Using Periodic Functions to Determine Primes, Composites, and Factors

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Abstract

This paper discusses connections between periodic functions and primes, composites, and factors. Specifically, it shows how to use periodic functions to construct formulas for the following: the number of factors of a number, the specific factors of a number, the exact prime counting function and distribution, the nth prime, primes of any size, "product polynomials" as periodic functions, primality and composite tests, prime gap finders, and "anti-pulses."

Contents

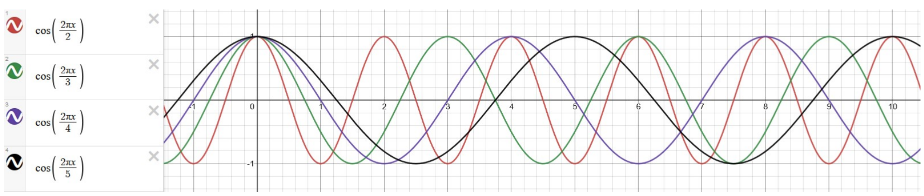
1 Introduction

This paper is an edit and revisionary overhaul to, and the fourth edition of, a paper written 8 years ago. It also includes additional related topics, as well as formulas from its development that had yet to make it into digital form, and it serves as both an expansion and consolidation of the material. As stated in the abstract, it describes how to use periodic functions to construct various formulas relating to prime numbers, composite numbers, and the factors of a number. The techniques revolve around using a periodic function to first create a formula for the number of factors of a number, often referred to as the "NumFac" function, and then further manipulating that function to get the desired results. It's important to note, that the initial periodic function used in the paper, namely the cosine function, was the one with which the methods were originally created, but that any periodic can be used as long as it meets the requirements given in each section. Sections 2 through 4 describe the original core of the main technique, however, a number of variations were later derived and explored, some of which may be deemed superior or more efficient. Therefore, section 8 is a discussion of some of those various alternatives to such.

While some of the results are not the most computationally efficient means for approaching certain problems, they are instead novel, deterministic equations that deliver exact answers with 100% accuracy, some can replace algorithms with formulas, some resolve questions regarding the nature of the primes, and others provide new functional relationships and alternatives to previous forms. As one example of such, and a preview of things to come, the paper provides a closed form algebraic recursive sequence that can generate the n-th prime, and a prime of any size, guaranteed. That alone serves as new options for the calculation and symbolic representations of primes. This opens up numerous possibilities for others to improve the field, further explore the methods within, and it provides options for certain processes to be tailored to meet specific needs. The paper also shows graphs between many key steps, helping to quickly visualize the reasoning and result of each maneuver, and it concludes with some brief afterthoughts and ongoing questions about the techniques.

2 The Number of Factors of a Number

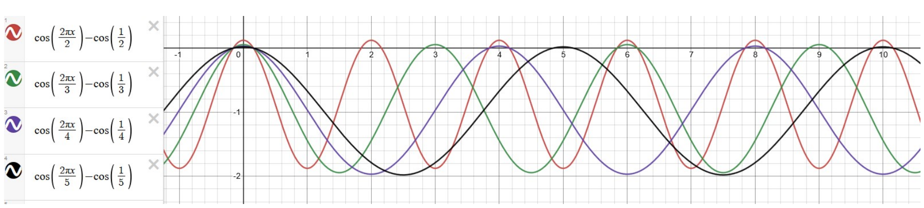
Begin with a simple set of cosine functions of the form $\cos \frac{2\pi x}{k}$, with natural wave number k, such that $k \geq 1$. Waves 2 through 5 are shown as a reference.



The 2π sets the peaks to the integers, and the wave number k selects the period. The first step in calculating the number of factors, called wave peak restricting, serves 2 purposes. It shifts the waves down, such that their values remain positive above the multiples of each's individual wave number, but become negative below integers that are not multiples of each's wave number. Secondly, it sets a constant width for each peak, such that wave crests that share an integer also share x-intercepts.

2.1 Wave Peak Restricting

To restrict the peaks, choose the crest half-width, d, evaluate each wave at x = d, and then slide it down the y-axis by that amount. Note, for the next stage of "de-noising" to work, d must meet the requirement that $0 < d \leq 1/2$. This will become clear momentarily. So, using $\cos \frac{2\pi d}{k}$, and the parameter requirement, convenient choices of d for simplification purposes are $1/2\pi$, $1/\pi$, and $1/2$. This paper uses d=$1/2\pi$, as it provides the most simplification. Again, waves 2 through 5 are shown as a reference.

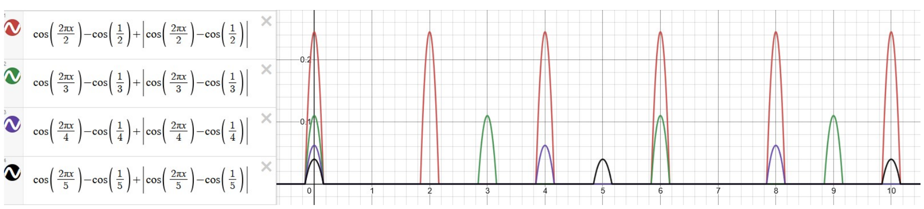


The reason for this restriction on d is due to the next 3 steps, and amounts to the facts that a choice of d = 0, and thus sliding the waves down by 1, leaves no information above the axis to work with, while a choice of d > 1/2 leaves information above the axis in places where constructive interference between waves during a summation creates noise.

2.2 De-noising

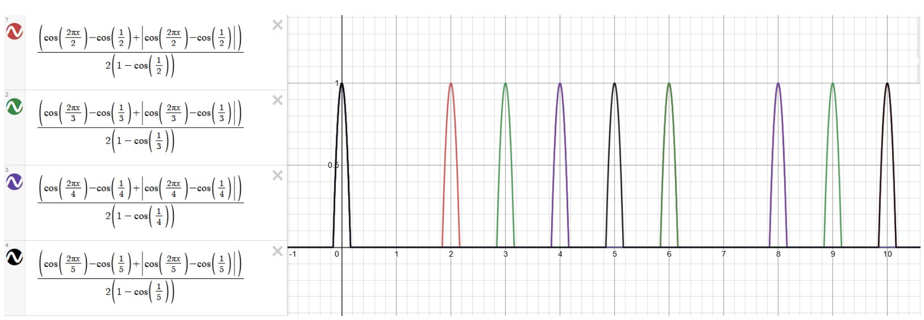
The goal is to add up the number of peaks at each integer, each factor of that integer contributing 1, while not adding values from other non factor waves that pass below that integer. During a subsequent summation, the information below the axis generates this unwanted interference. To accommodate this, it is removed ahead of time via addition of the absolute values of the functions to themselves. there are other means to this, some discussed later, yet this is one of the cleanest and most straightforward.

$$\cos\left(\frac{2\pi x}{k}\right) - \cos\left(\frac{1}{k}\right) + \left|\cos\left(\frac{2\pi x}{k}\right) - \cos\left(\frac{1}{k}\right)\right| \tag{1}$$



2.3 Re-normalizing Peaks to 1

Due to the process so far, the peaks no longer have a value of 1. To re-normalize all the values to 1, divide by 2 to counteract the amplitude doubling addition of absolute values during the de-noising, and divide by $1 - \cos(1/k)$ to counteract the sliding down during wave peak restricting. The result is equation 2.



