SOME DEFINITE INTEGRALS OVER A POWER MULTIPLIED BY FOUR MODIFIED BESSEL FUNCTIONS

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ABSTRACT. The definite integrals $\int_0^\infty x^j I_0^s(x) I_1^t(x) K_0^u(x) K_1^v(x) dx$ are considered for non-negative integer j and four integer exponents $s+t+u+v=4$, where I and K are Modified Bessel Functions. There are essentially 15 types of the 4-fold product. Partial integration of each of these types leads correlations between these integrals. The main result are (forward) recurrences of the integrals with respect to the exponent j of the power.

1. 15 Quadruple Products

1.1. **Classification.** The definite integrals of the form $\int_0^\infty x^j I_0^s(x) I_1^t(x) K_0^u(x) K_1^v(x) dx$ with integer exponents require $s+t \le u+v$ to converge at the upper limit, which leads to 22 4-tuples of the four integers s, t, u, and v if we exclusively consider products of 4 Bessel Functions, $s+t+u+v=4$. A further reduction of the set of the four integers s, t, u, and v to 15 cases is achieved, if any product $I_0(x)K_1(x)$ is substituted by [1, 9.6.15]

$$I_0(x)K_1(x) = \frac{1}{x} - I_1(x)K_0(x). \tag{1}$$

Out of these, the cases with $s+t=1$, which means the 5 cases with exponents $(s,t,u,v) = (1,0,3,0)$, $(0,1,3,0)$, $(0,1,2,1)$, $(0,1,1,1)$, and $(0,1,0,3)$ are considered in Section 2. The cases with $s+t=0$, which means the 5 cases with exponents $(0,0,4,0)$, $(0,0,3,1)$, $(0,0,2,2)$, $(0,0,1,4)$, and $(0,0,0,4)$ are considered in Section 3. The cases with $s+t=2$ are considered in Section 4.

To converge at the lower limit, the exponent of the power must be $j+t-v \ge 0$ according to the known limits.

1.2. **Reduction Strategy.** The generic approach in this manuscript to correlate integrals over powers and the Bessel Functions is to rise the exponent in the power and to differentiate the Bessel Functions by partial integration. The differentiation involves the product rule and [1, 9.6.26, 9.6.27]

$$I_0'(x) = I_1(x), \tag{2}$$

$$K_0'(x) = -K_1(x); \tag{3}$$

$$I_1'(x) = I_0(x) - \frac{I_1(x)}{x}, \tag{4}$$

$$K_1'(x) = -K_0(x) - \frac{K_1(x)}{x}. \tag{5}$$

Date: June 21, 2016.
2010 *Mathematics Subject Classification.* Primary 65D30; Secondary 33C10.
Key words and phrases. Modified Bessel Function; Definite Integral; Product.

We only deal with integrals in the limits between zero and infinity: the limits 0 and ∞ are usually omitted; the pre-integrated term will be indicated by triple dots to indicate if it becomes irrelevant when it vanishes once these limits are inserted. If a product $I_0(x)K_1(x)$ appears, it will be substituted via (1) to keep the types of integrals at a minimum.

The argument of the Bessel Functions is always (x) and omitted for brevity.

2. Group 1: One First Kind, Three Second Kind

2.1. Class $I_0K_0^3$.

$$\begin{aligned}\int x^j I_0K_0^3 dx &= \ldots - \int \frac{x^{j+1}}{j+1}[I_1K_0^3 - 3I_0K_0^2K_1]dx \\ &= \ldots - \int \frac{x^{j+1}}{j+1}I_1K_0^3dx + 3\int \frac{x^{j+1}}{j+1}I_0K_0^2K_1dx \\ &= \ldots - \int \frac{x^{j+1}}{j+1}I_1K_0^3dx + 3\int \frac{x^{j+1}}{j+1}(1/x - I_1K_0)K_0^2dx \\ &= \ldots - \int \frac{x^{j+1}}{j+1}I_1K_0^3dx + 3\int \frac{x^j}{j+1}K_0^2dx - 3\int \frac{x^{j+1}}{j+1}I_1K_0^3dx \\ &= \ldots + \frac{3\times 2^{j-2}}{j!(j+1)}\Gamma^4(\frac{1+j}{2}) - \frac{4}{1+j}\int x^{j+1}I_1K_0^3dx.\end{aligned} \tag{6}$$

We have replaced the integral over two Bessel Functions in the last step with [8, 6.576.4]:

Definition 1.

$$\int_0^\infty x^j K_0(x)^2 dx = 2^{j-2}\frac{1}{\Gamma(1+j)}\Gamma^4(\frac{1+j}{2}) \equiv \Gamma_{00}(j). \tag{7}$$

The first few special cases are

(8)

$$\Gamma_{00}(0) = \pi^2/4; \quad \Gamma_{00}(1) = 1/2; \quad \Gamma_{00}(2) = \pi^2/32; \quad \Gamma_{00}(3) = 1/3; \quad \Gamma_{00}(4) = 27\pi^2/512.$$

2.2. Class $I_1K_0^3$.

$$\begin{aligned}\int x^j I_1K_0^3 dx &= \ldots - \int \frac{x^{j+1}}{j+1}[I_0K_0^3 - \frac{1}{x}I_1K_0^3 - 3I_1K_0^2K_1]dx \\ &= \ldots + \int \frac{x^j}{j+1}I_1K_0^3dx - \int \frac{x^{j+1}}{j+1}[I_0K_0^3 - 3I_1K_0^2K_1]dx.\end{aligned} \tag{9}$$

Whenever an integral re-appears on the right hand side we combine it with the left hand side:

$$\frac{j}{j+1}\int x^jI_1K_0^3dx = \ldots - \int \frac{x^{j+1}}{j+1}I_0K_0^3dx + 3\int \frac{x^{j+1}}{j+1}I_1K_0^2K_1dx. \tag{10}$$

$$j\int x^jI_1K_0^3dx = \ldots - \int x^{j+1}I_0K_0^3dx + 3\int x^{j+1}I_1K_0^2K_1dx. \tag{11}$$

2.3. **Class $I_1K_0^2K_1$.**

(12)
$$\int x^j I_1K_0^2K_1 dx = \ldots - \int \frac{x^{j+1}}{j+1}[I_0K_0^2K_1 - \frac{2}{x}I_1K_0^2K_1 - 2I_1K_0K_1^2 - I_1K_0^3]dx$$
$$= \ldots - \int \frac{x^{j+1}}{j+1}[(1/x - I_1K_0)K_0^2 - \frac{2}{x}I_1K_0^2K_1 - 2I_1K_0K_1^2 - I_1K_0^3]dx$$
$$= \ldots - \int \frac{x^{j+1}}{j+1}[1/xK_0^2 - \frac{2}{x}I_1K_0^2K_1]dx - \int \frac{x^{j+1}}{j+1}[-I_1K_0K_0^2 - 2I_1K_0K_1^2 - I_1K_0^3]dx$$
$$= \ldots - \int \frac{x^j}{j+1}K_0^2dx + 2\int \frac{x^j}{j+1}I_1K_0^2K_1dx + 2\int \frac{x^{j+1}}{j+1}[I_1K_0^3 + I_1K_0K_1^2]dx.$$

Merging common integrals of both sides yields

$$\frac{j-1}{j+1}\int x^j I_1K_0^2K_1dx = \ldots - \int \frac{x^j}{j+1}K_0^2dx + 2\int \frac{x^{j+1}}{j+1}[I_1K_0^3 + I_1K_0K_1^2]dx. \tag{13}$$

Then with (7)

(14)
$$(j-1)\int x^j I_1K_0^2K_1dx = \ldots - 2^{j-2}\frac{1}{j!}\Gamma^4(\frac{1+j}{2}) + 2\int x^{j+1}I_1K_0^3dx + 2\int x^{j+1}I_1K_0K_1^2dx.$$

2.4. **Class $I_1K_0K_1^2$.**

(15)
$$\int x^j I_1K_0K_1^2dx = \ldots - \int \frac{x^{j+1}}{j+1}[I_0K_0K_1^2 - \frac{3}{x}I_1K_0K_1^2 - I_1K_1^3 - 2I_1K_0^2K_1]dx$$
$$= \ldots - \int \frac{x^{j+1}}{j+1}[(\frac{1}{x} - I_1K_0)K_0K_1 - \frac{3}{x}I_1K_0K_1^2 - I_1K_1^3 - 2I_1K_0^2K_1]dx$$
$$= \ldots - \int \frac{x^j}{j+1}[K_0K_1 - 3I_1K_0K_1^2]dx + \int \frac{x^{j+1}}{j+1}[I_1K_1^3 + 3I_1K_0^2K_1]dx.$$

Merging common integrals yields

$$\frac{j-2}{j+1}\int x^j I_1K_0K_1^2dx = \ldots - \int \frac{x^j}{j+1}K_0K_1dx + \int \frac{x^{j+1}}{j+1}[I_1K_1^3 + 3I_1K_0^2K_1]dx \tag{16}$$

With the notation [8, 6.576.4]

Definition 2.

$$\int_0^\infty x^j K_0K_1dx = 2^{j-4}\frac{j}{(j-1)!}\Gamma^4(\frac{j}{2}) \equiv \Gamma_{01}(j), \quad j > 0 \tag{17}$$

this simplifies to

(18)
$$(j-2)\int x^j I_1K_0K_1^2dx = \ldots - 2^{j-4}\frac{j}{(j-1)!}\Gamma^4(\frac{j}{2}) + \int x^{j+1}I_1K_1^3dx + 3\int x^{j+1}I_1K_0^2K_1dx.$$

The first few values of the constants defined above are

$$\Gamma_{01}(1) = \pi^2/8; \quad \Gamma_{01}(2) = 1/2; \quad \Gamma_{01}(3) = 3\pi^2/64; \quad \Gamma_{01}(3) = 2/3. \tag{19}$$

2.5. **Class $I_1K_1^3$.**

$$\int x^j I_1 K_1^3 dx = \ldots - \int \frac{x^{j+1}}{j+1}[I_0K_1^3 - \frac{4}{x}I_1K_1^3 - 3I_1K_0K_1^2]dx \tag{20}$$
$$= \ldots - \int \frac{x^{j+1}}{j+1}[(\frac{1}{x} - I_1K_0)K_1^2 - \frac{4}{x}I_1K_1^3 - 3I_1K_0K_1^2]dx$$
$$= \ldots - \int \frac{x^{j+1}}{j+1}[\frac{1}{x}K_1^2 - \frac{4}{x}I_1K_1^3]dx - \int \frac{x^{j+1}}{j+1}[-I_1K_0K_1^2 - 3I_1K_0K_1^2]dx$$
$$= \ldots - \int \frac{x^j}{j+1}K_1^2dx + 4\int \frac{x^j}{j+1}I_1K_1^3dx + 4\int \frac{x^{j+1}}{j+1}I_1K_0K_1^2dx.$$

Merging common integrals yields

$$\frac{j-3}{j+1}\int x^jI_1K_1^3dx = \ldots - \int \frac{x^j}{j+1}K_1^2dx + 4\int \frac{x^{j+1}}{j+1}I_1K_0K_1^2dx. \tag{21}$$

