New method to get the Kochen-Specker theorem

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June 9, 2016

Abstract

We derive new type of no-hidden-variables theorem based on the assumptions proposed by Kochen and Specker. We consider N spin-1/2 systems. The hidden results of measurement are either $+1$ or -1 (in $\hbar/2$ unit). We derive some proposition concerning a quantum expected value under an assumption about the existence of the Bloch sphere in N spin-1/2 systems. However, the hidden variables theory violates the proposition with a magnitude that grows exponentially with the number of particles. Therefore, we have to give up either the existence of the Bloch sphere or the hidden variables theory. Also we discuss two-dimensional no-hidden-variables theorem of the KS type. Especially, we systematically describe our assertion based on more mathematical analysis using raw data in a thoughtful experiment.

Keywords: 03.65.Ud (Quantum non locality), 03.65.Ta (Quantum measurement theory), 03.65.Ca (Formalism)

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1 Introduction

Quantum mechanics (cf. [1, 2, 3, 5, 4, 6]) gives accurate and at times remarkably accurate numerical predictions. Much experimental data has fit to the quantum predictions for long time.

Kochen and Specker present the no-hidden-variables theorem (the KS theorem) [7]. The KS theorem says the non-existence of a real-valued function which is multiplicative and linear on commuting operators. The proof of the KS theorem relies on intricate geometric argument. Greenberger, Horne, and Zeilinger discover [8, 9] the so-called GHZ theorem for four-partite GHZ state. And, the KS theorem becomes very simple form (see also Refs. [10, 11, 12, 13, 14]).

It is begun to research the validity of the KS theorem by using inequalities (see Refs. [15, 16, 17, 18]). To find such inequalities to test the validity of the KS theorem is particularly useful for experimental investigation [19]. One of authors derives an inequality [18] as tests for the validity of the KS theorem. The quantum predictions violate the inequality when the system is in an uncorrelated state. An uncorrelated state is defined in Ref. [20]. The quantum predictions by n-partite uncorrelated state violate the inequality by an amount that grows exponentially with n.

Recently, Leggett-type non-local variables theory [21] is experimentally investigated [22, 23, 24]. The experiments report that quantum mechanics does not accept Leggett-type non-local variables interpretation. However there are debates for the conclusions of the experiments. See Refs. [25, 26, 27].

As for the applications of quantum mechanics, implementation of a quantum algorithm to solve Deutsch's problem [28, 29, 30] on a nuclear magnetic resonance quantum computer is reported firstly [31]. Implementation of the Deutsch-Jozsa algorithm on an ion-trap quantum computer is also reported [32]. There are several attempts to use single-photon two-qubit states for quantum computing. Oliveira *et al.* implement Deutsch's algorithm with polarization and transverse spatial modes of the electromagnetic field as qubits [33]. Single-photon Bell states are prepared and measured [34]. Also the decoherence-free implementation of Deutsch's algorithm is reported by using such single-photon and by using two logical qubits [35]. More recently, a one-way based experimental implementation of Deutsch's algorithm is reported [36]. In 1993, the Bernstein-Vazirani algorithm was reported [37]. It can be considered as an extended Deutsch-Jozsa algorithm. In 1994, Simon's algorithm was reported [38]. Implementation of a quantum algorithm to solve the Bernstein-Vazirani parity problem without entanglement on an ensemble quantum computer is reported [39]. Fiber-optics implementation of the Deutsch-Jozsa and Bernstein-Vazirani quantum algorithms with three qubits is discussed [40]. A quantum algorithm for approximating the influences of Boolean functions and its applications is recently reported [41].

On the other hand, the double-slit experiment is an illustration of wave-particle duality. In it, a beam of particles (such as photons) travels through a barrier with two slits removed. If one puts a detector screen on the other side, the pattern of detected particles shows interference fringes characteristic of waves; however, the detector screen responds to particles. The system exhibits the behaviour of both waves (interference patterns) and particles (dots on the screen).

If we modify this experiment so that one slit is closed, no interference pattern is observed. Thus, the state of both slits affects the final results. We can also arrange to have a minimally invasive detector at one of the slits to detect which slit the particle went through. When we do that, the interference pattern disappears [42]. An analysis of a two-atom double-slit experiment based on environment-induced measurements is reported [43].

We try to implement the double-slit experiment. There is a detector just after each slit. Thus interference figure does not appear, and we do not consider such a pattern. The possible values of the result of measurements are ± 1 (in $\hbar/2$ unit). If a particle passes one side slit, then the value of the result of measurement is $+1$. If a particle passes through another slit, then the value of the result of measurement is -1. This is an easy detector model for a Pauli observable.

In this paper, we derive new type of no-hidden-variables theorem based on the assumptions proposed by Kochen and Specker. We consider N spin-1/2 systems. The hidden results of measurement are either $+1$ or -1 (in $\hbar/2$ unit). We derive some proposition concerning a quantum expected value under an assumption about the existence of the Bloch sphere in N spin-1/2 systems. However, the hidden variables theory violates the proposition with a magnitude that grows exponentially with the number of particles. Therefore, we have to give up either the existence of the Bloch sphere or the hidden variables theory. Also we discuss two-dimensional no-hidden-variables theorem of the KS type, by using the double-slit experiment. Especially, we systematically describe our assertion based on more mathematical analysis using raw data in a thoughtful experiment.