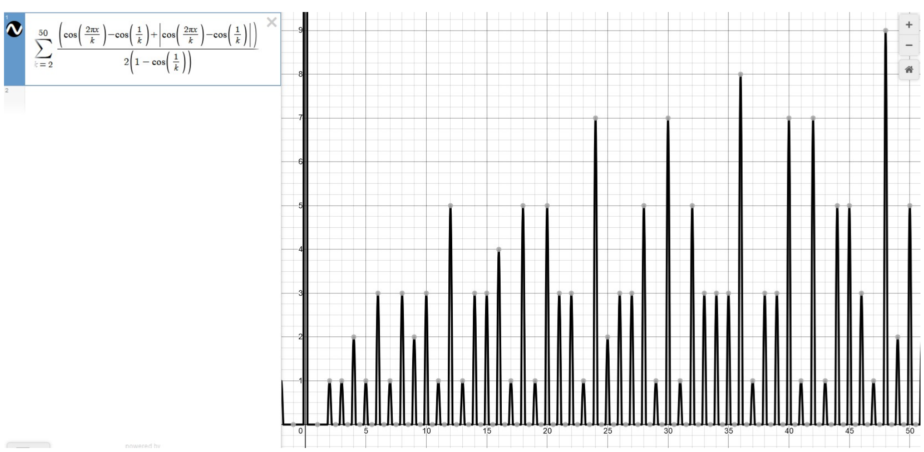
$$\frac{\cos\left(\frac{2\pi x}{k}\right)-\cos\left(\frac{1}{k}\right)+\left|\cos\left(\frac{2\pi x}{k}\right)-\cos\left(\frac{1}{k}\right)\right|}{2\left(1-\cos\frac{1}{k}\right)} \tag{2}$$

2.4 Summation

Now, every wave that is a factor of an integer x, or that x is a multiple of its wave number if one prefers to think of it that way, contributes a value of 1 above that integer, and is 0 elsewhere. Also, all peaks have a uniform base width. Therefore, since each factor contributes 1 wave, adding the number of waves at each integer gives the number of factors for that integer, not including the factor of 1 when the index is started at 2. This is defined as the NumFac function, F(x).

$$F(x)=\sum_{k=2}^{j}\frac{\cos\left(\frac{2\pi x}{k}\right)-\cos\left(\frac{1}{k}\right)+\left|\cos\left(\frac{2\pi x}{k}\right)-\cos\left(\frac{1}{k}\right)\right|}{2\left(1-\cos\frac{1}{k}\right)} \tag{3}$$

In order to guarantee function accuracy for an integer x, the summation must include all waves up to x. That is, j $\geq$ x. For that reason, it's also possible to set j = x, and alternatively, setting j equal to infinity automatically makes the function true for all x. Here is k from 2 to 50.



Notice that NumFac correctly gives the number of factors for all numbers, and that prime numbers accordingly have a value of 1, corresponding to the number itself being the only factor other than the number 1. Thus, it neatly shows all primes. NumFac acts as a Prime Sieve, such that F(x) = 1 over the integers for all primes x, where x $\leq$ j, and F(x) > 1 for all composites. Don't forget that the process can be repeated with any periodic function as long as the guidelines are followed. Specific considerations for d may be needed though, according to the symmetry or lack thereof of the periodic chosen, but the general premise is the same.

2.5 Factors Including 1

As mentioned, the value of the function at an integer is equal to the number of factors of the integer, including the integer, but not including 1. When required, there are 2 easy ways to make the value of the function equal to the number of factors including 1. The first, is to simply add a baseline of 1 to the function, that is, F(x)+1. The second, is to include the k = 1 wave, and run the lower index from 1 instead of 2. Doing so familiarly sets all primes equal to 2 factors. The reason the 1 wave is normally left off F(x), as described, is simply due to the fact that it regularly adds the need to subtract an extra 1 in many further constructions to compensate for it, and so it's easier to just leave it out.

3 Specific Factors and Factor Tagging

In order to find the specific factors of a number, take note that each factor contributes 1 to the value of F(x) at that integer. Next, include the fact that every number has a unique set of factors with no duplicates in that set. If each factor was to contribute a unique value instead of 1, and if the sums of the corresponding values assigned to the elements of every set was also unique, then the resulting output at any integer would uniquely correspond to the specific set. Actually, only the second part would be needed for a unique correspondence, but the first part, when chosen correctly, facilitates the ability to extract the factors from the sum without needing a lookup table. Setting the unique value for each factor becomes that factor's "Tag."

For each wave number k, consider a tag of $10^{(k-1)}$. That is, 2's tag is 10, 3's tag is 100, 4's tag is 1000, and so on. Adding each factor's tag value, instead of 1, now gives that exclusive output. This is due to the fact that each factor now controls its own decimal place. There are certainly other tags meeting the mentioned requirements, but this is one of the simpler ones, and as such, it is used in the following equation and example. Define a Factor Tagging Function, T(x), which is just F(x) with the factor tag included as a product in the sum.

$$T(x) = \sum_{k=1}^{j} \frac{10^{(k-1)} \left(\cos\left(\frac{2\pi x}{k}\right) - \cos\left(\frac{1}{k}\right) + \left|\cos\left(\frac{2\pi x}{k}\right) - \cos\left(\frac{1}{k}\right)\right|\right)}{2\left(1 - \cos\frac{1}{k}\right)} \tag{4}$$

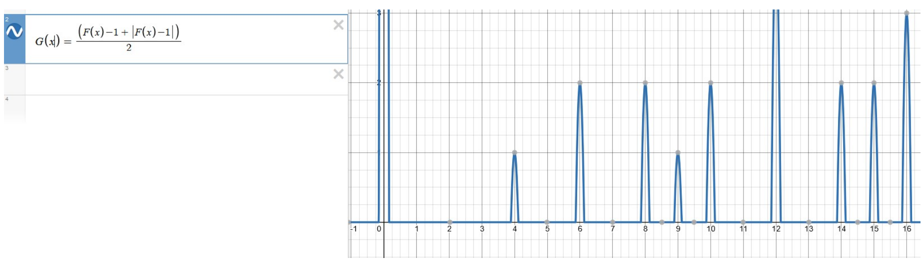
A visual plot of T(x) is of minimal value, since it's basically a plot of F(x) scaled by $10^{(x-1)}$, and the exact values are not visible on the log scale. Instead, it's better to just look at the output values directly. For example, the specific values of T(x), for x = 1 to 12, are as follows. [1], [11], [101], [1011], [10001], [100111], [1000001], [10001011], [100000101], [1000010011], [10000000001], and [100000101111].