Definition 3. [8, 6.576]

(22)
$$\int x^jK_1(x)^2dx = 2^{j-6}\frac{(j^2-1)(j-1)}{(j-2)!j}\Gamma^4(\frac{j-1}{2}) \equiv \Gamma_{11}(j) = \frac{1+j}{j-1}\Gamma_{00}(j), \quad j > 1.$$

(23)
$$(j-3)\int x^jI_1K_1^3dx = \ldots - 2^{j-6}\frac{(j^2-1)(j-1)}{(j-2)!j}\Gamma^4(\frac{j-1}{2}) + 4\int x^{j+1}I_1K_0K_1^2dx.$$

2.6. **Synthesis.** The partial integrations of the 5 classes of the previous sections are summarized by the matrix equation

(24)
$$\begin{pmatrix} \int x^jI_0K_0^3 \\ \int x^jI_1K_0^3 \\ \int x^jI_1K_0^2K_1 \\ \int x^jI_1K_0K_1^2 \\ \int x^jI_1K_1^3 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{4}{1+j} & 0 & 0 & 0 \\ -\frac{1}{j} & 0 & \frac{3}{j} & 0 & 0 \\ 0 & \frac{2}{j-1} & 0 & \frac{2}{j-1} & 0 \\ 0 & 0 & \frac{3}{j-2} & 0 & \frac{1}{j-2} \\ 0 & 0 & 0 & \frac{4}{j-3} & 0 \end{pmatrix} \cdot \begin{pmatrix} \int x^{j+1}I_0K_0^3 \\ \int x^{j+1}I_1K_0^3 \\ \int x^{j+1}I_1K_0^2K_1 \\ \int x^{j+1}I_1K_0K_1^2 \\ \int x^{j+1}I_1K_1^3 \end{pmatrix} + \begin{pmatrix} \frac{3}{j+1}\Gamma_{00}(j) \\ 0 \\ -\frac{1}{j-1}\Gamma_{00}(j) \\ -\frac{1}{j-2}\Gamma_{01}(j) \\ -\frac{1}{j-3}\Gamma_{11}(j) \end{pmatrix}.$$

The matrix is singular, which inhibits a standard inversion that aims at creating a forward recurrence $j \to j+1$ at this place. We substitute $j \to j+1$ and insert the result into the right hand side:

(25)

$$\begin{pmatrix} \int x^j I_0 K_0^3 \\ \int x^j I_1 K_0^3 \\ \int x^j I_1 K_0^2 K_1 \\ \int x^j I_1 K_0 K_1^2 \\ \int x^j I_1 K_1^3 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{4}{1+j} & 0 & 0 & 0 \\ -\frac{1}{j} & 0 & \frac{3}{j} & 0 & 0 \\ 0 & \frac{2}{j-1} & 0 & \frac{2}{j-1} & 0 \\ 0 & 0 & \frac{3}{j-2} & 0 & \frac{1}{j-2} \\ 0 & 0 & 0 & \frac{4}{j-3} & 0 \end{pmatrix} \cdot \Bigg[\begin{pmatrix} 0 & -\frac{4}{2+j} & 0 & 0 & 0 \\ -\frac{1}{1+j} & 0 & \frac{3}{1+j} & 0 & 0 \\ 0 & \frac{2}{j} & 0 & \frac{2}{j} & 0 \\ 0 & 0 & \frac{3}{j-1} & 0 & \frac{1}{j-1} \\ 0 & 0 & 0 & \frac{4}{j-2} & 0 \end{pmatrix} \cdot$$

$$\begin{pmatrix} \int x^{j+2} I_0 K_0^3 \\ \int x^{j+2} I_1 K_0^3 \\ \int x^{j+2} I_1 K_0^2 K_1 \\ \int x^{j+2} I_1 K_0 K_1^2 \\ \int x^{j+2} I_1 K_1^3 \end{pmatrix} + \begin{pmatrix} \frac{3}{j+2}\Gamma_{00}(j+1) \\ 0 \\ -\frac{1}{j}\Gamma_{00}(j+1) \\ -\frac{1}{j-1}\Gamma_{01}(j+1) \\ -\frac{1}{j-2}\Gamma_{11}(j+1) \end{pmatrix} \Bigg] + \begin{pmatrix} \frac{3}{j+1}\Gamma_{00}(j) \\ 0 \\ -\frac{1}{j-1}\Gamma_{00}(j) \\ -\frac{1}{j-2}\Gamma_{01}(j) \\ -\frac{1}{j-3}\Gamma_{11}(j) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{(1+j)^2} & 0 & -\frac{12}{(1+j)^2} & 0 & 0 \\ 0 & \frac{2(5j+6)}{j^2(2+j)} & 0 & \frac{6}{j^2} & 0 \\ -\frac{2}{(j-1)(1+j)} & 0 & \frac{12j}{(j-1)^2(1+j)} & 0 & \frac{2}{(j-1)^2} \\ 0 & \frac{6}{j(j-2)} & 0 & \frac{2(5j-6)}{(j-2)^2 j} & 0 \\ 0 & 0 & \frac{12}{(j-3)(j-1)} & 0 & \frac{4}{(j-3)(j-1)} \end{pmatrix} \cdot \begin{pmatrix} \int x^{j+2} I_0 K_0^3 \\ \int x^{j+2} I_1 K_0^3 \\ \int x^{j+2} I_1 K_0^2 K_1 \\ \int x^{j+2} I_1 K_0 K_1^2 \\ \int x^{j+2} I_1 K_1^3 \end{pmatrix}$$

$$+ \begin{pmatrix} 3\times 2^{j-2}\Gamma^4((1+j)/2)/(1+j)! \\ -3\times 2^{j-4} j\Gamma^4(j/2)/[(2+j)(j-1)!] \\ -2^{j-5}(j-1)\Gamma^4((j-1)/2)/(j-2)! \\ -2^{j-4}(3j^2-3j-2)j\Gamma^4(j/2)/[(1+j)(j-2)^2(j-1)!] \\ -3\times 2^{j-10}(j-3)^3(j-1)^2(1+j)\Gamma^4((j-3)/2)/[j(j-2)!] \end{pmatrix}.$$

Here $0! = 1$. This iteration has separated the subspace of the second and fourth equation with a 2×2 matrix,

$$\begin{pmatrix} \int x^j I_1 K_0^3 \\ \int x^j I_1 K_0 K_1^2 \end{pmatrix} = \begin{pmatrix} \frac{2(5j+6)}{j^2(2+j)} & \frac{6}{j^2} \\ \frac{6}{j(j-2)} & \frac{2(5j-6)}{(j-2)^2 j} \end{pmatrix} \cdot \begin{pmatrix} \int x^{j+2} I_1 K_0^3 \\ \int x^{j+2} I_1 K_0 K_1^2 \end{pmatrix} + \begin{pmatrix} -3\times 2^{j-4} j\Gamma^4(j/2)/[(2+j)(j-1)!] \\ -2^{j-4}(3j^2-3j-2)j\Gamma^4(j/2)/[(1+j)(j-2)^2(j-1)!] \end{pmatrix}. \tag{26}$$

That matrix is not singular and is inverted to support a calculation of integrands with a higher power x^{j+2} from integrands with a lower power x^j:

(27)

$$\begin{pmatrix} \int x^{j+2} I_1 K_0^3 \\ \int x^{j+2} I_1 K_0 K_1^2 \end{pmatrix} = \begin{pmatrix} \frac{(5j-6)(2+j)}{32} & -\frac{3(2+j)(j-2)^2}{32j} \\ -\frac{3(2+j)(j-2)}{32} & \frac{(5j+6)(j-2)^2}{32j} \end{pmatrix} \cdot \begin{pmatrix} \int x^j I_1 K_0^3 \\ \int x^j I_1 K_0 K_1^2 \end{pmatrix}$$

$$+ \begin{pmatrix} 3\times 2^{j-8}(2+j+j^3-2j^2)\Gamma^4(j/2)/[(1+j)(j-1)!] \\ 2^{j-8}(6j^2+3j^3-5j-6)\Gamma^4(j/2)/[(1+j)(j-1)!] \end{pmatrix}.$$

Algorithm 1. (1) *For the (finite) integrals for $j \le 2$ use (28)–(32):* [12, A255986]

$$\int_0^\infty I_1(x)K_0^3(x)dx = \zeta(3)/4 \approx 0.30051422578989857134993454038; \tag{28}$$

$$\int_0^\infty xI_1(x)K_0^3(x)dx \approx 0.10116007103409665114227760576; \tag{29}$$

$$\int_0^\infty x^2I_1(x)K_0^3(x)dx \approx 0.06657486246595754316142215625 34; \tag{30}$$

$$\int_0^\infty xI_1(x)K_0(x)K_1^2(x)dx \approx 0.54735455135794868179362042374; \tag{31}$$

$$\int_0^\infty x^2I_1(x)K_0(x)K_1^2(x)dx \approx 0.18342513753404245683857784375. \tag{32}$$

(2) *Compute* $\int_0^\infty x^jI_1(x)K_0^3(x)dx$ *and* $\int_0^\infty x^jI_1(x)K_0(x)K_1^2(x)dx$ *for* $j \geq 3$ *recursively via* (27)*, but for*

$$\int x^4I_1(x)K_0(x)K_1^2(x)dx = 1/6 \tag{33}$$

–obtained by inserting $j = 3$ *in* (23)*.*

The first, third and fifth line of (24) are unrolled as follows:

Algorithm 2. (1) *The integrals* $\int_0^\infty x^jI_0K_0^3dx$ *are evaluated recursively with* [3, 4]

$$(k+1)^5\int_0^\infty x^kI_0(x)K_0^3(x)dx - 4(k+2)(5k^2+20k+23)\int_0^\infty x^{k+2}I_0(x)K_0^3(x)dx + 64(k+3)\int_0^\infty x^{k+4}I_0(x)K_0^3(x)dx = 0. \tag{34}$$

starting at [12, A273951]

$$\int I_0(x)K_0^3(x)dx \approx 6.99756301668063235955675782 69; \tag{35}$$

[12, A222068][5, (89)]

$$\int xI_0K_0^3dx = \frac{1}{16}\pi^2; \tag{36}$$

$$\int x^2I_0(x)K_0^3(x)dx \approx 0.21790296563849283706539235962; \tag{37}$$

$$\int x^3I_0(x)K_0^3(x)dx = \frac{1}{64}\pi^2. \tag{38}$$

Remark 1. [12, A245058]

$$\int xI_1K_0^2K_1dx = \frac{1}{48}\pi^2, \tag{39}$$

is obviously correlated to (36) *by putting* $j = 0$ *in* (11)*.*

Remark 2. *According to the first line of* (24)*, the value in* (29) *is obtained from* (35) *by subtracting* $3\pi^2/4$ *and dividing through* -4*.*

Remark 3. *Recurrences equivalent to (34) are derived for the remaining 4 classes of integrals by insertion of the recurrence into (24). From the first line e.g.*

$$24\frac{k^2(3k^3+14k^2+18k+8)}{(k+1)(k+3)}\Gamma_{01}(k)-4k^4(k+2)(k+4)\int_0^\infty x^k I_1K_0^3dx \\ +16(k+1)(k+4)(5k^2+10k+8)\int_0^\infty x^{k+2}I_1K_0^3dx \\ -256(k+2)^2\int_0^\infty x^{k+4}I_1K_0^3dx=0. \tag{40}$$

