Mathematical Formulas : Part 3

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Abstract

In this paper we give some formulas related with the constant pi:

$$\pi = 4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots\right) = 3.1415926535\ldots$$

1. Introducción

Notación Básica:

La unidad imaginaria es: $i=\sqrt{-1}$, Si $z=x+i\,y\,,x,y\in\mathbb{R}$, entonces : $Re(z)=x\,,Im(z)=y$, $[x]=$ parte entera de x , E_n números de Euler , B_n números de Bernoulli , $P_n(x)$ polinomios de Legendre, $H_n(x)$ polinomios de Hermite , $T_n(x)$ polinomios de Chebyshev , $U_n(x)$ polinomios de Chebyshev de segunda clase , $\binom{n}{k}=\frac{n!}{k!(n-k)!}$ coeficiente binomial , $F(a,b,c,x)$ función hipergeométrica de Gauss.

Algunos números de Euler: $E_n=\{1,5,61,1385,50521,\ldots\}\,,E_0=1$

Algunos números de Bernoulli: $B_n=\left\{\frac{1}{6},\frac{1}{30},\frac{1}{42},\frac{1}{30},\frac{5}{66},\ldots\right\}$

Recordamos algunas fórmulas clásicas para la constante Pi:

(1) $$\frac{\pi\sqrt{3}}{6}=\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n(2n+1)}$$

(2) $$\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

(3) $$\frac{\pi}{2}=2\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)+\tan^{-1}\left(\frac{1}{2\sqrt{2}}\right)$$

(4) $$\pi = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}\left(\frac{2}{4n+1} + \frac{2}{4n+2} + \frac{1}{4n+3}\right)$$

(5) $$\frac{1}{\pi} = \sum_{n=0}^{\infty} \binom{2n}{n}^3 \frac{(42n+5)}{2^{12n+4}}$$

(6) $$\pi = 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)(2n+1)}$$

2. Otras Fórmulas

(7) $$\pi = 2\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \int_u^v \frac{x^{2n+1}\cosh x}{\sqrt{1-\sinh x}\;(\sinh x)^{3/2}}\,dx$$

donde

$$u = \ln\left(\frac{1+\sqrt{5}}{2}\right) \quad , v = \ln\left(1+\sqrt{2}\right)$$

(8) $$\pi = 6\sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^n 2^{-2n}}{2n+1} \int_{1/4}^{1/2} \frac{(\sinh x)^{2n+1}}{x\sqrt{x(1-x)}}\,dx$$

(9) $$\pi = 6\sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^n 2^{-2n}}{2n+1} \int_{1/2}^{3/4} \frac{(\sinh x)^{2n+1}}{x\sqrt{x(1-x)}}\,dx$$

(10) $$\pi = 6\sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^n 2^{-2n}}{2n+1} \int_{(2-\sqrt{3})/4}^{1/4} \frac{(\sinh x)^{2n+1}}{x\sqrt{x(1-x)}}\,dx$$

(11) $$\pi = 6\sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-2n}}{2n+1} \int_{1/4}^{1/2} \frac{(\sin x)^{2n+1}}{x\sqrt{x(1-x)}}\,dx$$

(12) $$\pi = 6\sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-2n}}{2n+1} \int_{1/2}^{3/4} \frac{(\sin x)^{2n+1}}{x\sqrt{x(1-x)}}\,dx$$

(13) $$\pi = 6\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\int_{(2-\sqrt{3})/4}^{1/4}\frac{(\sin x)^{2n+1}}{x\sqrt{x(1-x)}}dx$$

(14) $$\pi\ \ln\left(\frac{\left(2-\sqrt{3}\right)^{1/3}}{\left(\sqrt{2}-1\right)^{1/2}}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n E_n}{(n+1)(2n)!}\left(\left(\ln\left(2-\sqrt{3}\right)\right)^{2n+2}-\left(\ln\left(\sqrt{2}-1\right)\right)^{2n+2}\right) - 4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\left(\left(\sqrt{2}-1\right)^{2n+1}-\left(2-\sqrt{3}\right)^{2n+1}\right)$$

(15) $$\pi\ \ln\left(3^{1/3}\sqrt{\sqrt{2}-1}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n E_n}{(n+1)(2n)!}\left(\left(\ln\left(\sqrt{2}-1\right)\right)^{2n+2}-\left(\ln\left(\frac{1}{\sqrt{3}}\right)\right)^{2n+2}\right) - 4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\left(\left(\frac{1}{\sqrt{3}}\right)^{2n+1}-\left(\sqrt{2}-1\right)^{2n+1}\right)$$

(16) $$\pi = \frac{108}{13}\sum_{n=0}^{\infty}c_n\int_{-1}^{u}x^n\sqrt{1+x}\,dx = \frac{216}{13}\sum_{n=0}^{\infty}(-1)^n\,c_n\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k v^{2k+3}}{2k+3}$$

donde

$$u = \left(\frac{3\left(2-\sqrt{3}\right)}{2}\right)^{2/3}-1\quad, v = \left(\frac{3\left(2-\sqrt{3}\right)}{2}\right)^{1/3}$$

$$c_{n+3} = -\frac{12}{13}c_{n+2}-\frac{12}{13}c_{n+1}-\frac{4}{13}c_n\quad, c_0 = 1, c_1 = -\frac{12}{13}\ , c_2 = -\frac{12}{169}$$

(17) $$\pi = 6\sin\left(\frac{\ln 3}{2}\right)-6\sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n}}{(4n)!}\int_0^{\ln 3/2}\frac{x^{4n}}{\cosh x}dx$$

(18) $$\pi = 6\,Si\left(\frac{\ln 3}{2}\right)-6\sum_{n=1}^{\infty}\frac{(-1)^{[n/2]}2^n}{(2n+1)!}\int_0^{\ln 3/2}\frac{x^{2n}}{\cosh x}dx$$

donde

$$Si(x) = \int_0^x \frac{\sin t}{t} dt$$

(19) $$\pi = 12 \sum_{n=0}^{\infty} \frac{c_n}{2n+1} z^{2n+1}$$

donde

$$c_0 = 1 \ , c_n = \frac{(-1)^n}{(2n+1)!} - \sum_{k=1}^{n} (-1)^{k-1} c_{n-k} \left(\sum_{m=0}^{k-1} \frac{1}{(2k-2m-1)(2k-2m-1)!\,(2m+1)(2m+1)!} \right)$$

$z = 0.26902858 \ldots$, satisface la ecuación: $Si(z) = \int_0^z \frac{\sin t}{t} dt = 2 - \sqrt{3}$