The function generates an output that is binary in form, such that the 1s correspond to the factors of x from right to left. For example, T(4) = 1011 shows the factors of 4 as 1, 2, and 4. T(6) = 100111 gives 1, 2, 3, and 6. Primes begin and end in 1 with all zeroes in between. While not explored here, further associations now also exist between the converted decimal value of each binary string and its associated set. That is, between string [1] = value 1 = set [1], between string [11] = value 3 = set [1,2], [101] = 5 = [1,3], [1011] = 11 = [1,2,4], [10001] = 17 = [1,5], and so on. This generates a unique sequence that represents all factors of all numbers, namely [1,3,5,11,17,65,139,261,531,...].

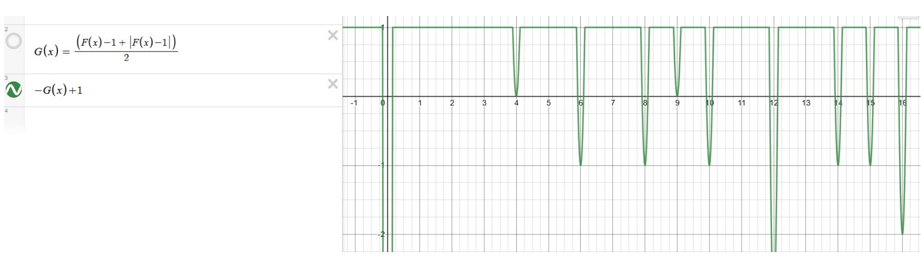
4 Flip-Flopping to Filter the Composites from the Naturals

Next, the function F(x) can be manipulated to separate the composites from the naturals. This is done similarly to before through a process of "flip-flopping" the wave back and forth across the x-axis through further wave peak restricting, de-noising, and re-normalizing. Shift F(x) down by 1, thus leaving information above the axis only above the composites. Remove the data below the axis as before, via adding a copy of the absolute value, and then divide by 2 to counter the amplitude change of that manipulation. Define that new function to be G(x).

$$G(x) = \frac{F(x) - 1 + |F(x) - 1|}{2} \tag{5}$$

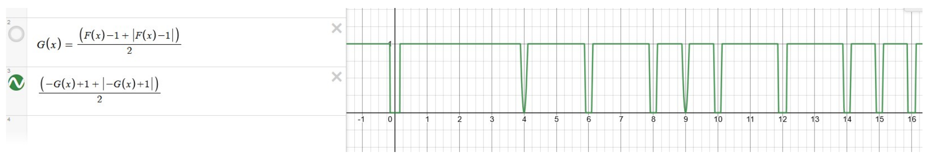


With the prime crests removed from the wave, the goal is to get all the remaining composite peaks to have the same value, namely 1. This is done by chopping off all the peaks greater than 1. Continue by again shifting the function down by 1, and then flipping it over the x axis by taking its negative, resulting in $-G(x) + 1$.



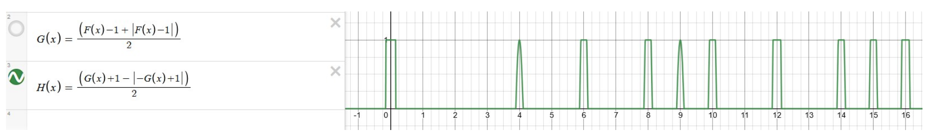
Yet again, add the absolute value, and divide by 2 to counter the magnitude change. This removes all the peaks which are now under the axis, which had values greater than 1, and effectively sets their heights to 1.

$$\frac{-G(x) + 1 + |-G(x) + 1|}{2} \tag{6}$$



Flip the function back over, and finally shift it back up by 1. This function now has a peak value of 1 for all composites, and only the composites, and is labeled H(x).

$$H(x) = \frac{G(x) + 1 - |-G(x) + 1|}{2} \tag{7}$$



H(x) can be used as a primality test, with integer solutions of 0, a composite test, with integer solutions of 1, and sums taken over this function give the number of composites $\leq$ a number. As such, it can be used to determine the exact prime distribution. H(x) is sometimes referred to s the composite wave. Note that this is how the technique was originally designed, however, it may be considered more efficient to jump directly from G(x) to a prime wave and it is discussed in section 9.

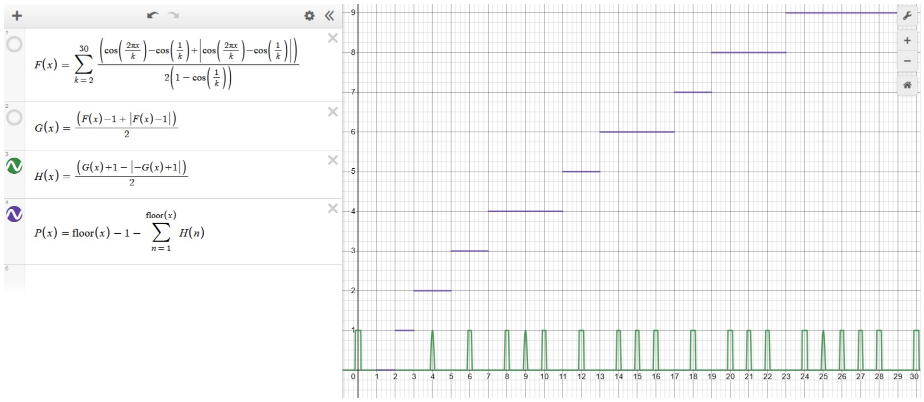
5 The Prime Distribution

Noting that the number of primes $\leq$ a number is equal to that number, minus the number of composites $\leq$ the number, minus 1 more for the number 1, the formula for the Prime Distribution, $\Pi(x)$, is:

$$\Pi(x) = x - 1 - \sum_{n=1}^{x} H(n) \tag{8}$$

Plotting $\Pi(x)$ shows the familiar Prime Distribution, and $\Pi(x)$ gives the exact distribution for all x, as long as one abides the initial restriction on j in F(x) throughout the calculation.

Note that the use of the floor function, as seen in the left panel of the graph version of the equation, was done solely due to how the graphing calculator



handled non-integer index summation, and to make the distribution appear as it normally is shown, with horizontal steps rather than slanted ones.