Supposed all the values of $\int_0^\infty x^jI_0(x)K_0^3(x)dx$ are known from Algorithm 2, one can lift the degeneracy in (24) and invert the 4×4 matrix:

$$\begin{pmatrix}\int x^jI_1K_0^3\\ \int x^jI_1K_0^2K_1\\ \int x^jI_1K_0K_1^2\\ \int x^jI_1K_1^3\end{pmatrix}=\begin{pmatrix}0&\frac{3}{j}&0&0\\ \frac{2}{j-1}&0&\frac{2}{j-1}&0\\ 0&\frac{3}{j-2}&0&\frac{1}{j-2}\\ 0&0&\frac{4}{j-3}&0\end{pmatrix}\cdot\begin{pmatrix}\int x^{j+1}I_1K_0^3\\ \int x^{j+1}I_1K_0^2K_1\\ \int x^{j+1}I_1K_0K_1^2\\ \int x^{j+1}I_1K_1^3\end{pmatrix}+\begin{pmatrix}-\frac{1}{j}\int x^{j+1}I_0K_0^3\\ -\frac{1}{j-1}\Gamma_{00}(j)\\ -\frac{1}{j-2}\Gamma_{01}(j)\\ -\frac{1}{j-3}\Gamma_{11}(j)\end{pmatrix}. \tag{41}$$

$$\begin{pmatrix}0&\frac{j-1}{2}&0&\frac{3-j}{4}\\ \frac{j}{3}&0&0&0\\ 0&0&0&\frac{j-3}{4}\\ -j&0&j-2&0\end{pmatrix}\cdot\left[\begin{pmatrix}\int x^jI_1K_0^3\\ \int x^jI_1K_0^2K_1\\ \int x^jI_1K_0K_1^2\\ \int x^jI_1K_1^3\end{pmatrix}+\begin{pmatrix}\frac{1}{j}\int x^{j+1}I_0K_0^3\\ \frac{1}{j-1}\Gamma_{00}(j)\\ \frac{1}{j-2}\Gamma_{01}(j)\\ \frac{1}{j-3}\Gamma_{11}(j)\end{pmatrix}\right]=\begin{pmatrix}\int x^{j+1}I_1K_0^3\\ \int x^{j+1}I_1K_0^2K_1\\ \int x^{j+1}I_1K_0K_1^2\\ \int x^{j+1}I_1K_1^3\end{pmatrix}. \tag{42}$$

Algorithm 3. *Given the $\int_0^\infty x^jI_0(x)K_0^3(x)dx$, compute the others in this group recursively with*

$$\begin{pmatrix}\int x^{j+1}I_1K_0^3\\ \int x^{j+1}I_1K_0^2K_1\\ \int x^{j+1}I_1K_0K_1^2\\ \int x^{j+1}I_1K_1^3\end{pmatrix}=\begin{pmatrix}0&\frac{j-1}{2}&0&\frac{3-j}{4}\\ \frac{j}{3}&0&0&0\\ 0&0&0&\frac{j-3}{4}\\ -j&0&j-2&0\end{pmatrix}\cdot\begin{pmatrix}\int x^jI_1K_0^3\\ \int x^jI_1K_0^2K_1\\ \int x^jI_1K_0K_1^2\\ \int x^jI_1K_1^3\end{pmatrix} \\ +\begin{pmatrix}\frac{1}{2}\Gamma_{00}(j)-\frac{1}{4}\Gamma_{11}(j)\\ \frac{1}{3}\int x^{j+1}I_0K_0^3\\ \frac{1}{4}\Gamma_{11}(j)\\ -\int x^{j+1}I_0K_0^3+\Gamma_{01}(j)\end{pmatrix}. \tag{43}$$

3. Group 2: Four Second Kind

The other five cases of integrals discussed in this manuscript contain four factors of K functions.

3.1. Class K_0^4.

$$\int x^jK_0^4dx=\ldots-\int\frac{x^{j+1}}{j+1}(-4K_0^3K_1)dx=\ldots+4\int\frac{x^{j+1}}{j+1}K_0^3K_1dx. \tag{44}$$

3.2. **Class** $K_0^3K_1$.

$$\int x^j K_0^3K_1dx = \ldots - \int \frac{x^{j+1}}{j+1}[-3K_0^2K_1^2 - K_0^4 - \frac{1}{x}K_0^3K_1]dx \tag{45}$$
$$= \ldots + \int \frac{x^j}{j+1}K_0^3K_1dx + \int \frac{x^{j+1}}{j+1}[3K_0^2K_1^2 + K_0^4]dx.$$

Merging common integrals yields

$$\frac{j}{j+1}\int x^j K_0^3K_1dx = \ldots + \int \frac{x^{j+1}}{j+1}[3K_0^2K_1^2 + K_0^4]dx. \tag{46}$$

$$j\int x^j K_0^3K_1dx = \ldots + 3\int x^{j+1}K_0^2K_1^2dx + \int x^{j+1}K_0^4dx \tag{47}$$

3.3. **Class** $K_0^2K_1^2$.

$$\int x^j K_0^2K_1^2dx = \ldots - \int \frac{x^{j+1}}{j+1}[-2K_0K_1^3 - 2K_0^3K_1 - \frac{2}{x}K_0^2K_1^2]dx \tag{48}$$
$$= \ldots - \int \frac{x^{j+1}}{j+1}[-\frac{2}{x}K_0^2K_1^2]dx - \int \frac{x^{j+1}}{j+1}[-2K_0K_1^3 - 2K_0^3K_1]dx$$
$$= \ldots + 2\int \frac{x^j}{j+1}K_0^2K_1^2dx + 2\int \frac{x^{j+1}}{j+1}[K_0K_1^3 + K_0^3K_1]dx.$$

Merging common integrals yields

$$\frac{j-1}{j+1}\int x^j K_0^2K_1^2dx = \ldots + 2\int \frac{x^{j+1}}{j+1}[K_0K_1^3 + K_0^3K_1]dx. \tag{49}$$

$$(j-1)\int x^j K_0^2K_1^2dx = \ldots + 2\int x^{j+1}K_0K_1^3dx + 2\int x^{j+1}K_0^3K_1dx. \tag{50}$$

3.4. **Class** $K_0K_1^3$.

$$\int x^j K_0K_1^3dx = \ldots - \int \frac{x^{j+1}}{j+1}[-K_1^4 - 3K_0^2K_1^2 - \frac{3}{x}K_0K_1^3]dx \tag{51}$$
$$= \ldots - \int \frac{x^{j+1}}{j+1}[-\frac{3}{x}K_0K_1^3]dx - \int \frac{x^{j+1}}{j+1}[-K_1^4 - 3K_0^2K_1^2]dx$$
$$= \ldots + 3\int \frac{x^j}{j+1}K_0K_1^3dx + \int \frac{x^{j+1}}{j+1}[K_1^4 + 3K_0^2K_1^2]dx.$$

Merging common integrals yields

$$\frac{j-2}{j+1}\int x^j K_0K_1^3dx = \ldots + \int \frac{x^{j+1}}{j+1}K_1^4dx + 3\int \frac{x^{j+1}}{j+1}K_0^2K_1^2dx. \tag{52}$$

$$(j-2)\int x^j K_0K_1^3dx = \ldots + \int x^{j+1}K_1^4dx + 3\int x^{j+1}K_0^2K_1^2dx. \tag{53}$$

3.5. **Class K_1^4.**

(54)
$$\int x^j K_1^4 dx = \ldots - \int \frac{x^{j+1}}{j+1}[-4K_0K_1^3 - \frac{4}{x}K_1^4]dx = \ldots + 4\int \frac{x^{j+1}}{j+1}[K_0K_1^3 + \frac{1}{x}K_1^4]dx$$
$$= \ldots + 4\int \frac{x^j}{j+1}K_1^4 dx + 4\int \frac{x^{j+1}}{j+1}K_0K_1^3 dx.$$

Merging common integrals yields

$$\frac{j-3}{j+1}\int x^j K_1^4 dx = \ldots + 4\int \frac{x^{j+1}}{j+1}K_0K_1^3 dx \tag{55}$$

$$(j-3)\int x^j K_1^4 dx = \ldots + 4\int x^{j+1}K_0K_1^3 dx \tag{56}$$

3.6. **Synthesis.** Gathering these 5 partial integrations forms a homogeneous system of equations very similar to (24):

$$\begin{pmatrix} \int x^j K_0^4 \\ \int x^j K_0^3 K_1 \\ \int x^j K_0^2 K_1^2 \\ \int x^j K_0 K_1^3 \\ \int x^j K_1^4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{4}{1+j} & 0 & 0 & 0 \\ \frac{1}{j} & 0 & \frac{3}{j} & 0 & 0 \\ 0 & \frac{2}{j-1} & 0 & \frac{2}{j-1} & 0 \\ 0 & 0 & \frac{3}{j-2} & 0 & \frac{1}{j-2} \\ 0 & 0 & 0 & \frac{4}{j-3} & 0 \end{pmatrix} \cdot \begin{pmatrix} \int x^{j+1} K_0^4 \\ \int x^{j+1} K_0^3 K_1 \\ \int x^{j+1} K_0^2 K_1^2 \\ \int x^{j+1} K_0 K_1^3 \\ \int x^{j+1} K_1^4 \end{pmatrix}. \tag{57}$$

This is also iterated:

(58)
$$\begin{pmatrix} \int x^j K_0^4 \\ \int x^j K_0^3 K_1 \\ \int x^j K_0^2 K_1^2 \\ \int x^j K_0 K_1^3 \\ \int x^j K_1^4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{4}{1+j} & 0 & 0 & 0 \\ \frac{1}{j} & 0 & \frac{3}{j} & 0 & 0 \\ 0 & \frac{2}{j-1} & 0 & \frac{2}{j-1} & 0 \\ 0 & 0 & \frac{3}{j-2} & 0 & \frac{1}{j-2} \\ 0 & 0 & 0 & \frac{4}{j-3} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \frac{4}{2+j} & 0 & 0 & 0 \\ \frac{1}{1+j} & 0 & \frac{3}{1+j} & 0 & 0 \\ 0 & \frac{2}{j} & 0 & \frac{2}{j} & 0 \\ 0 & 0 & \frac{3}{j-1} & 0 & \frac{1}{j-1} \\ 0 & 0 & 0 & \frac{4}{j-2} & 0 \end{pmatrix} \cdot \begin{pmatrix} \int x^{j+2} K_0^4 \\ \int x^{j+2} K_0^3 K_1 \\ \int x^{j+2} K_0^2 K_1^2 \\ \int x^{j+2} K_0 K_1^3 \\ \int x^{j+2} K_1^4 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{4}{(1+j)^2} & 0 & \frac{12}{(1+j)^2} & 0 & 0 \\ 0 & \frac{2(5j+6)}{j^2(2+j)} & 0 & \frac{6}{j^2} & 0 \\ \frac{2}{(j-1)(1+j)} & 0 & \frac{12j}{(j-1)^2(1+j)} & 0 & \frac{2}{(j-1)^2} \\ 0 & \frac{6}{j(j-2)} & 0 & \frac{2(5j-6)}{(j-2)^2 j} & 0 \\ 0 & 0 & \frac{12}{(j-3)(j-1)} & 0 & \frac{4}{(j-3)(j-1)} \end{pmatrix} \cdot \begin{pmatrix} \int x^{j+2} K_0^4 \\ \int x^{j+2} K_0^3 K_1 \\ \int x^{j+2} K_0^2 K_1^2 \\ \int x^{j+2} K_0 K_1^3 \\ \int x^{j+2} K_1^4 \end{pmatrix}.$$