(20) $$\frac{6}{\pi} = \frac{2}{\ln 3} + \sum_{n=1}^{\infty} c_n \left(\frac{\ln 3}{2} \right)^{2n-1}$$

donde

$$c_0 = 1 \ , c_n = - \sum_{k=1}^{n} \frac{(-1)^k E_k}{(2k+1)!} c_{n-k}$$

(21) $$\pi = 12 \sum_{n=0}^{\infty} \frac{(-1)^n \left(2-\sqrt{3}\right)^{2n+1}}{(2n+1)\left(1-\left(2-\sqrt{3}\right)^{2n+1}\right)} - 12 \sum_{n=1}^{\infty} c_n \left(2-\sqrt{3}\right)^{n+1}$$

donde

$$c_n = \sum_{m=0}^{\left[\frac{n-1}{4}\right]} \frac{(-1)^m}{2m+1} a_{m,n} \ , \qquad a_{m,n} = \begin{cases} 1 & si \ \ \dfrac{n+1}{2m+1} \in \mathbb{N} \\ 0 & en\ otro\ caso \end{cases}$$

$$c_n = \left\{ 1,1,1,1,\frac{2}{3},1,1,\frac{2}{3},\frac{6}{5},1,\frac{2}{3},1,\frac{6}{7},\frac{13}{15},1,1,\frac{7}{9},1,\frac{6}{5},\frac{11}{21},\frac{10}{11},1,\ldots \right\}$$

(22) $$\pi = 12 \sum_{n=0}^{\infty} a_n(x)\, b_n(x)$$

donde

$$a_n(x) = \int_0^{2-\sqrt{3}} t^n \, e^{-2xt+t^2} \, dt \quad , n \in \mathbb{N} \cup \{0\}, x \in \mathbb{R}$$

$$b_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k H_{n-2k}(x)}{(n-2k)!}$$

$H_n(x)$, son los polinomios de Hermite

(23) $$\pi = 12 \sum_{n=0}^{\infty} a_n(x) \, b_n(x)$$

donde

$$a_n(x) = \int_0^{2-\sqrt{3}} t^n \sqrt{1-2xt+t^2} \, dt \quad , n \in \mathbb{N} \cup \{0\}, |x| \leq 1$$

$$b_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \, P_{n-2k}(x)$$

$P_n(x)$, son los polinomios de Legendre

(24) $$\pi = 12 \sum_{n=0}^{\infty} a_n(x) \, b_n(x)$$

donde

$$a_n(x) = \left(2-\sqrt{3}\right)^{n+1} \left(\frac{1}{n+1} - \frac{2x\left(2-\sqrt{3}\right)}{n+2} + \frac{\left(2-\sqrt{3}\right)^2}{n+3} \right) \; , n \in \mathbb{N} \cup \{0\}, |x| \leq 1$$

$$b_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \, U_{n-2k}(x)$$

$U_n(x)$, son los polinomios de Chebyshev de segunda clase

(25) $$\pi = 6x \ln\left(8-4\sqrt{3}\right) + 12 \sum_{n=0}^{\infty} a_n(x) \, b_n(x)$$

donde

$$a_n(x) = \left(2-\sqrt{3}\right)^{n+1}\left(\frac{1}{n+1} - \frac{2x\left(2-\sqrt{3}\right)}{n+2} + \frac{\left(2-\sqrt{3}\right)^2}{n+3}\right) \ ,n \in \mathbb{N} \cup \{0\}, |x| \leq 1$$

$$b_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \, T_{n-2k}(x)$$

$T_n(x)$, son los polinomios de Chebyshev

(26) $$\pi = 3\sqrt{\frac{6}{5}}\left(1 + \sum_{n=1}^{\infty}\binom{2n}{n} 2^{-2n} \sum_{k=0}^{n}\binom{n}{k}\frac{(-3/5)^{n-k}(6/5)^k}{2k+1}\right)$$

(27) $$\pi = 12\sum_{n=0}^{\infty}\frac{(-1)^n\left(2-\sqrt{3}\right)^{2n+1}}{(2n+1)\left(1-\left(2-\sqrt{3}\right)^{2n+1}\right)} - 12\ \tan^{-1}\left(f\left(2-\sqrt{3}\right)\right)$$

donde

$$f(x) = \frac{p(x)}{q(x)} = \frac{x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 - x^{11} - \cdots}{1 - x^5 - x^6 - 2x^7 - 2x^8 - 3x^9 - 3x^{10} - 4x^{11} - \cdots}$$

$$p_n(x) = p_{n-1}(x) + x^{n+1}q_{n-1}(x) \quad ,n = 2,3,4,\ldots, |x| < 1$$

$$q_n(x) = q_{n-1}(x) - x^{n+1}p_{n-1}(x) \quad ,n = 2,3,4,\ldots, |x| < 1$$

$$p_1(x) = x^2 \ \ ,q_1(x) = 1$$

$$p_n(x) \to p(x) \quad , \qquad q_n(x) \to q(x)$$

(28) $$\pi = 8\sinh u \ \sin u + 8\sum_{k=0}^{\infty}\frac{(-1)^k 2^{2k+3}u^{4k+6}}{4k+6}\sum_{n=0}^{2k+1}\frac{(-1)^n E_{n+1}}{(2n+2)!\,(4k-2n+3)!}$$

donde

$u = 0.63537231\ldots$, es solución de la ecuación no lineal: $\tanh u \ \tan u = \sqrt{2} - 1$

(29) $$F(y) = \int_0^y \frac{\cosh(2x) - 1}{(2+x^2)\cosh(2x) - 2x\sinh(2x) + x^2}\,dx$$

$$= \int_0^y \frac{(-1+e^{2x})^2}{2 + 2x + x^2 + 2x^2e^{2x} + (2 - 2x + x^2)e^{4x}}\,dx$$

$$y = 1 + \tanh(1 + \tanh(1 + \tanh(1 + \cdots))) \;\Rightarrow\; F(y) = \frac{\pi}{4}$$

$$y = \frac{1}{\sqrt{3}} + \tanh\left(\frac{1}{\sqrt{3}} + \tanh\left(\frac{1}{\sqrt{3}} + \tanh\left(\frac{1}{\sqrt{3}} + \cdots\right)\right)\right) \;\Rightarrow\; F(y) = \frac{\pi}{6}$$

$$y = \sqrt{2} - 1 + \tanh\left(\sqrt{2} - 1 + \tanh\left(\sqrt{2} - 1 + \cdots\right)\right) \;\Rightarrow\; F(y) = \frac{\pi}{8}$$

$$y = 2 - \sqrt{3} + \tanh\left(2 - \sqrt{3} + \tanh\left(2 - \sqrt{3} + \cdots\right)\right) \;\Rightarrow\; F(y) = \frac{\pi}{12}$$

(30) $$\pi = 4\sum_{n=0}^{\infty} \frac{c_n}{2n+3}\left(12\alpha^{2n+3} + 8\beta^{2n+3} - 5\gamma^{2n+3}\right)$$

donde

$$c_0 = 1\,, c_1 = -\frac{2}{3}\,, \qquad c_n = \frac{2^{2n+1}}{(2n+2)!} - c_{n-1} - \sum_{k=2}^{n} \frac{2^{2k-2}(2k^2 - 5k + 4)c_{n-k}}{(2k)!}$$

$$\alpha = \frac{1}{18} + \tanh\left(\frac{1}{18} + \tanh\left(\frac{1}{18} + \cdots\right)\right) = 0.57342771\ldots$$

$$\beta = \frac{1}{57} + \tanh\left(\frac{1}{57} + \tanh\left(\frac{1}{57} + \cdots\right)\right) = 0.38190174\ldots$$

$$\gamma = \frac{1}{239} + \tanh\left(\frac{1}{239} + \tanh\left(\frac{1}{239} + \cdots\right)\right) = 0.23408798\ldots$$

