On The Exponential Mangoldt Function

ANTHONY J. BROWNE

June 14, 2016

ABSTRACT

A form of the exponential Mangoldt function is derived using indicator functions. The function's relationship to other important number theoretic functions are derived and discussed.

The Mangoldt function, $\wedge(n)$ is defined as

$$\wedge(n) = \begin{cases} \ln(p) \ \ if\ n = p^k,\ k \geq 1 \\ 0\ otherwise \end{cases}.$$

So, by the property of the natural logarithm, the exponential Mangoldt function is defined as

$$e^{\wedge(n)} = \begin{cases} p\ if\ n = p^k, \quad k \geq 1 \\ 1\ otherwise \end{cases}$$

The function can be represented as a sum using indicator functions as I shall prove in the following theorem.

Theorem:

$$e^{\wedge(n)} = 1 + \sum_{j=2}^{n} (j-1) \left(\left\lfloor \frac{(j-1)!+1}{j} \right\rfloor - \left\lfloor \frac{(j-1)!}{j} \right\rfloor \right) \left(\left\lfloor \frac{j^n}{n} \right\rfloor - \left\lfloor \frac{j^n - 1}{n} \right\rfloor \right).$$

Proof:

The two expressions within parenthesis that are a difference of floor functions are indicator functions of the form

$$\left\lfloor \frac{f}{g} \right\rfloor - \left\lfloor \frac{f-1}{g} \right\rfloor = \mathbf{1}_{g|f} = \begin{cases} 1\ \ if\ g|f \\ 0\ otherwise \end{cases}.$$

We can use this form to define the first indicator function in the sum, $\left\lfloor \frac{(j-1)!+1}{j} \right\rfloor - \left\lfloor \frac{(j-1)!}{j} \right\rfloor$. If we define f to be $(j-1)!+1$ and g to be j, then we have

$$\left\lfloor\frac{(j-1)!+1}{j}\right\rfloor-\left\lfloor\frac{(j-1)!}{j}\right\rfloor=\mathbf{1}_{j|(j-1)!+1.}$$

Wilson's theorem states $(n-1)!=-1\ (mod\ n)$. Using this we have

$$\left\lfloor\frac{(j-1)!+1}{j}\right\rfloor-\left\lfloor\frac{(j-1)!}{j}\right\rfloor=\mathbf{1}_{j|(j-1)!+1}=\mathbf{1}_p=\begin{cases}1\ if\ n=p\ prime\\ 0\ otherwise\end{cases}$$

which is the characteristic function of the primes $\mathbf{1}_p$. For the other indicator function in the sum $\left\lfloor\frac{j^n}{n}\right\rfloor-\left\lfloor\frac{j^n-1}{n}\right\rfloor=\mathbf{1}_{n|j^n}$, note that the exponents in the prime factorization of j^n are going to be larger than the exponents of the prime factorization of n, because j is raised to the power of n. So, only the prime factors of n and j matter in whether or not $n|j^n$. Considering the factors, n can only divide j^n if j is a multiple of $rad(n)$, the product of the distinct prime factors of n, which is defined as $\prod_{p|n}p$.

From this we have

$$\left\lfloor\frac{j^n}{n}\right\rfloor-\left\lfloor\frac{j^n-1}{n}\right\rfloor=\mathbf{1}_{n|j^n}=\mathbf{1}_{j=k\cdot rad(n)}=\begin{cases}1\ if\ j\ is\ a\ multiple\ of\ rad(n)\\ 0\ otherwise\end{cases}.$$

Thus, for the two indicator functions multiplied together in the sum we have

$$\left(\left\lfloor\frac{(j-1)!+1}{j}\right\rfloor-\left\lfloor\frac{(j-1)!}{j}\right\rfloor\right)\left(\left\lfloor\frac{j^n}{n}\right\rfloor-\left\lfloor\frac{j^n-1}{n}\right\rfloor\right)=\mathbf{1}_{j=k\cdot rad(n)}\cdot\mathbf{1}_{j=p}.$$

Summing this from 2 to n we have

$$\sum_{j=2}^{n}\left(\left\lfloor\frac{(j-1)!+1}{j}\right\rfloor-\left\lfloor\frac{(j-1)!}{j}\right\rfloor\right)\left(\left\lfloor\frac{j^n}{n}\right\rfloor-\left\lfloor\frac{j^n-1}{n}\right\rfloor\right)=\mathbf{1}_{p^k}=\begin{cases}1\ if\ n=p^k,\ k\geq 1\\ 0\quad otherwise\end{cases}.$$

So, if we multiply the argument of this sum by j we have

$$\sum_{j=2}^{n}j\left(\left\lfloor\frac{(j-1)!+1}{j}\right\rfloor-\left\lfloor\frac{(j-1)!}{j}\right\rfloor\right)\left(\left\lfloor\frac{j^n}{n}\right\rfloor-\left\lfloor\frac{j^n-1}{n}\right\rfloor\right)=p\cdot\mathbf{1}_{p^k}=\begin{cases}p\quad if\ n=p^k,\quad k\geq 1\\ 0\quad otherwise\end{cases}.$$

