up an input of a polynomial complexity into an output of exponential complexity, with no additional resources required, Fig. 8.

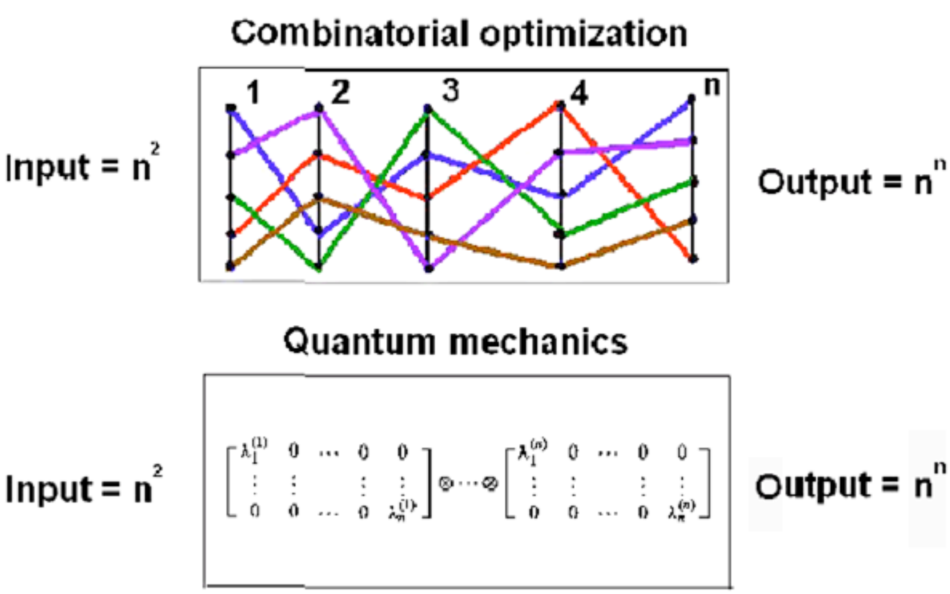


Figure 8. Mapping combinatorial optimization to quantum mechanics.

The challenge of our approach was in finding a ‘‘replacement” for the fundamental property of the Schrödinger equation in quantum-classical hybrids. It turns out that eigen-values of linear parabolic PDE possess similar property. Indeed, consider a linear n-dimensional parabolic PDE subject to boundary conditions. Then the eigen-values corresponding to each variable form a sequence of monotonously increasing positive numbers $\lambda_i^{(1)}...\lambda_i^{(n)}$.

However, each linear combination of these eigen-values represents another eigen-value of the solution, and that is the same ‘‘combinatorial explosion” that is illustrated in Fig. 8. Due to that property, for each n-string-number label, one can find an excitation force that activates the corresponding eigen-value. The second challenge was to satisfy a global (normalization) constraint imposed upon the probability density (in addition to boundary conditions). That was achieved via a special form of the excitation force. Finally these work ads a positive comment to a question posed in [7]: Can **NP-complete problems** be solved efficiently in the physical universe? The answer given by the author, Scott Aaronson, is negative. To our opinion, it could be positive if we complement the “physical world” with self-supervised systems capable to violate the second law of thermodynamics in order to find short cuts to solutions of combinatorial problems.

References.

1. Madelung, E. (1927). "Quantentheorie in hydrodynamischer Form". *Z. Phys.* **40**: 322–326

2. Zak, M., Randomness in physics revisited, Int. J. of quantum foundations, **2** (2016) 47 – 66.

3. Zak, M., Physics of Life from First Principles, EJTP 4, No. 16(II) (2007) 11–96 Electronic Journal of Theoretical Physics

4. Zak, M., Introduction to quantum-inspired intelligence, PHYSICS ESSAYS **23,** 1 2010

5. Zak,M., Non-Newtonian aspects of artificial intelligence. Found Phys , Vol.46, May 2016,

6. Zak, M., Entanglement in Livings, Journal of Quantum Information Science, 2012, 2, 66-77

7. Zak, M., "Terminal Model of Newtonian Dynamics," Int. J. of Theoretical Physics, No.32, 159-190, 1992

8. Risken,H., 1989, The Fokker-Planck Equation, Springer, NY.

9. Zak ,M., Quantum evolution as a nonlinear Markov process, Foundations of Physics Letters, vol. 15. No. 3, 2002

10. Whitham,G., *Linear and Nonlinear Waves* Wiley-Interscience, New York, 1974

11. Zak, M., Interference of probabilities in dynamics, AIP Advances **4**, 087130 (2014)

12. Zabusky N. J and Kruskal M. D, "Interaction of "Solitons" in a Collisionless Plasma and the Recurrence of Initial States", *Phys. Rev. Lett.*, **15** (6): 240–243, (1965

13. Aaronson, S., NP-complete Problems and Physical Reality, Institute for Advanced Study, Princeton,NJ

14. Bennett C.H., Bernstein E., Brassard G., Vazirani U. (1997). "The strengths and weaknesses of quantum computation". *SIAM Journal on Computing*. 26(5): 1510–1523

15. Zak, M,. Quantum-inspired resonance for associative memory, Chaos, Solitons and Fractals 41 (2009) 2306–2312

16. Zak,M.,Quantum-inspired maximizer, JOURNAL OF MATHEMATICAL PHYSICS **49,** 042702 ,2008

17. Landay, L. (1997), Quantum mechanics, Butterworth, 1997.

18. G. Salesi: *Spin and Madelung fluid*, submitted 23 June 2009, arXiv:quant-ph/0906.4147v1

19. Zak,M., (2016) Toward Quantum-inspired Model of Human Mind, Journal of Quantum Information Science, 4, 22-43