The subspace of lines 2 and 4 in (58) reads:

$$\begin{pmatrix} \int x^j K_0^3 K_1 \\ \int x^j K_0 K_1^3 \end{pmatrix} = \begin{pmatrix} \frac{2(5j+6)}{j^2(2+j)} & \frac{6}{j^2} \\ \frac{6}{j(j-2)} & \frac{2(5j-6)}{(j-2)^2 j} \end{pmatrix} \cdot \begin{pmatrix} \int x^{j+2} K_0^3 K_1 \\ \int x^{j+2} K_0 K_1^3 \end{pmatrix}. \tag{59}$$

By matrix inversion we obtain the forward recursion

$$\begin{pmatrix} \int x^{j+2} K_0^3 K_1 \\ \int x^{j+2} K_0 K_1^3 \end{pmatrix} = \begin{pmatrix} \frac{(5j-6)(2+j)}{32} & -\frac{3(j-2)^2(2+j)}{32j} \\ -\frac{3(j-2)(2+j)}{32} & \frac{(5j+6)(j-2)^2}{32j} \end{pmatrix} \cdot \begin{pmatrix} \int x^j K_0^3 K_1 \\ \int x^j K_0 K_1^3 \end{pmatrix}. \tag{60}$$

Algorithm 4. (1) *For the (finite) integrals $j \le 4$ use (61)–(66):*

$$\int_0^\infty x K_0^3 K_1 dx \approx 6.8103346044514933516774950661; \tag{61}$$

$$\int_0^\infty x^2 K_0^3 K_1 dx = 7\zeta(3)/16 \approx 0.52589989513232249986238544566 1; \tag{62}$$

$$\int_0^\infty x^3 K_0^3 K_1 dx \approx 0.146827968935465937913093562 46; \tag{63}$$

[12, A273841],*see the first line in* (57)

$$\int_0^\infty x^4 K_0^3 K_1 dx = [7\zeta(3)/2 - 3]/16 \approx 0.0754499475661612499311927228306; \tag{64}$$

$$\int_0^\infty x^3 K_0 K_1^3 dx \approx 0.955599360931901634496912601 35; \tag{65}$$

$$\int_0^\infty x^4 K_0 K_1^3 dx = 1/4. \tag{66}$$

(2) *For* $j \geq 5$ *calculate* $\int_0^\infty x^j K_0^3(x) K_1(x) dx$ *and* $\int_0^\infty x^j K_0(x) K_1(x)^3 dx$ *recursively with* (60).

Remark 4. (66) *follows from the partial integration*

$$\int_0^\infty x^j K_1^{j-1}(x) K_0(x) dx = \lim_{x\to 0} \frac{1}{j}(x^j K_1^j(x)) = 1/j. \tag{67}$$

(62) *is half of (70) and that is proved by setting* $j = 1$ *in (44).*

Algorithm 5. *Compute the* $\int_0^\infty x^j K_0^4(x) dx$ *recursively with* [3, 4]

$$(k+1)^5 \int_0^\infty x^k K_0^4(x) dx - 4(k+2)(5k^2+20k+23) \int_0^\infty x^{k+2} K_0^4(x) dx + 64(k+3) \int_0^\infty x^{k+4} K_0^4(x) dx = 0 \tag{68}$$

starting at [3, (73)][12, A273839]

$$\int_0^\infty K_0^4(x) dx = \frac{\pi^4}{4} {}_4F_3(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1, 1; 1) \approx \frac{\pi^4}{4} \cdot 1.1186363871641870683496192575256. \tag{69}$$

[5, (90)][12, A233091]

$$\int_0^\infty x K_0^4(x) dx = 7\zeta(3)/8 \approx 1.0517997902646449997247708913; \tag{70}$$

[12, A273840]

$$\int_0^\infty x^2 K_0^4(x) dx \approx 0.195770625247287917217458083 28; \tag{71}$$

[12, A273841]

$$\int_0^\infty x^3 K_0^4(x) dx = [7\zeta(3)/2 - 3]/16 \approx 0.075449947566161249931192722831. \tag{72}$$

To obtain $\int_0^\infty x^j K_0^2(x) K_1^2(x) dx$ and $\int_0^\infty x^j K_1^4(x) dx$ one may plug these values of Algorithm 5 into the first and last line of (58). The alternative is to eliminate the $\int_0^\infty x^j K_0^4 dx$ from (57) and to invert the remaining 4×4 matrix:

(73)
$$\begin{pmatrix} \int x^j K_0^3 K_1 \\ \int x^j K_0^2 K_1^2 \\ \int x^j K_0 K_1^3 \\ \int x^j K_1^4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{3}{j} & 0 & 0 \\ \frac{2}{j-1} & 0 & \frac{2}{j-1} & 0 \\ 0 & \frac{3}{j-2} & 0 & \frac{1}{j-2} \\ 0 & 0 & \frac{4}{j-3} & 0 \end{pmatrix} \cdot \begin{pmatrix} \int x^{j+1} K_0^3 K_1 \\ \int x^{j+1} K_0^2 K_1^2 \\ \int x^{j+1} K_0 K_1^3 \\ \int x^{j+1} K_1^4 \end{pmatrix} + \begin{pmatrix} -\frac{1}{j}\int x^{j+1} K_0^4 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

(74)
$$\begin{pmatrix} 0 & \frac{j-1}{2} & 0 & \frac{3-j}{4} \\ \frac{j}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{j-3}{4} \\ -j & 0 & j-2 & 0 \end{pmatrix} \cdot \left[\begin{pmatrix} \int x^j K_0^3 K_1 \\ \int x^j K_0^2 K_1^2 \\ \int x^j K_0 K_1^3 \\ \int x^j K_1^4 \end{pmatrix} + \begin{pmatrix} \frac{1}{j}\int x^{j+1} K_0^4 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} \int x^{j+1} K_0^3 K_1 \\ \int x^{j+1} K_0^2 K_1^2 \\ \int x^{j+1} K_0 K_1^3 \\ \int x^{j+1} K_1^4 \end{pmatrix}.$$

Algorithm 6. *Given the* $\int_0^\infty x^j K_0^4(x)dx$*, compute the others in this group recursively with*
(75)
$$\begin{pmatrix} \int x^{j+1} K_0^3 K_1 \\ \int x^{j+1} K_0^2 K_1^2 \\ \int x^{j+1} K_0 K_1^3 \\ \int x^{j+1} K_1^4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{j-1}{2} & 0 & \frac{3-j}{4} \\ \frac{j}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{j-3}{4} \\ -j & 0 & j-2 & 0 \end{pmatrix} \cdot \begin{pmatrix} \int x^j K_0^3 K_1 \\ \int x^j K_0^2 K_1^2 \\ \int x^j K_0 K_1^3 \\ \int x^j K_1^4 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{3}\int x^{j+1} K_0^4 \\ 0 \\ -\int x^{j+1} K_0^4 \end{pmatrix}.$$

4. Group 3: Two First Kind, Two Second Kind

In this group there are two Modified Bessel Functions of the First Kind in the integrand. The reductions are similar to the treatment in the other two groups, but (1) is *not* used, because this leaves diverging integrals of the mixed type $\int_0^\infty x^j I_\nu(x) K_\mu(x) dx$. In consequence, 9 coupled classes of integrals emerge.

4.1. Class $I_0^2 K_0^2$.

$$\int x^j I_0^2 K_0^2 dx = \ldots - 2\int \frac{x^{j+1}}{j+1}[I_0(x)I_1(x)K_0^2(x) - I_0^2(x)K_0(x)K_1(x)]dx \tag{76}$$
$$= \ldots - 2\int \frac{x^{j+1}}{j+1} I_0(x)I_1(x)K_0^2(x)dx + 2\int \frac{x^{j+1}}{j+1} I_0^2(x)K_0(x)K_1(x)dx.$$

4.2. Class $I_0^2 K_0 K_1$.

(77)
$$\int x^j I_0^2 K_0 K_1 dx = \ldots - \int \frac{x^{j+1}}{j+1}[2I_0 I_1 K_0 K_1 - I_0^2 K_1^2 - I_0^2 K_0^2 - \frac{1}{x} I_0^2 K_0 K_1]dx$$
$$= \ldots + \int \frac{x^j}{j+1} I_0^2 K_0 K_1 dx - \int \frac{x^{j+1}}{j+1}[2I_0 I_1 K_0 K_1 - I_0^2 K_1^2 - I_0^2 K_0^2]dx.$$

Merging common integrals yields

(78)
$$\frac{j}{j+1}\int x^j I_0^2 K_0 K_1 dx = \ldots - 2\int \frac{x^{j+1}}{j+1} I_0 I_1 K_0 K_1 dx + \int \frac{x^{j+1}}{j+1} I_0^2 K_1^2 dx + \int \frac{x^{j+1}}{j+1} I_0^2 K_0^2 dx.$$

(79)
$$j\int x^jI_0^2K_0K_1dx = \ldots -2\int x^{j+1}I_0I_1K_0K_1dx+\int x^{j+1}I_0^2K_1^2dx+\int x^{j+1}I_0^2K_0^2dx.$$

4.3. Class $I_0^2K_1^2$.

(80)
$$\int x^jI_0^2K_1^2dx = \ldots - \int \frac{x^{j+1}}{j+1}[2I_0I_1K_1^2 - 2I_0^2K_0K_1 - \frac{2}{x}I_0^2K_1^2]dx$$
$$= \ldots + 2\int \frac{x^j}{j+1}I_0^2K_1^2dx - \int \frac{x^{j+1}}{j+1}[2I_0I_1K_1^2 - 2I_0^2K_0K_1]dx.$$

Merging common integrals yields

(81)
$$\frac{j-1}{j+1}\int x^jI_0^2K_1^2dx \;=\; \ldots - 2\int \frac{x^{j+1}}{j+1}I_0I_1K_1^2dx + 2\int \frac{x^{j+1}}{j+1}I_0^2K_0K_1dx.$$

(82)
$$(j-1)\int x^jI_0^2K_1^2dx \;=\; \ldots - 2\int x^{j+1}I_0I_1K_1^2dx + 2\int x^{j+1}I_0^2K_0K_1dx.$$