This is just the exponential Mangoldt function with value 0 instead of 1 at non-prime powers. Thus, adding the characteristic function of non-prime powers will give the exponential Mangoldt function. But, this is just $1-\mathbf{1}_{p^k}$. Thus, using the two sums directly above we have

$$e^{\Lambda(n)}=p\cdot\mathbf{1}_{p^k}+1-\mathbf{1}_{p^k}=1+\sum_{j=2}^{n}(j-1)\left(\left\lfloor\frac{(j-1)!+1}{j}\right\rfloor-\left\lfloor\frac{(j-1)!}{j}\right\rfloor\right)\left(\left\lfloor\frac{j^n}{n}\right\rfloor-\left\lfloor\frac{j^n-1}{n}\right\rfloor\right).$$

End Proof.

We can also restrict this function from including any powers of 2 by beginning the sum at 3 instead of 2.

$$1+\sum_{j=3}^{n}(j-1)\left(\left\lfloor\frac{(j-1)!+1}{j}\right\rfloor-\left\lfloor\frac{(j-1)!}{j}\right\rfloor\right)\left(\left\lfloor\frac{j^n}{n}\right\rfloor-\left\lfloor\frac{j^n-1}{n}\right\rfloor\right)=e^{\Lambda(n)}\ (for\ odd\ primes).$$

It is also interesting to sum the Mangoldt function over indicator functions. Many important number theoretic functions can be derived. I will demonstrate this in the following theorems.

Theorem:

$$e^{\sum_{j=2}^{n}\left(\left\lfloor\frac{n}{j}\right\rfloor-\left\lfloor\frac{n-1}{j}\right\rfloor\right)\left(\left\lfloor\frac{(j-1)!+1}{j}\right\rfloor-\left\lfloor\frac{(j-1)!}{j}\right\rfloor\right)\Lambda(j)}=rad(n).$$

Proof:

$$e^{\sum_{j=2}^{n}\left(\left\lfloor\frac{n}{j}\right\rfloor-\left\lfloor\frac{n-1}{j}\right\rfloor\right)\left(\left\lfloor\frac{(j-1)!+1}{j}\right\rfloor-\left\lfloor\frac{(j-1)!}{j}\right\rfloor\right)\Lambda(j)}=e^{\sum_{j=2}^{n}\Lambda(j)\cdot\mathbf{1}_{j|n}\cdot\mathbf{1}_{j=p}}=\prod_{p|n}e^{\Lambda(p)}=\prod_{p|n}p=rad(n).$$

End Proof.

Theorem:

$$e^{\sum_{j=2}^{n}\left(\left\lfloor\frac{n}{j}\right\rfloor-\left\lfloor\frac{n-1}{j}\right\rfloor\right)\left(1-\left\lfloor\frac{(j-1)!+1}{j}\right\rfloor+\left\lfloor\frac{(j-1)!}{j}\right\rfloor\right)\Lambda(j)}=\frac{n}{rad(n)}.$$

Proof:

$$e^{\sum_{j=2}^{n}\left(\left\lfloor\frac{n}{j}\right\rfloor-\left\lfloor\frac{n-1}{j}\right\rfloor\right)\left(1-\left\lfloor\frac{(j-1)!+1}{j}\right\rfloor+\left\lfloor\frac{(j-1)!}{j}\right\rfloor\right)\Lambda(j)}=e^{\sum_{j=2}^{n}\Lambda(j)\cdot\mathbf{1}_{j|n}\cdot(1-\mathbf{1}_{j=p})}=\prod_{(c\neq p,c>1)|n}e^{\Lambda(c)}=\frac{n}{rad(n)}.$$

End Proof.

From this we can see that the multiplication of these two functions must be n. Testing this we have

$$\left(\prod_{p|n}e^{\Lambda(p)}\right)\left(\prod_{(c\neq p,c>1)|n}e^{\Lambda(c)}\right)=\prod_{(c,p,\ c>1)|n}e^{\Lambda(p)+\Lambda(c)}=\prod_{(d,\ d>1)|n}e^{\Lambda(d)}=n.$$

As can be seen, this is a product of the prime factors of n with multiplicity, the factorization.

The function $\frac{n}{rad(n)}$ can be calculated simply as $\sum_{j=1}^{n}\left(\left\lfloor\frac{j^n}{n}\right\rfloor-\left\lfloor\frac{j^n-1}{n}\right\rfloor\right)=\frac{n}{rad(n)}$. So, this is not meant for calculation efficiency purposes, the author only wishes to present new relationships with the Mangoldt function using indicator functions.