(31) $$\pi = -4\sum_{n=1}^{\infty} \frac{\ln\left((1 - 2^{-n})(1 - 3^{-n})\right)}{n} c_n$$

donde

$$c_n = \sum_{\substack{d/n \\ d\ impar}} (-1)^{(d-1)/2} \mu\left(\frac{n}{d}\right)$$

$$c_n = \{1, -1, -2, 0, 0, 2, -2, 0, 2, 0, \ldots\}$$

μ , es la función Mu de Moebius

$$\mu(k) = \begin{cases} 1 & , si\ k = 1 \\ (-1)^r & , si\ k = al\ producto\ de\ r\ primos\ distintos \\ 0 & , en\ otro\ caso \end{cases}$$

(32) $$\pi = 8\sum_{n=0}^{\infty}\frac{1}{2n+1}\cosh\left(\frac{x_n}{5^{2n+1}}\right)\sin\left(\frac{y_n}{5^{2n+1}}\right) - 4\sum_{n=1}^{\infty}\frac{1}{(2n+1)!}\tan^{-1}\left(\frac{2y_n}{5^{2n+1}-1}\right)$$

(33) $$\pi = 8\sum_{n=0}^{\infty}\frac{1}{2n+1}\cos\left(\frac{x_n}{5^{2n+1}}\right)\sinh\left(\frac{y_n}{5^{2n+1}}\right) + 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n+1)!}\tan^{-1}\left(\frac{2y_n}{5^{2n+1}-1}\right)$$

En las fórmulas (32)-(33) , $x_n = Re((1+2i)^{2n+1})$, $y_n = Im((1+2i)^{2n+1})$

(34) $$\pi = 8\sum_{n=0}^{\infty}\frac{1}{2n+1}\cosh\left(\frac{\left(\sqrt{3}-1\right)^{2n+1}}{2^{n+1}}\ x_n\right)\sin\left(\frac{\left(\sqrt{3}-1\right)^{2n+1}}{2^{n+1}}\ y_n\right) - 4\sum_{n=1}^{\infty}\frac{1}{(2n+1)!}\tan^{-1}\left(\frac{2^{n+1}\left(\sqrt{3}-1\right)^{2n+1}y_n}{2^{2n+1}-\left(\sqrt{3}-1\right)^{4n+2}}\right)$$

(35) $$\pi = 8\sum_{n=0}^{\infty}\frac{1}{2n+1}\cos\left(\frac{\left(\sqrt{3}-1\right)^{2n+1}}{2^{n+1}}\ x_n\right)\sinh\left(\frac{\left(\sqrt{3}-1\right)^{2n+1}}{2^{n+1}}\ y_n\right) + 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n+1)!}\tan^{-1}\left(\frac{2^{n+1}\left(\sqrt{3}-1\right)^{2n+1}y_n}{2^{2n+1}-\left(\sqrt{3}-1\right)^{4n+2}}\right)$$

En las fórmulas (34)-(35) , $x_n = Re\big((1+i)i^n\big)$, $y_n = Im\big((1+i)i^n\big)$

(36) $$\pi = 12\sum_{n=0}^{\infty}\frac{c_n}{n+1}u^{n+1}$$

donde

$u = 0.21591456\ldots$, es solución de la ecuación no lineal: $u\,e^u = 2-\sqrt{3}$

$$u = \left(2-\sqrt{3}\right)e^{-\left(2-\sqrt{3}\right)e^{-\left(2-\sqrt{3}\right)\cdots}}$$

$$c_0 = 1, c_1 = 2, c_n = \frac{1+n}{n!} - \sum_{k=2}^{n}\frac{2^{k-2}c_{n-k}}{(k-2)!} \quad , n = 2,3,4,\ldots$$

(37) $$\pi = 12 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(1 - \sqrt{1-4a^{2n+1}}\right) - 24 \sum_{n=1}^{\infty} c(n) \tan^{-1}(a^{n+1})$$

donde

$$a = \frac{2\sqrt{2-\sqrt{3}}-1}{2-\sqrt{3}}$$

(38) $$\pi = 24 \sum_{n=1}^{\infty} (-1)^{n-1} c(n) \tan^{-1}(a^{n+1}) - 12 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(1 - \sqrt{1+4a^{2n+1}}\right)$$

donde

$$a = \frac{2\sqrt{2-\sqrt{3}}-1}{2-\sqrt{3}}$$

(39) $$\pi = 6 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\sqrt{1+4a^{2n+1}} - \sqrt{1-4a^{2n+1}}\right) - 24 \sum_{n=1}^{\infty} c(2n) \tan^{-1}(a^{2n+1})$$

donde

$$a = \frac{2\sqrt{2-\sqrt{3}}-1}{2-\sqrt{3}}$$

En las fórmulas (37)-(38)-(39) , $c(n)$ son los números de Catalan:

$$c(n) = \frac{1}{n+1}\binom{2n}{n}$$

$$c(n+1) = \frac{2(2n+1)}{n+2}\, c(n) \ , \qquad c(0) = 1$$

$$c(2n+2) = \frac{2(4n+1)(4n+3)}{(n+1)(2n+3)}\, c(2n)$$

(40) $$\pi = 12 \sum_{n=1}^{\infty} \frac{a^n}{n} b_n + 24 \sum_{n=1}^{\infty} b_n\, 2^{-2n} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(-1)^k}{2k+3}\left(1 - (1-4a)^{k+\frac{3}{2}}\right)$$

donde

$$a = \frac{2\sqrt{2-\sqrt{3}}-1}{2-\sqrt{3}}$$

$$b_n = \sum_{k=0}^{[(n-1)/2]} (-1)^k \, c(n-2k-1)$$

$c(n)$, números de Catalan

(41) $$\pi = 4\tan^{-1}\left(\frac{\tanh 1}{\tan 1}\right) + 4\tan^{-1}\left(\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1}}{(4n+3)!}\Bigg/\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{(4n+1)!}\right)$$

(42) $$\pi = 4\tan^{-1}\left(\frac{\sin 1 \sinh 1 + \cos 1 \cosh 1 - 1}{\sin 1 \sinh 1 - \cos 1 \cosh 1 + 1}\right) + 4\tan^{-1}\left(\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1}}{(4n+4)!}\Bigg/\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{(4n+2)!}\right)$$