This is a novel an interesting result, in that not only does it give the full infinite distribution, but it shows that the prime counting function can be composed of basic algebraic summations in closed form, and it provides how to generate a finite equation for $\Pi(x)$ valid up to any given x. This is in contrast to traditional methods and discussions regarding the distribution that resort to invoking more exotic equations. As a brief comparison, take for example the explanation according to Wikipedia.[1] There is a section dedicated to approximations of the growth rate that calls upon the logarithmic integral and Big O notation. Then they discuss exact forms using the Mobius function, the Riemann Zeta function, branch cuts, and the exponential integral, all of which are more complicated. While a bit unwieldy looking due to the transformations in equations 5 and 7, the full expanded form of $\Pi(x)$ in terms of NumFac, F(x), can be found in the appendix for reference, eq. 30, so that the reader can get a sense of it, and to show that relatively speaking, it's not very complex.

6 Recursive Sequence for the N-th Prime

Using the formula for the exact distribution of the primes, $\Pi(x)$, a recursive sequence $Q_s(n)$ can be fashioned to determine the nth prime. This brings up an age old question, "how does one go about directly calculating the n-th prime?" First note that the value of the n-th prime is always greater than n, which means it can loosely be stated as n + something. What then is that something? For a moment, reverse the question, and imagine beginning with a prime and asking which prime is it. This is easy to answer in terms of the number of composite numbers less than a number. That is, take the prime, subtract the number of composites less than it, and subtract 1 for the number 1, and that is which prime it is. For example, take the number 11. 11 has 5 composites less than it, and $11 - 5 - 1 = 5$, making it the 5th prime. Now reverse the process.

Say one wants to find the 5th prime. Start with 5, add the number of composites less then 5, and add 1 for the number 1, giving $5+1+1=7$. From there, one now needs to add the number of additional composites between and including that new number and the previous number, in this case between 5 and 7, namely adding 1 more for 6. This gives $7+1=8$. This process continues until no new composites are added, and thus one finds the next prime. In this example, 8 adds 1 to go to 9, 9 adds 1 to become 10, 10 adds one to become 11, and 11 will see no new composites, and thus the 5th prime is 11. Try one more. Take the 8th prime for example. The sequence would be 8, 12, 15, 17, 18, 19, stop. Indeed, 19 is the 8th prime.

Something to notice about the sequence when finding a prime, is that it makes larger jumps at first, which then decrease in size, and it will always end by increasing by 1s from the previous (n-1)-th prime to the n-th prime. In the last example for n = 8, that would be the steps 17, 18, and 19. This sequence can be written in terms of $\Pi(x)$ as follows, with the initial term defined as $Q_0(n) = 0$.

The current term in terms of the previous term.

$$Q_s(n) = n + Q_{s-1}(n) - \Pi(Q_{s-1}(n)) \tag{9}$$

Or rearranging the subscripts, the next term in terms of the current term.

$$Q_{s+1}(n) = n + Q_s(n) - \Pi(Q_s(n)) \tag{10}$$

In words, eq.9 says that the current term is equal to n plus the previous term minus the number of primes less than the previous term. Note that the previous term, minus the number of primes less than the previous term, is the same as 1 plus the running total number of composites added to the sequence. Similarly, eq. 10 says that the next term is n plus 1 plus the running total number of composites through the current term, as was demonstrated in the examples. While infinite terms s of the sequence can be generated indefinitely, once it hits a prime number, no new values will be generated, it terminates, and every subsequent term will also be that prime. For example, for the 5th prime, the sequence is [5, 7, 8, 9, 10, 11, 11, 11, 11, ...]. While not shown here, it's fairly straightforward to show that the termination always occurs by some term s, when s $\leq$ n. That is, one never needs to calculate terms beyond $Q_n(n)$ in order to guarantee generating the n-th prime.

For those who have carefully followed and explored the technique, and who have walked through calculating some examples, they might notice that there is a matter of housekeeping that needs to be addressed regarding $Q_1(n)$ and $\Pi(0)$. As a final sample, and segue into the housekeeping, here is the sequence for the 20th prime, 71, from $Q_1(20)$ through $Q_{16}(20)$: [20, 32, 41, 48, 53, 57, 61, 63, 65, 67, 68, 69, 70, 71, 71, 71].

6.1 Some Housekeeping for $Q_1(n)$ and $\Pi(0)$

Using eq. 9, the $Q_1(n)$ term of the sequence is equal to $n + Q_0(n) - \Pi(Q_0(n))$. Given that $Q_0(n) = 0$ is defined as 0 leads to evaluating $\Pi(0)$. Logically, the function $\Pi(x)$ represents the number of primes $\leq$ a number x, and so it's known that $\Pi(0)$ should be 0. The $Q_1(n)$ term then always simplifies to being equal to n. However, the $\Pi(x)$ developed is only accurate for all x $\geq$ 1, and erroneously gives $\Pi(0)$ = -1, not 0. This can be addressed in at least 2 ways.

One, is to simply define both the first and second terms of the sequence rather than only the first, with $Q_0(n) = 0$ and $Q_1(n) = n$. Another method is to actually adjust $\Pi(x)$ such that it keeps all its values for x $\geq$ 1, but gains $\Pi(0)$ = 0. One way to do that, for example, is back at the G(x) stage of the process. Multiplying the original G(x), eq. 5, by x, and then continuing from there through the method, slightly complicates the equations, but gives the desired results. This is mentioned here for completeness and demonstrative reasons, but is only necessary dependent on the specific utility and purpose the user needs from the system. Thus, and similar to the fact that other periodic functions could be used, the entire process so far is not repeated with the adapted G(x), but rather only the new starting point for that branch is shown. That is, equation 5 would become the following, and the system continued from there.

$$G(x) = \frac{x * (F(x) - 1 + |F(x) - 1|)}{2} \tag{11}$$

6.2 Prime Numbers of Any Size

With the recursive sequence for the n-th prime in hand, it is now possible to generate a prime of any size. This is in stark contrast to needing to brute force verify a number to see if it is prime. Instead, the number can be calculated directly. Say you want a prime with a billion digits. This can be approached with generalities of the prime number theorem to get a prime of approximately 1 billion digits, or it can be achieved exactly using the prime distribution. Both ways are described.

To use the number theorem, which states $\Pi(n) \sim n/\ln n$, take a number with a billion digits, say $10^{billion}$, and insert it into the formula. This tells you that there are about 10ˆ(10ˆ8.999999995934042) primes with less than 1 billion digits. Take that number and plug it into eq. 9, in $Q_n(n)$, and voila, a 1 billion digit prime is guaranteed in less than n iterations.

If one doesn't want to rely on the prime number theorem, and instead wants an exact approach, the method is the same as before, but this time uses the exact counting function $\Pi(x)$, eq. 8, to determine the n to use in eq. 9. Granted, these calculations are not very efficient, but they are, in the 2nd version, 100% accurate and deterministic. Not only that, but they offer a starting point for others to improve the process, and as is briefly discussed in section 8, the ability to use different periodic functions, and the possibility of finding one that meets certain criteria, could drastically and profoundly improve the efficiency.