We can also derive the product of the primes $< n$ that do not divide n.

$$e^{\sum_{j=2}^{n}\left(1-\left\lfloor\frac{n}{j}\right\rfloor+\left\lfloor\frac{n-1}{j}\right\rfloor\right)\left(\left\lfloor\frac{(j-1)!+1}{j}\right\rfloor-\left\lfloor\frac{(j-1)!}{j}\right\rfloor\right)\Lambda(j)}=e^{\sum_{j=2}^{n}\Lambda(j)\cdot(1-\mathbf{1}_{j|n})\cdot\mathbf{1}_{j=p}}=\prod_{p\nmid n}e^{\Lambda(p)}.$$

Evaluating this function at the prime sequence, p_n gives the primorial numbers $p\#_n$.

$$e^{\sum_{j=2}^{p_n}\left(1-\left\lfloor\frac{p_n}{j}\right\rfloor+\left\lfloor\frac{p_n-1}{j}\right\rfloor\right)\cdot\left(\left\lfloor\frac{(j-1)!+1}{j}\right\rfloor-\left\lfloor\frac{(j-1)!}{j}\right\rfloor\right)\Lambda(j)}=e^{\sum_{j=2}^{n}\Lambda(j)\cdot(1-\mathbf{1}_{j|p_n})\cdot\mathbf{1}_{j=p}}=e^{\sum_{j=2}^{n}\Lambda(j)\cdot\mathbf{1}_{j=p}}=p\#_n.$$

As one may now see, indicator functions can be used to derive many interesting relationships involving the Mangoldt function. Consider the indicator function $\left(\left\lceil\frac{n^j}{j}\right\rceil-\left\lfloor\frac{n^j}{j}\right\rfloor\right)=\mathbf{1}_{j|n^j}$. This function is 1 if the set of prime factors of j are a subset of the set of prime factors of n and 0 otherwise. Interesting relationships can be derived when the Mangoldt function is summed over this indicator function.

$$e^{\sum_{j=1}^{n}\left(\left\lceil\frac{n^j}{j}\right\rceil-\left\lfloor\frac{n^j}{j}\right\rfloor\right)\cdot\Lambda(j)}=e^{\sum_{j=1}^{n}\mathbf{1}_{j|n^j}\cdot\Lambda(j)}=GCD(\,n!,n^n,LCM(1,2,\ldots,n)\,)$$

where LCM denotes the least common multiple and GCD denotes greatest common divisor.

This is sequence A064446 in the Online Encyclopedia of Integer Sequences, the OEIS.

Summing the Mangoldt function over the compliment of this indicator function, $1-\mathbf{1}_{j|n^j}$ gives

$$e^{\sum_{j=1}^{n}\left(1-\left\lceil\frac{n^j}{j}\right\rceil+\left\lfloor\frac{n^j}{j}\right\rfloor\right)\Lambda(j)}=e^{\sum_{j=1}^{n}(1-\mathbf{1}_{j|n^j})\cdot\Lambda(j)}=LCM\ of\ integers < and\ prime\ to\ n.$$

This is sequence A038610 in the OEIS.

The product of these two functions gives

$$e^{\sum_{j=1}^{n}\left(\left\lceil\frac{n^j}{j}\right\rceil-\left\lfloor\frac{n^j}{j}\right\rfloor\right)\Lambda(j)}\cdot e^{\sum_{j=1}^{n}\left(1-\left\lceil\frac{n^j}{j}\right\rceil+\left\lfloor\frac{n^j}{j}\right\rfloor\right)\Lambda(j)}=LCM(1,2,\ldots,n)=e^{\psi(n)}.$$

Which is of course the exponential of the second Chebyshev function.

This is sequence A003418 in the OEIS.

The power of indicator functions is of no surprise when you consider that there is absolutely no difference between sums over divisors and sums over $\mathbf{1}_{j|n}$. In other words, these two functions are identical,

$$\sum_{d|n} f(d) = \sum_{j=1}^{n} \left(\left\lfloor \frac{n}{j} \right\rfloor - \left\lfloor \frac{n-1}{j} \right\rfloor \right) f(j).$$

Dirichlet convolution can be defined with indicator functions as well.

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right) = \sum_{j=1}^{n} \left(\left\lfloor \frac{n}{j} \right\rfloor - \left\lfloor \frac{n-1}{j} \right\rfloor \right) f(j) g\left(\frac{n}{j}\right)$$

where $(f * g)(n)$ denotes the Dirichlet convolution of f and g at n.

Möbius inversion can be defined in the same way,

$$g(n) = \sum_{d|n} f(d) = \sum_{j=1}^{n} \left(\left\lfloor \frac{n}{j} \right\rfloor - \left\lfloor \frac{n-1}{j} \right\rfloor \right) f(j) \Rightarrow f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right) = \sum_{j=1}^{n} \left(\left\lfloor \frac{n}{j} \right\rfloor - \left\lfloor \frac{n-1}{j} \right\rfloor \right) \mu(j) g\left(\frac{n}{j}\right).$$

In conclusion, It can be seen that many interesting and possibly important relationships involving the Mangoldt function can be derived with the help of indicator functions. Further study in this area may turn up unexpected results and may give insight to potential factorization algorithms. Further study should be done in this area.