(43) $$\pi = 4 + 2\sqrt{2}\sum_{n=1}^{\infty} e^{-2n}\frac{1}{\sqrt{n}}\gamma\left(\frac{1}{2}, 2n\right) - 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}B_n}{(2n)!}\sum_{k=0}^{2n-1}\binom{2n-1}{k}\frac{1}{2k+1}$$

(44) $$\pi = 2\sqrt{3} + 3\sqrt{2}\sum_{n=1}^{\infty} e^{-2n}\frac{1}{\sqrt{n}}\gamma\left(\frac{1}{2}, \frac{2n}{3}\right) - 2\sqrt{3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}B_n}{(2n)!}\sum_{k=0}^{2n-1}\binom{2n-1}{k}\frac{3^{-k}}{2k+1}$$

(45) $$\pi = 8\left(\sqrt{2}-1\right) + 4\sqrt{2}\sum_{n=1}^{\infty} e^{-2n}\frac{1}{\sqrt{n}}\gamma\left(\frac{1}{2}, \left(6-4\sqrt{2}\right)n\right) - 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}B_n}{(2n)!}\sum_{k=0}^{2n-1}\binom{2n-1}{k}\frac{\left(\sqrt{2}-1\right)^{2k+1}}{2k+1}$$

(46) $$\pi = 12\left(2-\sqrt{3}\right) + 6\sqrt{2}\sum_{n=1}^{\infty} e^{-2n}\frac{1}{\sqrt{n}}\gamma\left(\frac{1}{2}, \left(14-8\sqrt{3}\right)n\right) - 12\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}B_n}{(2n)!}\sum_{k=0}^{2n-1}\binom{2n-1}{k}\frac{\left(2-\sqrt{3}\right)^{2k+1}}{2k+1}$$

(47) $$\pi = 4\sum_{n=0}^{\infty} e^{-(2n+1)}\frac{1}{\sqrt{2n+1}}\gamma\left(\frac{1}{2}, 2n+1\right) + 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2^{2n-1}-1)B_n}{(2n)!}\sum_{k=0}^{2n-1}\binom{2n-1}{k}\frac{1}{2k+1}$$

(48) $$\pi = 6\sum_{n=0}^{\infty} e^{-(2n+1)}\frac{1}{\sqrt{2n+1}}\gamma\left(\frac{1}{2}, \frac{2n+1}{3}\right) + 4\sqrt{3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2^{2n-1}-1)B_n}{(2n)!}\sum_{k=0}^{2n-1}\binom{2n-1}{k}\frac{3^{-k}}{2k+1}$$

(49) $$\pi = 8\sum_{n=0}^{\infty} e^{-(2n+1)}\frac{1}{\sqrt{2n+1}}\gamma\left(\frac{1}{2}, (2n+1)\left(3-2\sqrt{2}\right)\right) + 16\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2^{2n-1}-1)B_n}{(2n)!}\sum_{k=0}^{2n-1}\binom{2n-1}{k}\frac{\left(\sqrt{2}-1\right)^{2k+1}}{2k+1}$$

(50) $$\pi = 12\sum_{n=0}^{\infty} e^{-(2n+1)}\frac{1}{\sqrt{2n+1}}\gamma\left(\frac{1}{2}, (2n+1)\left(7-4\sqrt{3}\right)\right) + 24\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2^{2n-1}-1)B_n}{(2n)!}\sum_{k=0}^{2n-1}\binom{2n-1}{k}\frac{\left(2-\sqrt{3}\right)^{2k+1}}{2k+1}$$

En las fórmulas (43)-(44)-(45)-(46)-(47)-(48)-(49)-(50) , aparece la función Gamma incompleta:

$$\gamma\left(\frac{1}{2}, 2ny^2\right) = 2\sqrt{2n}\int_0^y e^{-2nx^2}\,dx$$

$$\gamma\left(\frac{1}{2}, (2n+1)y^2\right) = 2\sqrt{2n+1}\int_0^y e^{-(2n+1)x^2}\,dx$$

(51) $$\frac{10}{3\pi} = \sum_{n=0}^{\infty}\left(-\frac{1}{36}\right)^n c_n$$

donde

$$c_0 = 1\ , \qquad c_n = -\sum_{k=1}^{n}\frac{d_k}{10k+5}\,c_{n-k}\ , n \in \mathbb{N}$$

$$d_k = 2^{2k+1} + 3^{2k+1} \ , k \in \mathbb{N}$$

$$d_{k+2} = 13 d_{k+1} - 36 d_k \ , d_1 = 35 \, , d_2 = 275$$

(52) $$\frac{1}{\pi} = \frac{3^{-2/3}}{\Gamma(2/3)} \sum_{n=0}^{\infty} \frac{3^n (1/3)_n}{(3n)!} \ \alpha^{3n} - \frac{3^{-1/3}}{\Gamma(1/3)} \sum_{n=0}^{\infty} \frac{3^n (2/3)_n}{(3n+1)!} \ \alpha^{3n+1}$$

donde

$\alpha = 0.142495089 \ldots$, es el número real que satisface la ecuación:

$$\int_0^{\infty} \cos\left(\frac{x^3}{3} + \alpha \, x\right) dx = 1$$

$\Gamma(x)$,es la clásica función Gamma.

El número α , también satisface la ecuación:

$$\int_0^{\infty} exp\left(-\frac{x^3}{3} - \frac{\alpha^3}{3x^3}\right) dx = \frac{2}{\sqrt{3}}$$

(53) $$\frac{1}{\pi} = \frac{3^{-2/3}}{\Gamma(2/3)} \sum_{n=0}^{\infty} \frac{(-1)^n 3^n (1/3)_n}{(3n)!} \ \alpha_m^{3n} + \frac{3^{-1/3}}{\Gamma(1/3)} \sum_{n=0}^{\infty} \frac{(-1)^n 3^n (2/3)_n}{(3n+1)!} \ \alpha_m^{3n+1}$$

donde

$$m = 1,2,3,4,5,6,7$$

$$\alpha_1 = 1.842930 \ldots$$

$$\alpha_2 = 4.553732 \ldots$$

$$\alpha_3 = 5.081701 \ldots$$

$$\alpha_4 = 7.233030 \ldots$$

$$\alpha_5 = 7.510458 \ldots$$

$$\alpha_6 = 9.493291 \ldots$$

$$\alpha_7 = 9.577545 \ldots$$

Los números α_m , satisfacen la ecuación:

$$\int_0^\infty \cos\left(\frac{x^3}{3} - \alpha_m\ x\right) dx = 1\ \ , m = 1,2,3,4,5,6,7$$