7 Product Polynomials

One useful form for generating periodic functions is what I often call "product polynomials." What then are they? It is commonly known how to generate a polynomial with any desired roots by simply multiplying out the quantities $(x - r_n)$, where r_n are the roots. Making those roots all the multiples of a number, creates an infinite polynomial with periodic roots. Likewise, taking all the multiples within a given domain, creates a function with periodic roots over that domain. This is useful for 2 related reasons. One, it can act as a substitute for the trig functions in F(x). Two, the multiplication of a finite number of integer roots, and the resulting polynomial, are purely algebraic, using only the basic operations of adding, subtracting, and multiplying. This shows that all the techniques described within this paper can be carried out using a polynomial having a finite number of terms. Now of course it's often a big clunky polynomial, but it's finite none the less. One can even go further, and use the full infinite domain to represent the entire distribution or recursive sequence for the n-th prime. To begin, consider the following form.

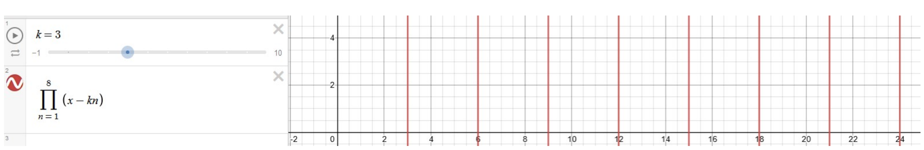
$$R(x) = \prod_{n=1}^{b} (x - kn) \tag{12}$$

Just as in the cosine function, let k be the wave number. The product represents and generates all the roots between 0 and k*b. Before continuing, it's informative to point out that the resulting product function has at least 2 alternative representations.

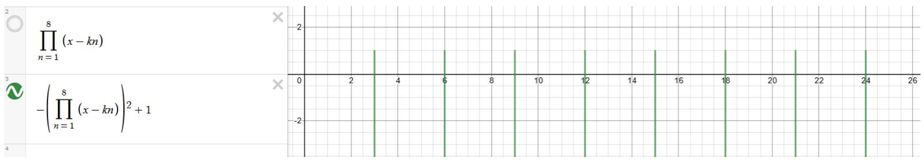
$$\prod_{n=1}^{b} (x - kn) = (-k)^b * (1 - \frac{x}{k})_b \tag{13}$$

$$\prod_{n=1}^{b} (x - kn) = \frac{(-k)^b * \Gamma(b - \frac{x}{k} + 1)}{\Gamma(1 - \frac{x}{k})} \tag{14}$$

In equation 13, the b in the subscript is using Pochhammer rising factorial notation, and in eq. 14, the product is given in terms of the Gamma function. Here is an example of the first 8 roots of the k = 3 wave; the curves in the function are far outside the snapshot.



Next comes the crafty part. Take the square of the function, thus keeping the roots and removing the negative portion within the domain. Next, take the negative to flip it over, and then shift the function up by adding 1. The result is now a periodic function within the domain, with all the peaks set to 1, and it can be treated just like the cosine was originally used. The peaks are very sharp and narrow, and so they appear as a spike or comb function on the graph.



Applying the results to F(x) as suggested, produces the following.

$$F(x) = \frac{1}{2}\sum_{k=2}^{x}\left[1 - \left(\prod_{n=1}^{b}(x-kn)\right)^2 + \left|1 - \left(\prod_{n=1}^{b}(x-kn)\right)^2\right|\right] \tag{15}$$

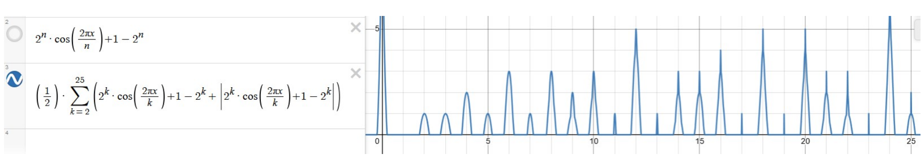
8 Alternate Functions

It's been mentioned that the presented techniques can be done with other periodic functions as the base. Throughout development of the methodology I have explored many such alternatives, usually in search of functions which could simplify the summation. The ultimate goal being finding a function that meets the periodic requirements for NumFac, but which also has an equatable closed form for the sum. That would lead to profound results. While I have not found said function, I have come across some notable alternatives and forms for various parts of the process, and as such is the focus of this section.

8.1 Replacing Trig Functions

It is possible to get rid of the trig functions added during the wave peak restricting and re-normalization phases, and to exchange them for an exponential-power function. The idea being that in order to avoid noise during summation, when the function is slid down, the peaks must not be wider than 1, and that the amount must also be based on the wave number. One such valid option is shifting down by $1 - (1/2^k)$, and doing so results in eq. 3, NumFac, becoming the following.

$$F(x) = \frac{1}{2}\sum_{k=2}^{j} 2^k(\cos(\frac{2\pi x}{k}) - 1) + 1 + |2^k(\cos(\frac{2\pi x}{k}) - 1) + 1| \tag{16}$$



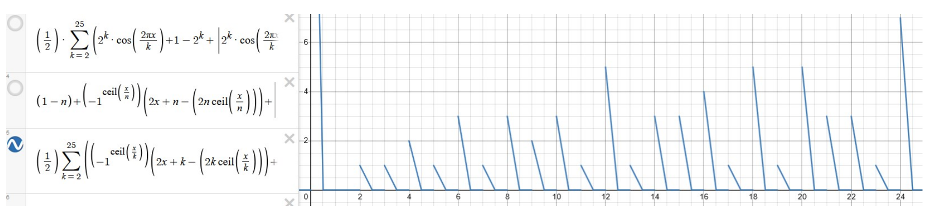
The only main difference in this version is that the wave peaks do not all have the same widths. Because the wave number in the power, they also get very thin and sharp rather quickly. Yet this function seems at first glance to be slightly simpler and therefore more computationally efficient.

The trig functions were used as the periodics specifically to eliminate the use of logic functions such as ceiling and floor. However, it's entirely possible to devise a periodic function, say using only ceilings, with no trig whatsoever.

$$-1^{\lceil \frac{x}{k} \rceil}(2x + k - (2k)\lceil \frac{x}{k} \rceil) + (1 - k) \tag{17}$$

This leads to the following NumFac, F(x), which is a sort of sloped comb function, and is probably even more computationally friendly.