4.4. Class $I_0I_1K_0^2$.

(83)
$$\int x^jI_0I_1K_0^2dx = \ldots -\int \frac{x^{j+1}}{j+1}[I_1^2(x)K_0^2(x)+I_0^2(x)K_0^2(x)-\frac{1}{x}I_0(x)I_1(x)(x)K_0^2(x)-2I_0(x)I_1(x)K_0(x)K_1(x)]dx$$
$$= \ldots +\int \frac{x^j}{j+1}I_0(x)I_1(x)(x)K_0^2(x)dx-\int \frac{x^{j+1}}{j+1}[I_1^2(x)K_0^2(x)+I_0^2(x)K_0^2(x)-2I_0(x)I_1(x)K_0(x)K_1(x)]dx$$

Merging common terms on both sides yields

(84)
$$\frac{j}{j+1}\int x^jI_0I_1K_0^2dx = \ldots -\int \frac{x^{j+1}}{j+1}[I_1^2(x)K_0^2(x)+I_0^2(x)K_0^2(x)-2I_0(x)I_1(x)K_0(x)K_1(x)]dx$$

(85)
$$j\int x^jI_0I_1K_0^2dx = \ldots -\int x^{j+1}I_1^2(x)K_0^2(x)dx-\int x^{j+1}I_0^2(x)K_0^2(x)dx+2\int x^{j+1}I_0(x)I_1(x)K_0(x)K_1(x)dx.$$

4.5. Class $I_0I_1K_0K_1$.

(86)
$$\int x^jI_0(x)I_1(x)K_0(x)K_1(x)dx = \ldots -\int \frac{x^{j+1}}{j+1}[I_1^2K_0K_1+I_0^2K_0K_1-\frac{2}{x}I_0I_1K_0K_1-I_0I_1K_1^2-I_0I_1K_0^2]dx$$
$$= \ldots -\int \frac{x^{j+1}}{j+1}[I_1^2(x)K_0K_1+I_0^2K_0K_1-I_0I_1K_1^2-I_0I_1K_0^2]dx+2\int \frac{x^j}{j+1}I_0I_1K_0K_1dx.$$

Merging common integrals yields

(87)
$$\frac{j-1}{j+1}\int x^jI_0(x)I_1(x)K_0(x)K_1(x)dx = \ldots -\int \frac{x^{j+1}}{j+1}[I_1^2K_0K_1+I_0^2K_0K_1-I_0I_1K_1^2-I_0I_1K_0^2]dx.$$

$$(j-1)\int x^j I_0(x)I_1(x)K_0(x)K_1(x)dx = \tag{88}$$
$$\ldots - \int x^{j+1}I_1^2K_0K_1dx - \int x^{j+1}I_0^2K_0K_1dx + \int x^{j+1}I_0I_1K_1^2dx + \int x^{j+1}I_0I_1K_0^2dx.$$

4.6. Class $I_0I_1K_1^2$.

$$\int x^j I_0I_1K_1^2dx = \ldots - \int \frac{x^{j+1}}{j+1}[I_1^2K_1^2 + I_0^2K_1^2 - \frac{3}{x}I_0I_1K_1^2 - 2I_0I_1K_0K_1]dx. \tag{89}$$

Merging common integrals yields

$$\frac{j-2}{j+1}\int x^j I_0I_1K_1^2dx \;=\; \ldots - \int \frac{x^{j+1}}{j+1}[I_1^2K_1^2 + I_0^2K_1^2 - 2I_0I_1K_0K_1]dx. \tag{90}$$

$$(j-2)\int x^j I_0I_1K_1^2dx = \ldots - \int x^{j+1}I_1^2K_1^2dx - \int x^{j+1}I_0^2K_1^2dx + 2\int x^{j+1}I_0I_1K_0K_1dx. \tag{91}$$

4.7. Class $I_1^2K_0^2$.

$$\int x^j I_1^2(x)K_0^2(x)dx = \ldots - \int \frac{x^{j+1}}{j+1}[2I_0I_1K_0^2 - \frac{2}{x}I_1^2K_0^2 - 2I_1^2K_0K_1]dx \tag{92}$$
$$= \ldots + 2\int \frac{x^j}{j+1}I_1^2K_0^2dx - \int \frac{x^{j+1}}{j+1}[2I_0I_1K_0^2 - 2I_1^2K_0K_1]dx.$$

$$\frac{j-1}{j+1}\int x^j I_1^2(x)K_0^2(x)dx \;=\; \ldots - 2\int \frac{x^{j+1}}{j+1}I_0I_1K_0^2dx + 2\int \frac{x^{j+1}}{j+1}I_1^2K_0K_1dx. \tag{93}$$

$$(j-1)\int x^j I_1^2(x)K_0^2(x)dx = \ldots - 2\int x^{j+1}I_0I_1K_0^2dx + 2\int x^{j+1}I_1^2K_0K_1dx. \tag{94}$$

4.8. Class $I_1^2K_0K_1$.

$$\int x^j I_1^2K_0K_1dx = \ldots - \int \frac{x^{j+1}}{j+1}[2I_0I_1K_0K_1 - \frac{3}{x}I_1^2K_0K_1 - I_1^2K_1^2 - I_1^2K_0^2]dx \tag{95}$$
$$= \ldots + 3\int \frac{x^j}{j+1}I_1^2K_0K_1dx - \int \frac{x^{j+1}}{j+1}[2I_0I_1K_0K_1 - I_1^2K_1^2 - I_1^2K_0^2]dx.$$

$$\frac{j-2}{j+1}\int x^j I_1^2K_0K_1dx = \ldots - 2\int \frac{x^{j+1}}{j+1}I_0I_1K_0K_1dx + \int \frac{x^{j+1}}{j+1}I_1^2K_1^2dx + \int \frac{x^{j+1}}{j+1}I_1^2K_0^2dx. \tag{96}$$

$$(j-2)\int x^j I_1^2K_0K_1dx = \ldots - 2\int x^{j+1}I_0I_1K_0K_1dx + \int x^{j+1}I_1^2K_1^2dx + \int x^{j+1}I_1^2K_0^2dx. \tag{97}$$

4.9. Class $I_1^2K_1^2$.

$$\int x^j I_1^2 K_1^2 dx = \ldots - \int \frac{x^{j+1}}{j+1}[2I_0I_1K_1^2 - \frac{4}{x}I_1^2K_1^2 - 2I_1^2K_0K_1]dx$$
$$= \ldots + 4\int \frac{x^j}{j+1} I_1^2K_1^2 dx - \int \frac{x^{j+1}}{j+1}[2I_0I_1K_1^2 - 2I_1^2K_0K_1]dx. \tag{98}$$

$$\frac{j-3}{j+1}\int x^j I_1^2 K_1^2 dx \;=\; \ldots - 2\int \frac{x^{j+1}}{j+1} I_0I_1K_1^2 dx \,+\, 2\int \frac{x^{j+1}}{j+1} I_1^2K_0K_1 dx. \tag{99}$$

$$(j-3)\int x^j I_1^2 K_1^2 dx \;=\; \ldots - 2\int x^{j+1} I_0I_1K_1^2 dx \,+\, 2\int x^{j+1} I_1^2K_0K_1 dx. \tag{100}$$

4.10. Synthesis.

The 9 partial integrations may be summarized as:

$$\begin{pmatrix} \int x^j I_0^2K_0^2 \\ \int x^j I_0^2K_0K_1 \\ \int x^j I_0^2K_1^2 \\ \int x^j I_0I_1K_0^2 \\ \int x^j I_0I_1K_0K_1 \\ \int x^j I_0I_1K_1^2 \\ \int x^j I_1^2K_0^2 \\ \int x^j I_1^2K_0K_1 \\ \int x^j I_1^2K_1^2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{j+1} & 0 & -\frac{2}{j+1} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{j} & 0 & \frac{1}{j} & 0 & -\frac{2}{j} & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{j-1} & 0 & 0 & 0 & -\frac{2}{j-1} & 0 & 0 & 0 \\ -\frac{1}{j} & 0 & 0 & 0 & \frac{2}{j} & 0 & -\frac{1}{j} & 0 & 0 \\ 0 & -\frac{1}{j-1} & 0 & \frac{1}{j-1} & 0 & \frac{1}{j-1} & 0 & -\frac{1}{j-1} & 0 \\ 0 & 0 & -\frac{1}{j-2} & 0 & \frac{2}{j-2} & 0 & 0 & 0 & -\frac{1}{j-2} \\ 0 & 0 & 0 & -\frac{2}{j-1} & 0 & 0 & 0 & \frac{2}{j-1} & 0 \\ 0 & 0 & 0 & 0 & -\frac{2}{j-2} & 0 & \frac{1}{j-1} & 0 & \frac{1}{j-2} \\ 0 & 0 & 0 & 0 & 0 & -\frac{2}{j-3} & 0 & \frac{2}{j-3} & 0 \end{pmatrix} \cdot \begin{pmatrix} \int x^{j+1} I_0^2K_0^2 \\ \int x^{j+1} I_0^2K_0K_1 \\ \int x^{j+1} I_0^2K_1^2 \\ \int x^{j+1} I_0I_1K_0^2 \\ \int x^{j+1} I_0I_1K_0K_1 \\ \int x^{j+1} I_0I_1K_1^2 \\ \int x^{j+1} I_1^2K_0^2 \\ \int x^{j+1} I_1^2K_0K_1 \\ \int x^{j+1} I_1^2K_1^2 \end{pmatrix}. \tag{101}$$

The matrix has 3 vanishing eigenvalues, and the basic strategy needs 3 independent anchor values before we can start to invert the matrix. One iteration decouples the equations into two groups of $5+4$:

$$
\begin{pmatrix} \int x^j I_0^2 K_0^2 \\ \int x^j I_0^2 K_1^2 \\ \int x^j I_0 I_1 K_0 K_1 \\ \int x^j I_1^2 K_0^2 \\ \int x^j I_1^2 K_1^2 \end{pmatrix}
= \begin{pmatrix}
\frac{4}{(1+j)^2} & \frac{2}{(1+j)^2} & -\frac{8}{(1+j)^2} & \frac{2}{(1+j)^2} & 0 \\
\frac{2}{(j-1)(1+j)} & \frac{4j}{(j-1)^2(1+j)} & -\frac{8j}{(j-1)^2(1+j)} & 0 & \frac{2}{(j-1)^2} \\
-\frac{2}{(j-1)(1+j)} & -\frac{2j}{(j-1)^2(1+j)} & \frac{8j}{(j-1)^2(1+j)} & -\frac{2j}{(j-1)^2(1+j)} & -\frac{2}{(j-1)^2} \\
\frac{2}{(j-1)(1+j)} & 0 & -\frac{8j}{(j-1)^2(1+j)} & \frac{4j}{(j-1)^2(1+j)} & \frac{2}{(j-1)^2} \\
0 & \frac{2}{(j-3)(j-1)} & -\frac{8}{(j-1)^2(1+j)} & \frac{2}{(j-3)(j-1)} & \frac{4}{(j-3)(j-1)}
\end{pmatrix} \cdot
\begin{pmatrix} \int x^{j+2} I_0^2 K_0^2 \\ \int x^{j+2} I_0^2 K_1^2 \\ \int x^{j+2} I_0 I_1 K_0 K_1 \\ \int x^{j+2} I_1^2 K_0^2 \\ \int x^{j+2} I_1^2 K_1^2 \end{pmatrix}. \tag{102}
$$