(54) $$\frac{1}{\pi} = -\frac{3^{-2/3}}{\Gamma(2/3)} \sum_{n=0}^{\infty} \frac{(-1)^n 3^n (1/3)_n}{(3n)!}\ \beta_m^{3n} - \frac{3^{-1/3}}{\Gamma(1/3)} \sum_{n=0}^{\infty} \frac{(-1)^n 3^n (2/3)_n}{(3n+1)!}\ \beta_m^{3n+1}$$

donde

$$m = 1,2,3,4,5,6$$

$$\beta_1 = 2.847171\ldots$$

$$\beta_2 = 3.633617\ldots$$

$$\beta_3 = 5.971042\ldots$$

$$\beta_4 = 6.353608\ldots$$

$$\beta_5 = 8.394892\ldots$$

$$\beta_6 = 8.581739\ldots$$

Los números β_m , satisfacen la ecuación:

$$\int_0^\infty \cos\left(\frac{x^3}{3} - \beta_m\ x\right) dx = -1\ \ , m = 1,2,3,4,5,6$$

(55) $$\pi = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_1^\infty \left(e^{\left(1+x^2\right)^{-1}} - 1\right)^n dx$$

(56) $$\pi = 12 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_1^{\sqrt{3}} \left(e^{\left(1+x^2\right)^{-1}} - 1\right)^n dx$$

(57) $$\pi = 12 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_{\sqrt{3}}^{2+\sqrt{3}} \left(e^{\left(1+x^2\right)^{-1}} - 1\right)^n dx$$

(58) $$\pi = 12 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_{2+\sqrt{3}}^{\infty} \left(e^{\left(1+x^2\right)^{-1}} - 1\right)^n dx$$

(59) $$\pi = \frac{4}{\sqrt{1-x^2}}\left(\frac{x\ \ln 2}{2} + \sum_{n=0}^{\infty}\frac{T_n(x)}{n+1}\left(\left(x+\sqrt{1-x^2}\right)^{n+1} - x^{n+1}\right)\right)$$

donde

$$-1 < x < 0$$

$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)\ \ , T_0(x) = 1, T_1(x) = x$$

$T_n(x)$, son los polinomios de Chebyshev

(60) $$\pi = 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\ a^{2n}\int_0^1 \frac{(1+x^2)^{n-1}}{\ln(1+a^2+a^2x^2)}dx$$

$$0 < a < 1/\sqrt{2}$$

(61) $$\pi = 2\sqrt{3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\ a^{2n}\int_0^1 \frac{\left(1+\frac{1}{3}\ x^2\right)^{n-1}}{\ln\left(1+a^2+\frac{1}{3}a^2x^2\right)}dx$$

$$0 < a < \sqrt{3}/2$$

(62) $$\pi = \frac{12}{\sinh 1}\sum_{n=0}^{\infty}\frac{c_n}{2n+1}\left(2-\sqrt{3}\right)^{2n+1} + 24\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2^{2n-1}-1)B_n}{(2n)!}\sum_{k=0}^{2n-1}\binom{2n-1}{k}\frac{\left(2-\sqrt{3}\right)^{2k+1}}{2k+1}$$

(63) $$\pi = \frac{8}{\sinh 1}\sum_{n=0}^{\infty}\frac{c_n}{2n+1}\left(\sqrt{2}-1\right)^{2n+1} + 16\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2^{2n-1}-1)B_n}{(2n)!}\sum_{k=0}^{2n-1}\binom{2n-1}{k}\frac{\left(\sqrt{2}-1\right)^{2k+1}}{2k+1}$$

En las fórmulas (62)-(63) , se tiene:

$$c_0 = 1\ ,\qquad c_n = -\sum_{k=1}^{n} a_k\, c_{n-k}\ \ , n \in \mathbb{N}$$

$$a_{2k} = \frac{1}{(2k)!}\ ,\qquad a_{2k-1} = \frac{\coth 1}{(2k-1)!}\ ,\qquad k \in \mathbb{N}$$

$$c_1 = -\coth 1 \ , c_2 = -\frac{1}{2} + (\coth 1)^2 \ , c_3 = \frac{5}{6}\coth 1 - (\coth 1)^3 \ , \ldots$$

(64) $$\pi \ \ln(1+a^2) = 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\ a^{2n}\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{1}{2k+1} - 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n(2n+1)}\left(\frac{a^2}{1+a^2}\right)^n F\left(1,1,n+\frac{3}{2},\frac{1}{2}\right)$$

$$0 < a < 1/\sqrt{2}$$

(65) $$\pi\sqrt{3}\ \ln(1+a^2) = 6\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\ a^{2n}\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{3^{-k}}{2k+1} - \frac{9}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}3^{-n}}{n(2n+1)}\left(\frac{a^2}{1+a^2}\right)^n F\left(1,1,n+\frac{3}{2},\frac{1}{4}\right)$$

$$0 < a < \sqrt{3}/2$$

(66) $$\pi = 12\sum_{n=0}^{\infty}(-1)^n\ e^{-(2n+1)\sqrt{3}/2}\ I_n + 3\sum_{n=1}^{\infty}\frac{(-1)^{n-1}E_n}{n\ (2n)!}\ \sin\left(\frac{n\pi}{3}\right)$$

donde

$$I_n = \int_0^1 \frac{\sqrt{3}\ \cos\left((2n+1)\frac{x}{2}\right) - x\ \sin\left((2n+1)\frac{x}{2}\right)}{3+x^2}dx \ \ , n = 0,1,2,3,\ldots$$

(67) $$\pi = 8\sum_{n=0}^{\infty}(-1)^n\ e^{-(2n+1)/\sqrt{2}}\ I_n + 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}E_n}{n\ (2n)!}\ \sin\left(\frac{n\pi}{2}\right)$$

donde

$$I_n = \int_0^1 \frac{\cos\left((2n+1)\frac{x}{\sqrt{2}}\right) - x\ \sin\left((2n+1)\frac{x}{\sqrt{2}}\right)}{1+x^2}dx \ \ , n = 0,1,2,3,\ldots$$

(68) $$\pi = 6\sqrt{3}\sum_{n=0}^{\infty}(-1)^n\ e^{-(2n+1)/2}\ I_n + \frac{3}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}E_n}{n\ (2n)!}\ \sin\left(\frac{2n\pi}{3}\right)$$

donde

$$I_n = \int_0^1 \frac{\cos\left((2n+1)\frac{\sqrt{3}\,x}{2}\right) - x\sqrt{3}\ \sin\left((2n+1)\frac{\sqrt{3}\,x}{2}\right)}{1+3x^2} dx \ \ , n = 0,1,2,3, \dots$$

(69) $$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^1 Re\left(\frac{(1+i\,x)^{2n}}{\sin(1+i\,x)}\right) dx$$

(70) $$\pi = 6\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^1 Re\left(\frac{\left(\sqrt{3}+i\,x\right)^{2n}}{\sin\left(\sqrt{3}+i\,x\right)}\right) dx$$

(71) $$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^1 Im\left(-\frac{(x+i)^{2n}}{\sin(x+i)}\right) dx$$

(72) $$\pi = 6\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^1 Im\left(-\frac{\left(x+i\,\sqrt{3}\right)^{2n}}{\sin\left(x+i\,\sqrt{3}\right)}\right) dx$$

(73) $$\pi = 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!} \int_0^1 Re\left(\frac{(1+i\,x)^{2n-1}}{1-\cos(1+i\,x)}\right) dx$$