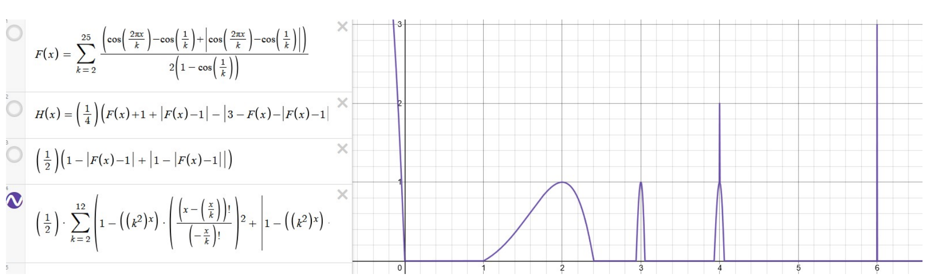
$$F(x) = \frac{1}{2}\sum_{k=2}^{j} -1^{\lceil \frac{x}{k} \rceil}(2x+k-(2k)\lceil \frac{x}{k} \rceil)+(1-k)+|-1^{\lceil \frac{x}{k} \rceil}(2x+k-(2k)\lceil \frac{x}{k} \rceil)+(1-k)| \tag{18}$$



Another version can be crafted using factorials.

$$F(x) = \frac{1}{2}\sum_{k=2}^{j} 1 - (k^{2x})(\frac{(x - (\frac{x}{k}))!}{(\frac{-x}{k})!})^2 + |1 - (k^{2x})(\frac{(x - (\frac{x}{k}))!}{(\frac{-x}{k})!})^2| \tag{19}$$

It turns out to be a bit odd in that it will call the Gamma function for calculation of negative factorials, and actually relies on the program used to ignore "does not exist" calculations when Gamma of a negative integer is called. However, one of its benefits is absolutely, ridiculously tight and narrow wave spikes. So much so, that the graphing program already starts to truncate the width to 0 by around x = 5, unless you zoom in prompting it to use greater precision. Still it's a unique form, and demonstrates the options provided by the technique.

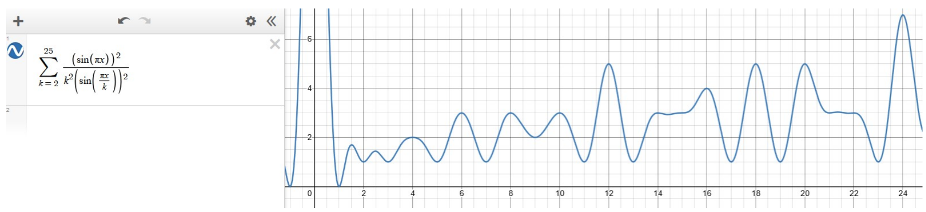


8.2 Replacing Absolute Values

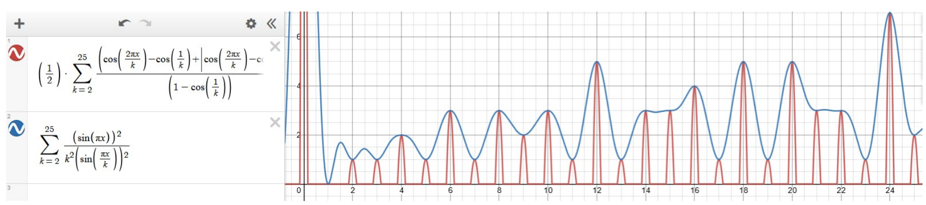
Another approach in the search for a reducible sum is to try and replace the absolute values. The simplest way to do so is to use the square root of the square of the values. However when doing so, it's important to only use the traditional positive roots for evaluation, and to also not make the mistake of symbolically combining that quantity with its copy in the first half of the numerator where it resides. As such, it doesn't really allow for further simplification.

There are however other methods to avoid using absolute values. A beautiful and concise example is found using the sine function.

$$F(x) = \sum_{k=2}^{j} \frac{(sin(\pi x))^2}{k^2(sin(\frac{\pi x}{k}))^2} \tag{20}$$



Interestingly, this version also acts as an extremely good bound for the original F(x) itself. Here, the two are shown in comparison.



An almost identical version graphically, yet much more complicated, is generated using the product function. It is mentioned as a contribution toward the study of equivalent forms to the NumFac function, F(x).

$$F(x) = \sum_{k=2}^{j} \frac{\prod_{n=1}^{k-1}(\cos(\frac{\pi(2x+k+2n)}{k}) + 1)}{\prod_{n=1}^{k-1}(\cos(\frac{\pi(k+2n)}{k}) + 1)} \tag{21}$$

8.3 Replacing Forms of G(x), H(x), and Other Tests

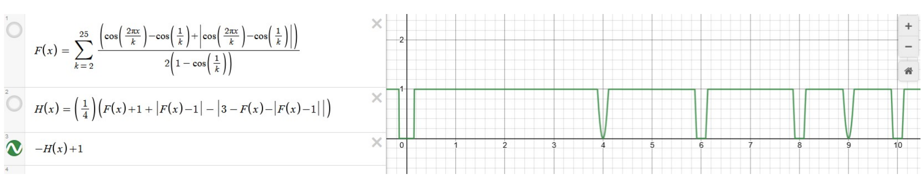
There are numerous ways to shift, flip, and flop the periodic functions during the G(x) and H(x) stage. Some of these notable alternative transformations are included in the appendix, as well as an additional primality test.

9 Prime Waves and Quasi-periodicity

What is a "Prime Wave," and what is meant by quasi-periodicity? While not normally thought of as periodic, it has been shown how to create the prime-composite distribution from periodic waves. Consider the idea that the integers are periodic among the real numbers by default; that is, the integers can be though of as the wave $\cos \frac{2\pi x}{k}$ with period equal to 1. With this in mind, and since the prime numbers are all integers, they occupy positions and spacings that are some subset of a periodic spacing. This is what is meant by quasi-periodicity. The primes are quasi-periodic among the reals, since they are all integers.

This concept can be taken further to conceive the ideas of "primal frequency" and "primal period." By definition, the idea of periodicity is based around something that is constant or repeats. Simple waves have a fixed period. However, as soon as one starts to examine things like forced-dampened waves, dispersion, and Doppler effects, ideas such as variable frequency and modulation surface. A question arises. "If you plug something non periodic into the period of a wave function, is the overall thing still periodic?" From the standpoint of it still being a wave, and that you know its oxymoronic non-periodic period, it could be considered periodic. That is, whatever you plugged in literally is its period. From the other standpoint, it's not periodic, as it displays no simple repetition. This quickly devolves into topics of chaos theory, and of what constitutes a pattern or chaos. However, even the most chaotic pattern, note the oxymoron again, can be considered a pattern, if one labels it as the de facto name of said pattern. The pattern of all the sand on the beach is the pattern of all the sand on the beach.

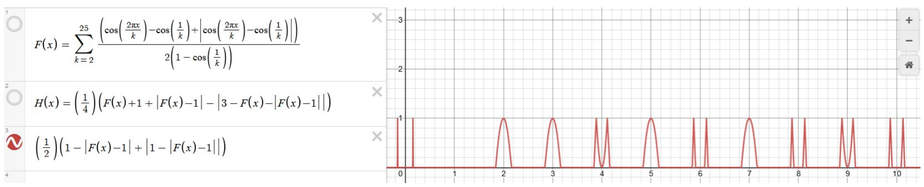
Using the idea of quasi-periodicity, it can then be said, that the prime distribution as a whole, exhibits and represents its own primal frequency. That is, "The" primal frequency, with a capital T. Once the method reaches H(x), eq. 7, the composite wave has a value of 1 for all composites, and 0 for all primes. It is then of particular interest to swap all the values, resulting in The Prime Wave, with its primal period and frequency, and there are numerous ways to do so. Probably the most direct way is to run it through the linear transformation $y = -x + 1$, resulting in -H(x)+1.