This 5×5 matrix has two vanishing eigenvalues.

$$
\begin{pmatrix} \int x^j I_0^2 K_0 K_1 \\ \int x^j I_0 I_1 K_0^2 \\ \int x^j I_0 I_1 K_1^2 \\ \int x^j I_1^2 K_0 K_1 \end{pmatrix}
= \begin{pmatrix}
\frac{2(3j+4)}{j^2(2+j)} & -\frac{4(1+j)}{j^2(2+j)} & -\frac{4}{j^2} & \frac{2}{j^2} \\
-\frac{4(1+j)}{j^2(2+j)} & \frac{2(3j+4)}{j^2(2+j)} & \frac{2}{j^2} & -\frac{4}{j^2} \\
-\frac{4}{j(j-2)} & \frac{2}{j(j-2)} & \frac{2(3j-4)}{(j-2)^2 j} & -\frac{4(j-1)}{(j-2)^2 j} \\
\frac{2}{j(j-2)} & -\frac{4}{j(j-2)} & -\frac{4(j-1)}{(j-2)^2 j} & \frac{2(3j-4)}{(j-2)^2 j}
\end{pmatrix} \cdot
\begin{pmatrix} \int x^{j+2} I_0^2 K_0 K_1 \\ \int x^{j+2} I_0 I_1 K_0^2 \\ \int x^{j+2} I_0 I_1 K_1^2 \\ \int x^{j+2} I_1^2 K_0 K_1 \end{pmatrix}. \tag{103}
$$

This 4×4 matrix has one vanishing eigenvalue. Overall the recurrences are of no importance, because there are only a few cases with small integers j in the range -2…0, where the 9 integrals converge.

4.11. **Converging Cases.** The Taylor Expansions of Modified Bessel Functions of the first kind are Hypergeometric Series [1, 9.6.47]

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1+\nu)} {}_0F_1(\nu+1; z^2/4). \tag{104}$$

Their products are also Hypergeometric Series [6]:

$$I_0^2(x) = \sum_{j \ge 0} i_{00}(j) \frac{x^j}{2^j j!} \tag{105}$$

where $i_{00}(j) = 0$ if j is odd, and [12, A002894]

$$i_{00}(2j) = [\binom{2j}{j}]^2. \tag{106}$$

$$I_0(x) I_1(x) = \sum_{j \ge 0} i_{01}(j) \frac{x^j}{2^j j!} \tag{107}$$

where $i_{01}(j) = 0$ if j is even, and [12, A060150]

$$i_{01}(2j-1) = [\binom{2j-1}{j-1}]^2. \tag{108}$$

$$I_1^2(x) = \sum_{j\ge 0} i_{11}(j)\frac{x^j}{2^j j!} \tag{109}$$

where $i_{11}(j) = 0$ if j is odd, and [12, A135389]

$$i_{11}(2j) = \binom{2j}{j-1}\binom{2j}{j}. \tag{110}$$

We exchange the order of summation and integration in these series and insert the integrals (7), (17) and (22) to end up with representations as generalized hypergeometric functions [11]:

$$\int_0^\infty x^j I_0^2(x)K_0^2(x)dx = \sum_{k=0,2,4,6,\ldots} \int_0^\infty x^j i_{00}(k)\frac{x^k}{2^k k!}K_0^2(x)dx \tag{111}$$
$$= \sum_{k\ge 0}\int_0^\infty x^j [\binom{2k}{k}]^2 \frac{x^{2k}}{2^{2k}(2k)!}K_0^2(x)dx = \sum_{k\ge 0}[\binom{2k}{k}]^2\frac{1}{2^{2k}(2k)!}\Gamma_{00}(j+2k)$$
$$= \sum_{k\ge 0}[\binom{2k}{k}]^2\frac{1}{2^{2k}(2k)!}2^{j+2k-2}\frac{1}{\Gamma(1+j+2k)}\Gamma^4(\frac{1+j+2k}{2})$$
$$= \frac{1}{4}\frac{\Gamma(1/2)\Gamma^3((1+j)/2)}{\Gamma(1+j/2)}\sum_{k\ge 0}\frac{\Gamma(k+1/2)\Gamma^3(k+(1+j)/2)\Gamma(1+j/2)}{\Gamma^2(1+k)\Gamma(j/2+k+1)\Gamma^3((1+j)/2)\Gamma(1/2)}\frac{1}{k!}$$
$$= \frac{1}{4}\frac{\Gamma(1/2)\Gamma^3((1+j)/2)}{\Gamma(1+j/2)}\,{}_4F_3(\frac{1}{2},\frac{1+j}{2},\frac{1+j}{2},\frac{1+j}{2};1,1,1+\frac{j}{2};1).$$

This converges only at $j = 0$, where it equals (69) divided by π^2:

$$\int_0^\infty I_0^2(x)K_0^2(x)dx = \frac{\pi^2}{4}\,{}_4F_3(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1,1;1). \tag{112}$$

$$\int_0^\infty x^j I_0(x)I_1(x)K_0^2(x)dx = \sum_{k=1,3,5,7\ldots}\int_0^\infty x^j i_{10}(k)\frac{x^k}{2^k k!}K_0^2(x)dx = \tag{113}$$
$$\sum_{k\ge 0}\int_0^\infty x^j[\binom{2k+1}{k}]^2\frac{x^{2k+1}}{2^{2k+1}(2k+1)!}K_0^2(x)dx = \sum_{k\ge 0}[\binom{2k+1}{k}]^2\frac{1}{2^{2k+1}(2k+1)!}\Gamma_{00}(j+2k+1)$$
$$= \sum_{k\ge 0}[\binom{2k+1}{k}]^2\frac{1}{2^{2k+1}(2k+1)!}2^{j+2k-1}\frac{1}{\Gamma(j+2k+2)}\Gamma^4(k+1+j/2)$$
$$= \frac{1}{4}\frac{\Gamma(3/2)\Gamma^3(1+j/2)}{\Gamma((3+j)/2)}\sum_{k\ge 0}\frac{(3/2)_k(1+j/2)_k(1+j/2)_k(1+j/2)_k}{k!(2)_k(2)_k(3/2+j/2)_k}$$
$$= \frac{1}{4}\frac{\Gamma(3/2)\Gamma^3(1+j/2)}{\Gamma((3+j)/2)}\,{}_4F_3(3/2,1+j/2,1+j/2,1+j/2;2,2,(3+j)/2;1).$$

The two converging cases are at $j = -1$

$$\int_0^\infty \frac{1}{x}I_0(x)I_1(x)K_0^2(x)dx = \frac{\pi^2}{8}\,{}_4F_3(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2};2,2,1;1) \tag{114}$$
$$\approx \frac{\pi^2}{8}\cdot 1.0678210230301728720492382229 51.$$

and at $j=0$ [2]

$$(115)\quad \int_0^\infty I_0(x)I_1(x)K_0^2(x)dx = \frac{1}{4}{}_3F_2(1,1,1;2,2;1) = \frac{1}{4}\zeta(2) = \frac{\pi^2}{24} \approx 0.41123351671205660911810379166.$$

(116)

$$\int_0^\infty x^j I_0(x)I_1(x)K_0(x)K_1(x)dx = \sum_{k=1,3,5,7,\ldots}\int_0^\infty x^j i_{10}(k)\frac{x^k}{2^k k!}K_0(x)K_1(x)dx =$$

$$\sum_{k\ge 0}\int_0^\infty x^j [\binom{2k+1}{k}]^2 \frac{x^{2k+1}}{2^{2k+1}(2k+1)!}K_0(x)K_1(x)dx = \sum_{k\ge 0}[\binom{2k+1}{k}]^2\frac{1}{2^{2k+1}(2k+1)!}\Gamma_{01}(j+2k+1)$$

$$= \sum_{k\ge 0}[\binom{2k+1}{k}]^2\frac{1}{2^{2k+1}(2k+1)!}2^{j+2k-3}\frac{j+2k+1}{(j+2k)!}\Gamma^4(k+1/2+j/2)$$

$$= \frac{1}{4}\frac{\Gamma(3/2)\Gamma^3(1/2+j/2)}{\Gamma(1+j/2)}\sum_{k\ge 0}(1/2+j/2)(3/2+j/2)_k\frac{(3/2)_k(1/2+j/2)_k}{k!(2)_k(2)_k(1+j/2)_k(1/2+j/2)_k}$$

$$= \frac{1+j}{8}\frac{\Gamma(3/2)\Gamma^3(1/2+j/2)}{\Gamma(1+j/2)}{}_5F_4(\frac{3+j}{2},\frac{3}{2},\frac{1+j}{2},\frac{1+j}{2},\frac{1+j}{2};2,2,1+\frac{j}{2},\frac{1+j}{2};1).$$

At $j=0$ this is

$$(117)\quad \int_0^\infty I_0(x)I_1(x)K_0(x)K_1(x)dx = \frac{1}{8}\frac{\pi^2}{2}{}_4F_3(\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2};2,2,1;1) \approx \frac{\pi^2}{16}\cdot 1.3200141469579356241340239985190.$$

Remark 5. *The contiguous relations imply* [7, 10]

$$(118)\quad {}_4F_3(\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2};2,2,1;1) = {}_4F_3(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2};2,2,1;1) - 4{}_4F_3(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2};2,1,1;1) + 4{}_4F_3(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1,1;1).$$

$$(119)\quad \int_0^\infty x^j I_1^2(x)K_0^2(x)dx = \sum_{k=2,4,6,8,\ldots}\int_0^\infty x^j i_{11}(k)\frac{x^k}{2^k k!}K_0^2(x)dx$$

$$= \sum_{k\ge 1}\int_0^\infty x^j\binom{2k}{k-1}\binom{2k}{k}\frac{x^{2k}}{2^{2k}(2k)!}K_0^2(x)dx = \sum_{k\ge 1}\binom{2k}{k-1}\binom{2k}{k}\frac{1}{2^{2k}(2k)!}\Gamma_{00}(j+2k)$$

$$= \sum_{k\ge 1}\binom{2k}{k-1}\binom{2k}{k}\frac{1}{2^{2k}(2k)!}2^{j+2k-2}\frac{1}{\Gamma(j+2k+1)}\Gamma^4(k+1/2+j/2)$$

$$= \frac{1}{4}\frac{\Gamma(3/2)\Gamma^3(3/2+j/2)}{\Gamma(3)\Gamma(j/2+2)}\sum_{k\ge 0}\frac{(3/2)_k(3/2+j/2)_k(3/2+j/2)_k(3/2+j/2)_k}{k!(3)_k(2)_k(j/2+2)_k}$$

$$= \frac{1}{8}\frac{\Gamma(3/2)\Gamma^3(3/2+j/2)}{\Gamma(j/2+2)}{}_4F_3(\frac{3}{2},\frac{3+j}{2},\frac{3+j}{2},\frac{3+j}{2};3,2,2+\frac{j}{2};1).$$