(74) $$\pi = 6\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!} \int_0^1 Re\left(\frac{\left(\sqrt{3}+i\,x\right)^{2n-1}}{1-\cos\left(\sqrt{3}+i\,x\right)}\right) dx$$

(75) $$\pi = 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!} \int_0^1 Im\left(-\frac{(x+i\,)^{2n-1}}{1-\cos(x+i)}\right) dx$$

(76) $$\pi = 6\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!} \int_0^1 Im\left(-\frac{\left(x+i\,\sqrt{3}\right)^{2n-1}}{1-\cos\left(x+i\,\sqrt{3}\right)}\right) dx$$

(77) $$\pi = 4\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \int_0^1 Re\left(\frac{(1+i\,x)^{2n}}{\sinh(1+i\,x)}\right) dx$$

(78) $$\pi = 6\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \int_0^1 Re\left(\frac{\left(\sqrt{3}+i\,x\right)^{2n}}{\sinh\left(\sqrt{3}+i\,x\right)}\right) dx$$

(79) $$\pi = 4\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\int_0^1 Im\left(-\frac{(x+i)^{2n}}{\sinh(x+i)}\right)dx$$

(80) $$\pi = 6\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\int_0^1 Im\left(-\frac{\left(x+i\sqrt{3}\right)^{2n}}{\sinh\left(x+i\sqrt{3}\right)}\right)dx$$

(81) $$\pi = 4\sum_{n=1}^{\infty}\frac{1}{(2n)!}\int_0^1 Re\left(\frac{(1+i\,x)^{2n-1}}{\cosh(1+i\,x)-1}\right)dx$$

(82) $$\pi = 6\sum_{n=1}^{\infty}\frac{1}{(2n)!}\int_0^1 Re\left(\frac{\left(\sqrt{3}+i\,x\right)^{2n-1}}{\cosh\left(\sqrt{3}+i\,x\right)-1}\right)dx$$

(83) $$\pi = 4\sum_{n=1}^{\infty}\frac{1}{(2n)!}\int_0^1 Im\left(-\frac{(x+i\,)^{2n-1}}{\cosh(x+i)-1}\right)dx$$

(84) $$\pi = 6\sum_{n=1}^{\infty}\frac{1}{(2n)!}\int_0^1 Im\left(-\frac{\left(x+i\sqrt{3}\right)^{2n-1}}{\cosh\left(x+i\sqrt{3}\right)-1}\right)dx$$

(85) $$\pi = 4\sum_{n=0}^{\infty}\binom{m+n}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{r=0}^{\left[\frac{m+k}{2}\right]}\binom{m+k}{2r}\frac{(-1)^r\,t^{m+k+1}}{2r+1}$$

$$0 < t < 1\ , \qquad m \in \mathbb{N}$$

(86) $$\pi = 2\sqrt{3}\sum_{n=0}^{\infty}\binom{m+n}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{r=0}^{\left[\frac{m+k}{2}\right]}\binom{m+k}{2r}\frac{(-1)^r\,3^{-r}}{2r+1}$$

$$m \in \mathbb{N}$$

(87) $$\pi$$
$$= 6\sum_{n=0}^{\infty}\binom{m+n}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{r=0}^{\left[\frac{m+k-1}{2}\right]}\binom{m+k}{2r+1}\frac{(-1)^{r+1}\left(\left(\sqrt{3}\right)^{m+k-2r}-1\right)}{(m+k-2r)\,2^{m+k}}$$

$$m \in \mathbb{N}$$

(88) $$\pi = 4\sum_{n=1}^{\infty}\frac{1}{n!}\int_0^1 Re\left(\frac{(1+i\,x)^{n-1}}{e^{1+i\,x}-1}\right)dx$$

(89) $$\pi = 6\sum_{n=1}^{\infty}\frac{1}{n!}\int_0^1 Re\left(\frac{\left(\sqrt{3}+i\,x\right)^{n-1}}{e^{\sqrt{3}+i\,x}-1}\right)dx$$

(90) $$\pi = 4\sum_{n=1}^{\infty}\frac{1}{n!}\int_0^1 Im\left(-\frac{(x+i\,)^{n-1}}{e^{x+i}-1}\right)dx$$

(91) $$\pi = 6\sum_{n=1}^{\infty}\frac{1}{n!}\int_0^1 Im\left(-\frac{\left(x+i\,\sqrt{3}\right)^{n-1}}{e^{x+i\,\sqrt{3}}-1}\right)dx$$

(92) $$\pi = 12\sum_{n=1}^{\infty} c_n\ Si\left(\left(2-\sqrt{3}\right)^{2n-1}\right)$$

donde

$$Si(x) = \int_0^x \frac{\sin t}{t}\,dt$$

$$c_1 = 1\ ,\qquad c_m = \frac{(-1)^{m-1}}{2m-1} - \sum_{\substack{1\le n\le m-1\\ 2\le k\le m\\ (2n-1)(2k-1)=2m-1}} \frac{(-1)^{k-1}c_n}{(2k-1)(2k-1)!}$$

$$c_m = \left\{1, -\frac{5}{18}, \frac{119}{600}, -\frac{5039}{35280}, \frac{312479}{3265920}, \dots\right\}$$

(93) $$\pi = 6\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\int_0^{1/2}\frac{x^{2n+1}}{\sqrt{1-x^2}\ \sin x}\,dx$$

(94) $$\pi = 6\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\int_0^{1/2}\frac{x^{2n+1}}{\sqrt{1-x^2}\ \sinh x}\,dx$$

(95) $$\pi = 6\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\int_0^{1/2}\frac{x^{2n}}{\sqrt{1-x^2}\ \cos x}\,dx$$

(96) $$\pi = 6\sum_{n=0}^{\infty}\frac{1}{(2n)!}\int_{0}^{1/2}\frac{x^{2n}}{\sqrt{1-x^2}\ \cosh x}dx$$

(97) $$\pi = 6\int_{0}^{1/2}\frac{\cos x}{\sqrt{1-x^2}}dx + 6\sum_{n=1}^{\infty}\frac{E_n}{(2n)!}\int_{0}^{1/2}\frac{x^{2n}\cos x}{\sqrt{1-x^2}}dx$$

(98) $$\pi = 6\int_{0}^{1/2}\frac{\cosh x}{\sqrt{1-x^2}}dx + 6\sum_{n=1}^{\infty}\frac{(-1)^n E_n}{(2n)!}\int_{0}^{1/2}\frac{x^{2n}\cosh x}{\sqrt{1-x^2}}dx$$

(99) $$\pi = 6\int_{0}^{1/2}\frac{\sin x}{x\sqrt{1-x^2}}dx + 12\sum_{n=1}^{\infty}\frac{(2^{2n-1}-1)\,B_n}{(2n)!}\int_{0}^{1/2}\frac{x^{2n-1}\sin x}{\sqrt{1-x^2}}dx$$