The only concern here may be that the resulting "wave" has plateaus for the prime peaks instead of curves, and all the composites are in valleys. Still, it works fine, and one could take a summation over this prime wave to get the prime counting function $\Pi(x)$ as an alternative to eq. 8. If needed, one could also use a different transformation, or simply sculpt the plateaus afterwards.

Another prime wave construction is the following, which results in nice wave peaks for the primes and "fanged" valleys for the composites.

$$P(x) = \frac{1 - |F(x) - 1| + |1 - |F(x) - 1||}{2} \tag{22}$$



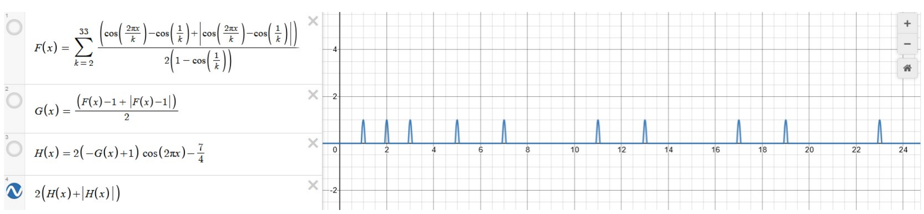
An almost identical fanged version is:

$$P(x) = \frac{F(x) - 2G(x) + |F(x) - 2G(x)|}{2} \tag{23}$$

Finally, here is a completely sculpted version using an adapted H(x). Note it includes 1 as a prime, which is negligible, or it could also be removed with further adaptation.

$$H(x) = 2(-G(x) + 1)\cos(2\pi x) - \frac{7}{4} \tag{24}$$

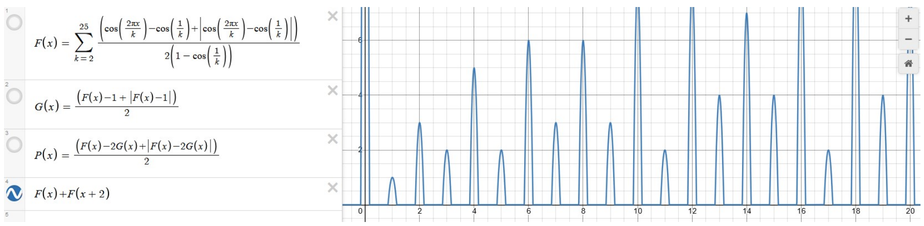
$$P(x) = 2(H(x) + |H(x)|) \tag{25}$$



10 Twin Primes and a Prime Gaps Finder

The NumFac function can also be used to plot, explore, and check for Twin Primes. Since primes are equal to 1 on F(x), an easy application of such is then looking at F(x) + F(x+2) = 2 over the integers, which shows all twin pairs (x,x+2).

In this instance, all other integer outputs are greater than 2. Similar to the various constructions for the functions F, G, H, and P, one can create more specific waves for the twin primes that are 1 at the twins, and 0 everywhere else. These "Twin Prime Waves" make it easier to visualize the pattern of the twins. Additionally, this can be applied to any size gap, not just the twins, and

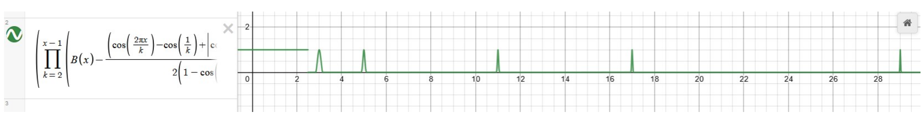


leads to "Prime Gaps Waves." While there are simple constructs that result in plateau and fanged versions, similar to other waves previously described, a more complex option, but very clean visually, is the following.

$$T(x) = \frac{\cos 2\pi x - \cos 1 + |\cos 2\pi x - \cos 1|}{2(1 - \cos 1)} \tag{26}$$

$$\left(\prod_{k=2}^{x-1}(T(x) - F(x))\right) * \left(\prod_{k=2}^{x-1}(T(x) - F(x+n))\right) \tag{27}$$

In equation 27, n represents the gap size, and is set to n = 2 for the twin primes in the graph below. The peaks in the graph show the location of the first twin of each pair: 3, 5, 11, 17, 29, and so on.

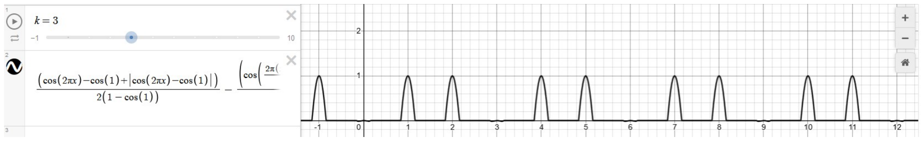


11 Anti-pulses

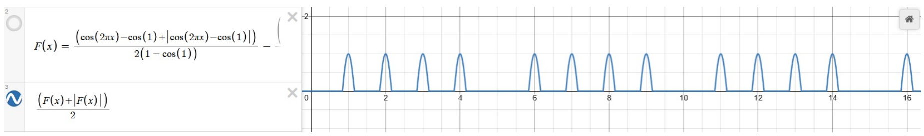
While working with T(x) and the prime gaps finder, it was discovered that T(x) could be used to create an "anti-pulse." Waves typically have crests and troughs spaced according to their wave number with the body of the wave in between. In the case of the waves developed in section 2, due to the absolute values, the body of the waves are 0 in between the pulses. Now consider the reverse; that is a wave which is 0 where the pulses were, according to wave number, but filled with pulses in between. These anti-pulses, or "valley waves," can be generated with the following formula, where k is the wave number.

$$T(x) - \frac{\cos(\frac{2\pi(x-k)}{k}) - \cos(\frac{1}{k}) + |\cos(\frac{2\pi(x-k)}{k}) - \cos(\frac{1}{k})|}{2(1 - \cos\frac{1}{k})} \tag{28}$$

The example is for the k = 3 anti-pulse. There are still small fringe humps around the anti-pulse wave value, yet the value will be 0. Even those can be removed if desired, by using the absolute value trick. That is, by adding a copy of the absolute value of the wave to itself, and then dividing by 2. In this way,



one can produce true anti-pulses. Here is the k = 5 anti-pulse with the noise cleaned up.