The three converging cases are at $j=-2$

(120)
$$\int_0^\infty \frac{1}{x^2} I_1^2(x)K_0^2(x)dx = \frac{\pi^2}{16}{}_4F_3(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2};3,2,1;1) \approx \frac{\pi^2}{16}\cdot 1.0398951971477615654901270587830,$$

at $j=-1$

(121)
$$\int_0^\infty \frac{1}{x} I_1^2(x)K_0^2(x)dx = \frac{1}{8}{}_3F_2(1,1,1;3,2;1) \approx \frac{1}{8}\cdot 1.2898681336964528729448303332 92,$$

and at $j=0$

(122)
$$\int_0^\infty I_1^2(x)K_0^2(x)dx = \frac{\pi^2}{128}{}_4F_3(\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2};3,2,2;1) \approx \frac{\pi^2}{128}\cdot 2.4089983305557497517443793026930.$$

Remark 6. *There is a weak link to* (69) *via* [9]

$$(123)\quad {}_4F_3(\frac{1}{2},\frac{1}{2},\frac{3}{2},-\frac{1}{2};1,1,1;1) = 8/\pi^2 - \frac{1}{32}{}_4F_3(\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2};2,2,3;1)$$
$$= 2{}_4F_3(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2};1,1,1;1) - {}_4F_3(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1,1;1).$$

$$(124)\quad \int_0^\infty x^j I_1^2(x)K_0(x)K_1(x)dx = \sum_{k=2,4,6,8\ldots} \int_0^\infty x^j i_{11}(k)\frac{x^k}{2^k k!}K_0(x)K_1(x)dx$$
$$= \sum_{k\ge 1}\int_0^\infty x^j \binom{2k}{k-1}\binom{2k}{k}\frac{x^{2k}}{2^{2k}(2k)!}K_0(x)K_1(x)dx$$
$$= \sum_{k\ge 1}\binom{2k}{k-1}\binom{2k}{k}\frac{1}{2^{2k}(2k)!}\Gamma_{01}(2k+j)$$
$$= \sum_{k\ge 1}\binom{2k}{k-1}\binom{2k}{k}\frac{1}{2^{2k}(2k)!}2^{2k+j-4}\frac{2k+j}{(2k+j-1)!}\Gamma^4(k+j/2)$$
$$= \frac{1}{4}\frac{\Gamma(3/2)\Gamma^3(1+j/2)}{\Gamma(3)\Gamma(3/2+j/2)}\sum_{k\ge 0}(j/2+1)\frac{(j/2+2)_k}{(j/2+1)_k}\frac{(3/2)_k(1+j/2)_k(1+j/2)_k(1+j/2)_k}{k!(3)_k(2)_k(3/2+j/2)_k}$$
$$= \frac{1}{8}(1+j/2)\frac{\Gamma(3/2)\Gamma^3(1+j/2)}{\Gamma(3/2+j/2)}{}_4F_3(2+\frac{j}{2},\frac{3}{2},1+\frac{j}{2},1+\frac{j}{2};3,2,\frac{j+3}{2};1).$$

The two converging cases are at $j=-1$

(125)
$$\int_0^\infty \frac{1}{x} I_1^2(x)K_0(x)K_1(x)dx = \frac{\pi^2}{32}{}_4F_3(\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2};3,2,1;1) \approx \frac{\pi^2}{32}\cdot 1.1515985006774067917265717 15453,$$

and at $j=0$

$$(126)\quad \int_0^\infty I_1^2(x)K_0(x)K_1(x)dx = \frac{1}{8}{}_2F_1(1,1;3;1) = 1/4.$$

$$\int_0^\infty x^j I_1^2(x)K_1^2(x)dx = \sum_{k=2,4,6,8,\ldots}\int_0^\infty x^j i_{11}(k)\frac{x^k}{2^k k!}K_1^2(x)dx \tag{127}$$
$$= \sum_{k\ge 1}\int_0^\infty x^j \binom{2k}{k-1}\binom{2k}{k}\frac{x^{2k}}{2^{2k}(2k)!}K_1^2(x)dx$$
$$= \sum_{k\ge 1}\binom{2k}{k-1}\binom{2k}{k}\frac{1}{2^{2k}(2k)!}\Gamma_{11}(2k+j)$$
$$= \sum_{k\ge 1}\binom{2k}{k-1}\binom{2k}{k}\frac{1}{2^{2k}(2k)!}2^{2k+j-6}\frac{(2k+j-1)^2(2k+j+1)}{(2k+j-2)!(2k+j)}\Gamma^4(k+j/2-1/2)$$
$$= \frac{1}{4}\frac{\Gamma(3/2)\Gamma(j/2+5/2)\Gamma^2(j/2+3/2)}{\Gamma(3)(1/2+j/2)\Gamma(2+j/2)}\sum_{k\ge 0}(1/2+j/2)_k\frac{(3/2)_k}{k!(3)_k(2)_k(2+j/2)_k}(j/2+5/2)_k(j/2+3/2)_k$$
$$= \frac{1}{4}\frac{\Gamma(3/2)\Gamma(j/2+5/2)\Gamma^2(j/2+3/2)}{(1+j)\Gamma(2+j/2)}{}_4F_3(\frac{1+j}{2},\frac{3}{2},\frac{j+5}{2},\frac{j+3}{2};3,2,2+\frac{j}{2};1).$$

The only converging case occurs at $j=0$:

$$\int_0^\infty I_1^2(x)K_1^2(x)dx = \frac{3\pi^2}{128}{}_4F_3(\frac{1}{2},\frac{3}{2},\frac{3}{2},\frac{5}{2};2,2,3;1) \approx \frac{3\pi^2}{128}\cdot 1.63068200421411291122704802939\,07. \tag{128}$$

The values (114), (112), (117) and (122) are coupled via the 4th line of (101) at $j=-1$.

5. Summary

Algorithms 2, 3, 5 and 6 compute 10 different types of products of four Modified Bessel Functions multiplied by a power x^j recursively for increasing exponent j.

Appendix A. Numerical Integration

For small upper limits, a series expansion of the product of the Bessel Functions in the integrand leads to stable numerical representations. The basic series expansions are [1, 9.6.10]

$$I_\nu(z) = \sum_{l\ge 0}\frac{1}{l!\Gamma(\nu+l+1)}(z/2)^{2l+\nu} \tag{129}$$

and

$$K_0(z) = -[\ln\frac{z}{2}+\gamma]I_0(z)+\sum_{k\ge 1}H(k)\frac{(z^2/4)^k}{(k!)^2} = -\ln\frac{z}{2}\sum_{k\ge 0}\frac{(z^2/4)^k}{(k!)^2}+\sum_{k\ge 0}(H(k)-\gamma)\frac{(z^2/4)^k}{(k!)^2}, \tag{130}$$

where $H(k)=\sum_{l=1}^k 1/l$, $H(0)=0$, are the Harmonic Numbers.

The multinomial expansion is

(131)
$$(z/2)^l I_\nu(z) K_0^3(z) = -\ln^3\frac{z}{2}\sum_{v\ge 0}\sum_{n=0}^{v}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{1}{(k!)^2(m-k)!^2}\frac{1}{((n-m)!)^2}\frac{1}{(v-n)!\Gamma(\nu+v-n+1)}(z/2)^{2v+l+\nu}$$
$$-3\ln\frac{z}{2}\sum_{v\ge 0}\sum_{n=0}^{v}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{H(k)-\gamma}{k!^2}\frac{H(m-k)-\gamma}{(m-k)!^2}\frac{1}{((n-m)!)^2}\frac{1}{(v-n)!\Gamma(\nu+v-n+1)}(z/2)^{2v+l+\nu}$$
$$+3\ln^2\frac{z}{2}\sum_{v\ge 0}\sum_{n=0}^{v}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{1}{(k!)^2}\frac{H(m-k)-\gamma}{((m-k)!)^2}\frac{1}{((n-m)!)^2}\frac{1}{(v-n)!\Gamma(\nu+v-n+1)}(z/2)^{2v+l+\nu}$$
$$+\sum_{v\ge 0}\sum_{n=0}^{v}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{H(k)-\gamma}{k!^2}\frac{H(m-k)-\gamma}{(m-k)!^2}\frac{H(n-m)-\gamma}{((n-m)!)^2}\frac{1}{(v-n)!\Gamma(\nu+v-n+1)}(z/2)^{2v+l+\nu}.$$

The presence of the $\log(z/2)$ factor proposes to use $z=2$ as an upper limit, [8, 2.722]

$$\int_0^2 \ln^j\frac{z}{2}(z/2)^{2v+l}dz = (-)^j\frac{2j!}{(2v+1+l)^{j+1}},\quad j=0,1,\ldots. \tag{132}$$

(133)
$$\int_0^2 (z/2)^l I_\nu(z)K_0^3(z)dz = \sum_{v\ge 0}\frac{1}{v!(v+\nu)!}\sum_{n=0}^{v}\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{m}{k}^2\binom{n}{m}^2\binom{v}{n}\binom{v+\nu}{n}\frac{12}{(2v+1+l+\nu)^4}$$
$$+\sum_{v\ge 0}\frac{1}{v!(v+\nu)!}\sum_{n=0}^{v}\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{m}{k}^2\binom{n}{m}^2\binom{v}{n}\binom{v+\nu}{n}(H(k)-\gamma)(H(m-k)-\gamma)\frac{6}{(2v+1+l+\nu)^2}$$
$$+\sum_{v\ge 0}\frac{1}{v!(v+\nu)!}\sum_{n=0}^{v}\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{m}{k}^2\binom{n}{m}^2\binom{v}{n}\binom{v+\nu}{n}(H(m-k)-\gamma)\frac{12}{(2v+1+l+\nu)^3}$$
$$+\sum_{v\ge 0}\frac{1}{v!(v+\nu)!}\sum_{n=0}^{v}\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{m}{k}^2\binom{n}{m}^2\binom{v}{n}\binom{v+\nu}{n}(H(k)-\gamma)(H(m-k)-\gamma)(H(n-m)-\gamma)\frac{2}{2v+1+l+\nu}.$$

In an equivalent way [1, 9.6.11]

(134)
$$K_1(z) = \frac{1}{2}\frac{1}{z/2}+\frac{z}{2}\ln\frac{z}{2}\sum_{k\ge 0}\frac{1}{k!(1+k)!}(z/2)^{2k}-\frac{1}{2}\frac{z}{2}\sum_{k\ge 0}[2H(k+1)-\frac{1}{k+1}-2\gamma]\frac{(z/2)^{2k}}{k!(k+1)!}$$

where we have replaced

$$\psi(k+1)+\psi(k+2) = H(k)+H(k+1)-2\gamma = 2H(k+1)-\frac{1}{k+1}-2\gamma,\quad k\ge 0. \tag{135}$$

Definition 4.