(100) $$\pi = 6\int_{0}^{1/2}\frac{\sinh x}{x\sqrt{1-x^2}}dx + 12\sum_{n=1}^{\infty}\frac{(-1)^n(2^{2n-1}-1)\,B_n}{(2n)!}\int_{0}^{1/2}\frac{x^{2n-1}\sinh x}{\sqrt{1-x^2}}dx$$

(101) $$\pi = 12\sum_{n=0}^{\infty}c_n\int_{0}^{2-\sqrt{3}}\frac{x^n}{\cos x+\sin x}dx$$

donde

$$c_{2n} = (-1)^n\sum_{k=0}^{n}\frac{1}{(2k)!}\ ,\qquad c_{2n+1} = (-1)^n\sum_{k=0}^{n}\frac{1}{(2k+1)!}$$

$$c_n = \left\{1,1,-\frac{3}{2},-\frac{7}{6},\frac{37}{24},\frac{47}{40},\dots.\right\}$$

(102) $$\pi = 8\sum_{n=0}^{\infty}\frac{1}{n+1}\ Im\left(\frac{1}{P_n(1-2i)\,P_{n+1}(1-2i)}\right)$$

(103) $$\pi = 12\sum_{n=0}^{\infty}\frac{1}{n+1}\ Im\left(\frac{1}{P_n\left(1-2\sqrt{3}\,i\right)P_{n+1}\left(1-2\sqrt{3}\,i\right)}\right)$$

(104) $$\pi = 16\sum_{n=0}^{\infty}\frac{1}{n+1}\ Im\left(\frac{1}{P_n\left(1-2\left(\sqrt{2}+1\right)i\right)P_{n+1}\left(1-2\left(\sqrt{2}+1\right)i\right)}\right)$$

(105) $$\pi = 24 \sum_{n=0}^{\infty} \frac{1}{n+1}\, Im\left(\frac{1}{P_n\big(1-2(2+\sqrt{3})i\big)\, P_{n+1}\big(1-2(2+\sqrt{3})i\big)}\right)$$

(106) $$\pi = 8 \sum_{n=0}^{\infty} \frac{1}{n+1}\, Im\left(\frac{1}{P_n(1-4i)\, P_{n+1}(1-4i)} + \frac{1}{P_n(1-6i)\, P_{n+1}(1-6i)}\right)$$

En las fórmulas (102)-(103)-(104)-(105)-(106) , $P_n(x)$, son los polinomios de Legendre.

(107) $$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^1 \frac{x^{2n+1}}{(1+x^2)\ \sin x}\, dx$$

(108) $$\pi = 4 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \int_0^1 \frac{x^{2n+1}}{(1+x^2)\ \sinh x}\, dx$$

(109) $$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int_0^1 \frac{x^{2n}}{(1+x^2)\ \cos x}\, dx$$

(110) $$\pi = 4 \sum_{n=0}^{\infty} \frac{1}{(2n)!} \int_0^1 \frac{x^{2n}}{(1+x^2)\ \cosh x}\, dx$$

(111) $$\pi = 12 \sum_{n=0}^{\infty} c_n \int_0^{2-\sqrt{3}} \frac{x^n}{\cos x - \sin x}\, dx$$

donde

$$c_{2n} = (-1)^n \sum_{k=0}^{n} \frac{1}{(2k)!}\ , \qquad c_{2n+1} = (-1)^{n-1} \sum_{k=0}^{n} \frac{1}{(2k+1)!}$$

(112) $$\pi\ \ln 2 = 8 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \int_0^1 \frac{\sin\big((2n+1)x\big)}{1+x^2}\, dx$$

(113) $$\pi = 3 \sum_{n=1}^{\infty} (-1)^{n-1} H_n \int_0^{\alpha} \frac{x^n(1-\ln(1+x))}{(\ln(1+x))^{3/2}\sqrt{1+x-\ln(1+x)}}\, dx$$

donde

$\alpha = 0.4296118247$ ... , es solución de la ecuación no lineal:

$$\ln(1+\alpha) = \frac{1}{4}(1+\alpha)$$

(114) $$\pi = 6\sum_{n=1}^{\infty}(-1)^{n-1} H_n \int_0^{\beta} \frac{x^n(1-\ln(1+x))}{(\ln(1+x))^{3/2}\sqrt{1+x-\ln(1+x)}}\,dx$$

donde

$\beta = 0.0746417465 \ldots$, es solución de la ecuación no lineal:

$$\left(8+4\sqrt{3}\right)\ln(1+\beta) = 1+\beta$$

En las fórmulas (113)-(114) , se tiene:

$$H_n = \sum_{k=1}^{n}\frac{1}{k}$$

(115) $$\frac{1}{\pi} = \frac{1}{3}\sum_{n=0}^{\infty} c_n\, 2^{-4n}$$

(116) $$\frac{\sqrt{2}}{\pi} = \frac{1}{2}\sum_{n=0}^{\infty} c_n\, 2^{-3n}$$

(117) $$\frac{\sqrt{2-\sqrt{2}}}{\pi} = \frac{1}{4}\sum_{n=0}^{\infty} c_n\left(\frac{2-\sqrt{2}}{16}\right)^n$$

En las fórmulas (115)-(116)-(117) , se tiene:

$$c_0 = 1 \;, c_n = -\sum_{k=1}^{n}\binom{2k}{k}\frac{c_{n-k}}{2k+1} \;, n \in \mathbb{N}$$

$$c_n = \left\{1, -\frac{2}{3}, -\frac{34}{45}, -\frac{1468}{945}, -\frac{55718}{14175}, \ldots\right\}$$

(118) $$\pi^2 = 32\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^n - 32\sum_{n=1}^{\infty}\frac{H_n}{n}\left(1-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^n$$

(119) $$\pi^2 = 72\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1-\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^n - 72\sum_{n=1}^{\infty}\frac{H_n}{n}\left(1-\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^n$$

(120) $$\pi^2 = 128 \sum_{n=1}^{\infty} \frac{1}{n^2}\left(1-\frac{\sqrt{2+\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^n - 128 \sum_{n=1}^{\infty} \frac{H_n}{n}\left(1-\frac{\sqrt{2+\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^n$$

(121) $$\pi^2 = 288 \sum_{n=1}^{\infty} \frac{1}{n^2}\left(1-\frac{\sqrt{6}+\sqrt{2}}{4}-\frac{i\left(\sqrt{6}-\sqrt{2}\right)}{4}\right)^n - 288 \sum_{n=1}^{\infty} \frac{H_n}{n}\left(1-\frac{\sqrt{6}+\sqrt{2}}{4}-\frac{i\left(\sqrt{6}-\sqrt{2}\right)}{4}\right)^n$$

(122) $$\pi \ln 2 = 16 \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(\sum_{k=1}^{n} \frac{1}{2k-1}\right) Im\left(\left(\frac{3-4i}{25}\right)^n\right)$$

(123) $$\pi^2 - 4(\ln 2)^2 = 64 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(\sum_{k=1}^{n} \frac{1}{2k-1}\right) Re\left(\left(\frac{3-4i}{25}\right)^n\right)$$

(124) $$\pi = -2 \sum_{n=0}^{\infty} \frac{4n+3}{(n+1)(2n+1)P_{2n}(i)P_{2n+2}(i)}$$

donde $P_n(x)$, son los polinomios de Legendre.