12 Questions

Last but not least are some questions about the method, and foremost among them is in regards to the periodic function. There is the general consideration of the pros and cons of stating with different periodic functions, but more specifically, what periodic function can be used that leads to a sum with a closed form for F(x)? Finding such would likely breakthrough any calculation efficiency issues. To this end, could summation's connection to integration, or metric theory, be used to find a closed form for any of the sums presented? Is it possible to find an equation that equivalently bounds the peaks of F(x), such as eq. 20, but which is not a sum itself, and that could be used instead?

Next, is in regards to the functional transformations. Is there a better choice for d? Can the same results be achieved with less flip flops or better transformations?

Contemplating the factor tags, are there more useful tags for T(x) than 10^{k-1}? As a note, I tried one, basically the reciprocal of the one used, which output all the ones to the right side of the decimal point, but found the whole number version more intuitive for simplicity and explanation.

What are the Time Complexity Spaces of F(x), $\Pi(x)$, and $Q_s(n)$ for the varied versions of those equations? What is the formula for the number of terms, s, of the sequence $Q_s(n)$ that are needed to converge to the prime?

Which open problems can F(x), or the method in general, be used to help solve? For example, showing whether the system $F(x) = F(x+2) = 1$ has infinite solutions, confirms or disproves the Twin Prime Conjecture, and similarly, so does showing the same for $F(2^m - 1) = 1$ for the Mersenne Conjecture.

13 Conclusion

This concludes the general methods for using periodic functions and summation to create formulas for the number of factors of a number, the specific factors of a number, the exact prime counting function and distribution, the nth prime, primes of any size, product polynomials as periodic functions, primality and composite tests, prime gap finders, and anti-pulses.

I hope the reader enjoyed the topics and constructions, and encourage those interested to graph, explore, and become familiar with the functions themselves. I feel these topics and approaches provide many options and jump off points to improve the methods within, and the study of primes in general. If anyone can shed light on these considerations, finds or knows a closed form for F(x), or wishes to discuss the method further, please let me know.

A Appendix

A.1 Expanded $\Pi(x)$

Functionally, this is H(x) in terms of F(x) using eqs. 5 and 7.

$$H(x) = \frac{F(x) + 1 + |F(x) - 1| - |3 - F(x) - |F(x) - 1||}{4} \tag{29}$$

Remembering that the following is only one example, and that other periodic functions can be used, the full form for the counting function is then:

$$\begin{aligned} \Pi(x) = x - 1 - \frac{1}{4}\sum_{n=2}^{x}\Bigg[&\Big(\sum_{k=2}^{n}\frac{\cos\frac{2\pi n}{k} - \cos\frac{1}{k} + |\cos\frac{2\pi n}{k} - \cos\frac{1}{k}|}{2*(1-\cos\frac{1}{k})}\Big) \\ &+1 + \left|\Big(\sum_{k=2}^{n}\frac{\cos\frac{2\pi n}{k} - \cos\frac{1}{k} + |\cos\frac{2\pi n}{k} - \cos\frac{1}{k}|}{2*(1-\cos\frac{1}{k})}\Big) - 1\right| \\ &-\Bigg|3 - \Big(\sum_{k=2}^{n}\frac{\cos\frac{2\pi n}{k} - \cos\frac{1}{k} + |\cos\frac{2\pi n}{k} - \cos\frac{1}{k}|}{2*(1-\cos\frac{1}{k})}\Big) \\ &-\left|\Big(\sum_{k=2}^{n}\frac{\cos\frac{2\pi n}{k} - \cos\frac{1}{k} + |\cos\frac{2\pi n}{k} - \cos\frac{1}{k}|}{2*(1-\cos\frac{1}{k})}\Big) - 1\right|\Bigg|\Bigg] \end{aligned} \tag{30}$$

Here the outer sum is started at 2 since the number of composites < 2 is 0, and also to insure that the nested NumFac function has an upper index $\geq$ its lower index. In other words, the function is not valid for the trivial case when x = 1.

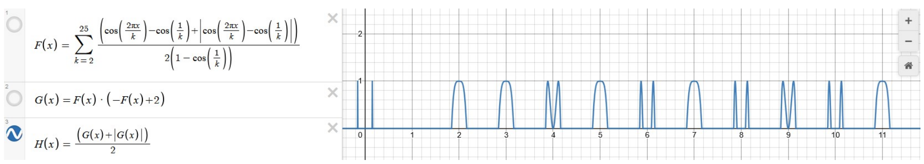
A.2 G(x) and H(x)

One unique and concise option to replace G(x) that does not rely on absolute values is the following. Note that it is not an exact analog of the original, it produces negative values for composites, but when combined with an adapted H(x) then leads to the same results.

$$G(x) = F(x) * (2 - F(x)) \tag{31}$$

The adapted H(x) still relies on absolute values, yet requires no other shifting during the flip-flop process, also making it very concise.

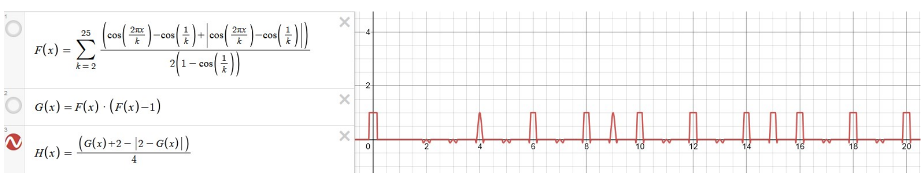
$$H(x) = \frac{G(x) + |G(x)|}{2} \tag{32}$$



Another very similar version is:

$$G(x) = F(x) * (F(x) - 1) \tag{33}$$

$$H(x) = \frac{G(x) + 2 - |2 - G(x)|}{4} \tag{34}$$



A.3 Other Tests

Throughout development, different tests for primality arose while exploring some of the equivalencies. While these are not very efficient calculation wise when run in full, and generate huge numbers, they are very concise symbolically, and are provided for reference. Depending on how a given algorithm recognizes values that become too large to calculate, there could be some programming methods that take advantage of their forms. These 2 are identical, except for the square inside the product. The subscript is again Pochhammer notation.

$$\prod_{k=2}^{\lceil \frac{x}{2} \rceil} \left(1 - \frac{x}{k}\right)_{\frac{x}{2}} <> 0 \tag{35}$$

This first one above is 0 when x is composite, and $< or > 0$ when x is prime.

$$\prod_{k=2}^{\lceil \frac{x}{2} \rceil} \left(\left(1 - \frac{x}{k}\right)_{\frac{x}{2}}\right)^2 > 0 \tag{36}$$

By taking the square it stays 0 when x is composite, but becomes > 0 when x is prime.

Bibliography

[1] Wikipedia contributors. *Prime-counting function — Wikipedia, The Free Encyclopedia.* https://en.wikipedia.org/w/index.php?title=Prime-counting_function&oldid=1022006295. [Online; accessed 11-May-2021]. 2021.