$$H_2(k)\equiv H(k)+H(k+1)-2\gamma. \tag{136}$$

The multinomial expansion of the product is

$$(137)\quad (z/2)^l I_\nu(z) K_1^3(z) = \frac{1}{2^3} \sum_{k\ge 0} \frac{1}{k!\Gamma(\nu+k+1)} (z/2)^{2k+\nu-3+l}$$

$$+\ln^3 \frac{z}{2} \sum_{v\ge 0} \sum_{n=0}^{v} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{1}{k!(1+k)!} \frac{1}{(m-k)!(1+m-k)!} \frac{1}{(n-m)!(1+n-m)!} \frac{1}{(v-n)!\Gamma(\nu+v-n+1)} (z/2)^{2v+\nu+}$$

$$-\frac{1}{2^3} \sum_{v\ge 0} \sum_{n=0}^{v} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{H_2(k)}{k!(k+1)!} \frac{H_2(m-k)}{(m-k)!(m-k+1)!} \frac{H_2(n-m)}{(n-m)!(n-m+1)!} \frac{1}{(v-n)!\Gamma(\nu+v-n+1)} (z/2)^{2v+\nu+3+}$$

$$+\frac{3}{2^2} \ln \frac{z}{2} \sum_{m\ge 0} \sum_{k=0}^{m} \frac{1}{k!(1+k)!} \frac{1}{(m-k)!\Gamma(\nu+m-k+1)} (z/2)^{2m+\nu-1+l}$$

$$-\frac{3}{2^2} \frac{1}{2} \sum_{m\ge 0} \sum_{k=0}^{m} [H_2(k)] \frac{1}{k!(k+1)!} \frac{1}{(m-k)!\Gamma(\nu+m-k+1)} (z/2)^{2m+\nu-1+l}$$

$$+\frac{3}{2} \ln^2 \frac{z}{2} \sum_{n\ge 0} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{1}{k!(1+k)!} \frac{1}{(m-k)!(1+m-k)!} \frac{1}{(n-m)!\Gamma(\nu+n-m+1)} (z/2)^{2n+\nu+1+l}$$

$$+\frac{3}{2} \frac{1}{2^2} \sum_{n\ge 0} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{H_2(k)}{k!(k+1)!} \frac{H_2(m-k)}{(m-k)!(m-k+1)!} \frac{1}{(n-m)!\Gamma(\nu+n-m+1)} (z/2)^{2n+\nu+1+l}$$

$$-\frac{3}{2} \ln^2 \frac{z}{2} \sum_{v\ge 0} \sum_{n=0}^{v} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{1}{k!(k+1)!} \frac{1}{(m-k)!(m-k+1)!} \frac{H_2(n-m)}{(n-m)!(n-m+1)!} \frac{1}{(v-n)!\Gamma(\nu+v-n+1)} (z/2)^{2v+}$$

$$+\frac{3}{2^2} \ln \frac{z}{2} \sum_{v\ge 0} \sum_{n=0}^{v} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{1}{k!(k+1)!} \frac{H_2(m-k)}{(m-k)!(m-k+1)!} \frac{H_2(n-m)}{(n-m)!(n-m+1)!} \frac{1}{(v-n)!\Gamma(\nu+v-n+1)} (z/2)^{2v+}$$

$$-\frac{6}{2^2} \ln \frac{z}{2} \sum_{n\ge 0} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{1}{k!(k+1)!} \frac{H_2(m-k)}{(m-k)!(m-k+1)!} \frac{1}{(n-m)!\Gamma(\nu+n-m+1)} (z/2)^{2n+\nu+1+l}.$$

The definite integral is

$$\int_0^2 (z/2)^l I_\nu(z) K_1^3(z) = \frac{1}{2^2}\sum_{k\ge 0}\frac{1}{k!\Gamma(\nu+k+1)}\frac{1}{2k+\nu-2+l} \tag{138}$$

$$-2{\cdot}6\sum_{v\ge 0}\sum_{n=0}^{v}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{1}{k!(1+k)!}\frac{1}{(m-k)!(1+m-k)!}\frac{1}{(n-m)!(1+n-m)!}\frac{1}{(v-n)!\Gamma(\nu+v-n+1)}\frac{1}{(2v+\nu+4-}$$

$$-\frac{1}{2^2}\sum_{v\ge 0}\sum_{n=0}^{v}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{H_2(k)}{k!(k+1)!}\frac{H_2(m-k)}{(m-k)!(m-k+1)!}\frac{H_2(n-m)}{(n-m)!(n-m+1)!}\frac{1}{(v-n)!\Gamma(\nu+v-n+1)}\frac{1}{2v+\nu+4+}$$

$$-\frac{3}{2}\sum_{m\ge 0}\sum_{k=0}^{m}\frac{1}{k!(1+k)!}\frac{1}{(m-k)!\Gamma(\nu+m-k+1)}\frac{1}{(2m+\nu+l)^2}$$

$$-\frac{3}{2}\frac{1}{2}\sum_{m\ge 0}\sum_{k=0}^{m}H_2(k)\frac{1}{k!(k+1)!}\frac{1}{(m-k)!\Gamma(\nu+m-k+1)}\frac{1}{2m+\nu+l}$$

$$+3{\cdot}2\sum_{n\ge 0}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{1}{k!(1+k)!}\frac{1}{(m-k)!(1+m-k)!}\frac{1}{(n-m)!\Gamma(\nu+n-m+1)}\frac{1}{(2n+\nu+2+l)^3}$$

$$+3\frac{1}{2^2}\sum_{n\ge 0}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{H_2(k)}{k!(k+1)!}\frac{H_2(m-k)}{(m-k)!(m-k+1)!}\frac{1}{(n-m)!\Gamma(\nu+n-m+1)}\frac{1}{2n+\nu+2+l)}$$

$$-3{\cdot}2\sum_{v\ge 0}\sum_{n=0}^{v}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{1}{k!(k+1)!}\frac{1}{(m-k)!(m-k+1)!}\frac{H_2(n-m)}{(n-m)!(n-m+1)!}\frac{1}{(v-n)!\Gamma(\nu+v-n+1)}\frac{1}{(2v+\nu+4-}$$

$$-\frac{3}{2}\sum_{v\ge 0}\sum_{n=0}^{v}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{1}{k!(k+1)!}\frac{H_2(m-k)}{(m-k)!(m-k+1)!}\frac{H_2(n-m)}{(n-m)!(n-m+1)!}\frac{1}{(v-n)!\Gamma(\nu+v-n+1)}\frac{1}{(2v+\nu+4+}$$

$$+\frac{6}{2}\sum_{n\ge 0}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{1}{k!(k+1)!}\frac{H_2(m-k)}{(m-k)!(m-k+1)!}\frac{1}{(n-m)!\Gamma(\nu+n-m+1)}\frac{1}{(2n+\nu+2+l)^2}.$$

Appendix B. Higher Derivatives

In conjunction with Euler-Maclaurin formulas, higher derivatives of the Modified Bessel Functions are needed. They both obey a differential equation of the form [1, 9.6.1]

$$\frac{d^2}{dz^2}\mathcal{L}_\nu = (1+\nu^2/z^2)\mathcal{L}_\nu - \frac{1}{z}\frac{d}{dz}\mathcal{L}_\nu. \tag{139}$$

By repeated differentiation, these derivatives may be bootstrapped from the value and *first* derivative at the desired argument z, where the prime denotes derivation with respect to the argument z:

$$\frac{d^3}{dz^3}\mathcal{L}_0 = \frac{1}{z^2}[(2+z^2)\mathcal{L}_0' - z\mathcal{L}_0]. \tag{140}$$

$$\frac{d^4}{dz^4}\mathcal{L}_0 = \frac{1}{z^3}[(-6-2z^2)\mathcal{L}_0' + (3z+z^3)\mathcal{L}_0]. \tag{141}$$

$$\frac{d^5}{dz^5}\mathcal{L}_0 = \frac{1}{z^4}[(24+7z^2+z^4)\mathcal{L}_0' + (-12z-2z^3)\mathcal{L}_0]. \tag{142}$$

$$\frac{d^6}{dz^6}\mathcal{L}_0 = \frac{1}{z^5}[(-120 - 33z^2 - 3z^4)\mathcal{L}_0' + (60z + 9z^3 + z^5)\mathcal{L}_0]. \tag{143}$$

$$\frac{d^7}{dz^7}\mathcal{L}_0 = \frac{1}{z^6}[(720 + 192z^2 + 15z^4 + z^6)\mathcal{L}_0' + (-360z - 51z^3 - 3z^5)\mathcal{L}_0]. \tag{144}$$

This defines reduction polynomials p

$$\frac{d^n}{dz^n}\mathcal{L}_0 = \frac{1}{z^{n-1}}[p_{01n}(z)\mathcal{L}_0' + p_{00n}(z)\mathcal{L}_0]. \tag{145}$$

with recurrences

$$p_{01n} = (1-n)p_{01n-1} + zp_{01n-1}' + zp_{00n-1}; \tag{146}$$

$$p_{00n} = (2-n)p_{00n-1} + zp_{00n-1}' + zp_{01n-1}. \tag{147}$$

In the equivalent manner for index $\nu = 1$

$$\frac{d^3}{dz^3}\mathcal{L}_1 = \frac{1}{z^3}[(3z + z^3)\mathcal{L}_1' + (-z^2 - 3)\mathcal{L}_1]. \tag{148}$$

$$\frac{d^4}{dz^4}\mathcal{L}_1 = \frac{1}{z^4}[(-12z - 2z^3)\mathcal{L}_1' + (5z^2 + 12 + z^4)\mathcal{L}_1]. \tag{149}$$

$$\frac{d^5}{dz^5}\mathcal{L}_1 = \frac{1}{z^5}[(60z + 9z^3 + z^5)\mathcal{L}_1' + (-24z^2 - 60 - 2z^4)\mathcal{L}_1]. \tag{150}$$

$$\frac{d^6}{dz^6}\mathcal{L}_1 = \frac{1}{z^6}[(-360z - 51z^3 - 3z^5)\mathcal{L}_1' + (141z^2 + 360 - 12z^4 + z^6)\mathcal{L}_1]. \tag{151}$$

$$\frac{d^7}{dz^7}\mathcal{L}_1 = \frac{1}{z^7}[(2520z + 345z^3 + 18z^5 + z^7)\mathcal{L}_1' + (-975z^2 - 2520 - 78z^4 - 3z^6)\mathcal{L}_1]. \tag{152}$$

$$\frac{d^n}{dz^n}\mathcal{L}_1 = \frac{1}{z^n}[p_{11n}(z)\mathcal{L}_1' + p_{10n}(z)\mathcal{L}_1]. \tag{153}$$

$$p_{11n} = -np_{11n-1} + zp_{11n-1}' + zp_{10n-1}. \tag{154}$$

$$p_{10n} = (1-n)p_{10n-1} + zp_{10n-1}' + (z + \frac{1}{z})p_{11n-1}. \tag{155}$$

If a break point is set at $z = 2$ for a Euler-Maclaurin expansion, the starting values are [12, A070910,A096789]

$$I_0(2) \approx 2.279585302336067267437204440 8; \tag{156}$$

$$I_0'(z) = I_1(2) \approx 1.5906368546373290633822544250; \tag{157}$$

$$I_1'(z) \approx 1.4842668750174027357460772283; \tag{158}$$

$$K_0(2) \approx 0.1138938727495334356527195749 3; \tag{159}$$

$$K_1(2) = -K_0'(2) \approx 0.1398658818165224272845988070 3; \tag{160}$$

$$K_1'(2) \approx -0.1838268136577946492950189784 5. \tag{161}$$

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