(125) $$\frac{\pi}{2\sqrt{3}} = \lim_{n\to\infty} s_n$$

donde

$$s_{n+2} = \frac{4n+12}{6n+15}\, s_{n+1} + \frac{2n+3}{6n+15}\, s_n \quad , s_0 = 1, s_1 = \frac{8}{9}$$

(126) $$\frac{2\sqrt{3}}{\pi} = \lim_{n\to\infty} s_n$$

donde

$$s_{n+2} = \frac{(6n+15)s_n s_{n+1}}{(4n+12)s_n + (2n+3)s_{n+1}} \quad , s_0 = 1, s_1 = \frac{9}{8}$$

(127) $$\pi = 24 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\, G\left(\left(2-\sqrt{3}\right)^{2n+1}\right) - 12 \sum_{n=1}^{\infty} c_n \tan^{-1}\left(\left(2-\sqrt{3}\right)^{n+1}\right)$$

donde

$$G(x)=\frac{x}{1-x+x^2+\sqrt{1-2x-x^2-2x^3+x^4}}$$

$$c_n=\sum_{k=1}^{n}\frac{1}{n+1-k}\binom{n+1-k}{k}\binom{n+1-k}{k-1}\quad ,n\in\mathbb{N}$$

(128) $$\pi=3\sum_{n=1}^{\infty}\frac{c_n}{n^2}$$

donde

$$c_1=1, c_2=\frac{1}{\binom{4}{2}}=\frac{1}{6}$$

$$c_m=\frac{1}{\binom{2m}{m}}-\frac{1}{2}\sum_{\substack{2\le n,k\le m-1\\ n\,k=m}}c_n\,c_k\quad ,m=3,4,5,\ldots$$

$$c_m=\frac{1}{\binom{2m}{m}}-\frac{1}{2}\sum_{n=2}^{[m/2]}\sum_{k=2}^{[m/2]}e(m,n,k)c_n c_k$$

$$e(m,n,k)=\begin{cases}1 & si\ \ nk=m\\ 0 & en\ otro\ caso\end{cases}$$

$$c_p=\frac{1}{\binom{2p}{p}}\quad ,p\ \text{número primo}$$

$$c_m=\left\{1,\frac{1}{6},\frac{1}{20},\frac{1}{2520},\frac{1}{252},\ldots\right\}$$

(129) $$\pi=\frac{9}{4}+\frac{1}{4}\sqrt{9+96\sum_{n=1}^{\infty}\frac{1}{n^2}(\cos n)^3}$$

(130) $$\pi^2=16\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\cos\left(\frac{n\pi}{6}\right)\right)^3$$

(131) $$\pi^2=-48\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\cos\left(\frac{n\pi}{2}\right)\right)^3$$

(132) $$\pi^2 = \frac{192}{5} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\cos\left(\frac{n\pi}{4}\right)\right)^3$$

(133) $$\pi + 2\ln 2 = 4 - 4 \sum_{n=1}^{\infty} \frac{a_n}{2^n n(n+1)}$$

$$a_n = Re((1+i)^{n+1}) = \frac{(1+i)^{n+1} + (1-i)^{n+1}}{2}$$

$$a_{n+2} = 2a_{n+1} - 2a_n \ \ , a_1 = 0, a_2 = -2$$

(134) $$\pi - 2\ln 2 = 4 - 4 \sum_{n=1}^{\infty} \frac{b_n}{2^n n(n+1)}$$

$$b_n = Im((1+i)^{n+1}) = \frac{(1+i)^{n+1} - (1-i)^{n+1}}{2i}$$

$$b_{n+2} = 2b_{n+1} - 2b_n \ \ , b_1 = 2, b_2 = 2$$

(135) $$\pi = 4 - 2 \sum_{n=1}^{\infty} \frac{c_n}{2^n n(n+1)}$$

$$c_n = (1+i)^n + (1-i)^n$$

$$c_{n+2} = 2c_{n+1} - 2c_n \ \ , c_1 = 2, c_2 = 0$$

(136) $$\pi = 4\, Im(i^m (1-i)^{m-1}) - 4 \sum_{k=2}^{m} a_{k,m} \ Im(i^m (1-i)^{m-k})$$
$$+ 4(-1)^m m! \sum_{n=1}^{\infty} \frac{c_n}{2^n n(n+1)\dots(n+m)}$$

donde

$$m = 2,3,4,\dots$$

$$a_{2,m} = m - \frac{1}{2} \ \ , m \geq 2$$

$$a_{k,m} = \frac{1}{k}(-1)^k \binom{m-1}{k-1} - \frac{m}{k} \ a_{k-1,m-1} \ \ , k \geq 3$$

$$c_n = Im(i^m (1+i)^n) = \frac{i^{m-1}((1+i)^n - (-1)^m (1-i)^n)}{2}$$

$$c_{n+2} = 2c_{n+1} - 2c_n \ \ , n \in \mathbb{N}$$

$$c_1 = i^{m-1}\big((1+i) - (-1)^m(1-i)\big)$$

$$c_2 = 2i^m(1+(-1)^m)$$

Ejemplos: $m = 2,3,4,5$:

$$\pi = 4 - 8 \sum_{n=1}^{\infty} \frac{Im((1+i)^n)}{2^n n(n+1)(n+2)}$$

$$\pi = \frac{8}{3} + 24 \sum_{n=1}^{\infty} \frac{Im(i(1+i)^n)}{2^n n(n+1)(n+2)(n+3)}$$

$$\pi = \frac{8}{3} + 96 \sum_{n=1}^{\infty} \frac{Im((1+i)^n)}{2^n n(n+1)(n+2)(n+3)(n+4)}$$

$$\pi = \frac{52}{15} - 480 \sum_{n=1}^{\infty} \frac{Im(i(1+i)^n)}{2^n n(n+1)(n+2)(n+3)(n+4)(n+5)}$$

(137) $$\pi = 16 \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{\sqrt{6}-\sqrt{2}}{2}\right)^{2n+1} \left(\sin\left(\frac{(2n+1)\pi}{4}\right)\right)^3$$

(138) $$\pi = 4 \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n \left(\sqrt{3}-1\right)^{4n+1} \left(\frac{1}{4n+1} + \frac{2-\sqrt{3}}{4n+3}\right)$$

(139) $$\pi = 24 \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^{2n+1} \left(\sin\left(\frac{(2n+1)\pi}{4}\right)\right)^3$$

(140) $$\pi = 6 \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n \left(\sqrt{5}-\sqrt{3}\right)^{4n+1} \left(\frac{1}{4n+1} + \frac{4-\sqrt{15}}{4n+3}\right)$$

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