Throughout this paper, we confine ourselves to the two-level (e.g., electron spin, photon polarizations, and so on) and the discrete eigenvalue case.

2 Notations and preparation to get new type of no-hidden-variables theorem of the KS type

We consider a two-dimensional space H. Let $\mathbf{N}$ denote a set of the numbers

$$\{1, 2, \ldots, +\infty\} \tag{2.1}$$

that contains the countably infinite. Let S be $\{\pm 1\}$. We assume that every result of measurements lies in S. We assume that every time t lies in $\mathbf{N}$. Let $\mathbf{N}_1$ denote a set of the numbers

$$\{1, 5, 9, \ldots, +\infty\} \tag{2.2}$$

that contains the countably infinite. Here we introduce $t_1 \in \mathbf{N}_1$. Let $\mathbf{N}_2$ denote a set of the numbers

$$\{2, 6, 10, \ldots, +\infty\} \tag{2.3}$$

that contains the countably infinite. Here we introduce $t_2 \in \mathbf{N}_2$. Let $\mathbf{N}_3$ denote a set of the numbers

$$\{3, 7, 11, \ldots, +\infty\} \tag{2.4}$$

that contains the countably infinite. Here we introduce $t_3 \in \mathbf{N}_3$. Let $\mathbf{N}_4$ denote a set of the numbers

$$\{4, 8, 12, \ldots, +\infty\} \tag{2.5}$$

that contains the countably infinite. Here we introduce $t_4 \in \mathbf{N}_4$. Let $\vec{\sigma}$ be

$$(\sigma_x, \sigma_y, \sigma_z), \tag{2.6}$$

the vector of Pauli operators. The measurements (observables) of $\vec{n} \cdot \vec{\sigma}$ are parameterized by a unit vector $\vec{n}$ (its direction along which the spin component is measured). Here, $\cdot$ is the scalar product in $\mathbf{R}^3$. One measures an observable $\vec{n} \cdot \vec{\sigma}$. We define a notation $\theta(t)$ which represents predetermined result of measurements at time t. We assume that measurement of an observable $\vec{n} \cdot \vec{\sigma}$ at time t for a physical system in a state ψ yields a value $\theta(\psi, \vec{n} \cdot \vec{\sigma}, t) \in S$.

We consider the following:

Assumption: M (*predetermined measurement outcome*),

$$\theta(\psi, \vec{n} \cdot \vec{\sigma}, t) \in S. \tag{2.7}$$

Assumption: E (*quantum expected value*),

$$\mathrm{Tr}[\psi \vec{n} \cdot \vec{\sigma}] = \lim_{m \to \infty} \frac{\sum_{t=1}^{m} \theta(\psi, \vec{n} \cdot \vec{\sigma}, t)}{m}. \tag{2.8}$$

Assumption: T
If

$$\mathrm{Tr}[\psi \vec{n} \cdot \vec{\sigma}] = \lim_{m \to \infty} \frac{\sum_{t=1}^{m} \theta(\psi, \vec{n} \cdot \vec{\sigma}, t)}{m}, \tag{2.9}$$

then

$$\mathrm{Tr}[\psi \vec{n} \cdot \vec{\sigma}] = \lim_{m_1 \to \infty} \frac{\sum_{t_1=1}^{m_1} \theta(\psi, \vec{n} \cdot \vec{\sigma}, t_1)}{m_1} = \lim_{m_2 \to \infty} \frac{\sum_{t_2=2}^{m_2} \theta(\psi, \vec{n} \cdot \vec{\sigma}, t_2)}{m_2} \tag{2.10}$$

and

$$\mathrm{Tr}[\psi \vec{n} \cdot \vec{\sigma}] = \lim_{m_3 \to \infty} \frac{\sum_{t_3=3}^{m_3} \theta(\psi, \vec{n} \cdot \vec{\sigma}, t_3)}{m_3} = \lim_{m_4 \to \infty} \frac{\sum_{t_4=4}^{m_4} \theta(\psi, \vec{n} \cdot \vec{\sigma}, t_4)}{m_4}. \tag{2.11}$$

3 New type of no-hidden-variables theorem of the KS type

In this section, we give new type of no-hidden-variables theorem of the KS type.

3.1 The existence of the Bloch sphere

We assume a pure spin-1/2 state ψ lying in the x-y plane. Let $\vec{\sigma}$ be $(\sigma_x, \sigma_y, \sigma_z)$, the vector of Pauli operators. The measurements (observables) on a spin-1/2 state lying in the x-y plane of $\vec{n} \cdot \vec{\sigma}$ are parameterized by a unit vector $\vec{n}$ (its direction along which the spin component is measured). Here, $\cdot$ is the scalar product in $\mathbf{R}^3$.

We have a quantum expected value $E_{\mathrm{QM}}^k,\ k=1,2$ as

$$E_{\mathrm{QM}}^k \equiv \mathrm{Tr}[\psi \vec{n}_k \cdot \vec{\sigma}],\ k=1,2. \tag{3.1}$$

We have $\vec{x} \equiv \vec{x}^{(1)}$, $\vec{y} \equiv \vec{x}^{(2)}$, and $\vec{z} \equiv \vec{x}^{(3)}$. They are the Cartesian axes relative to which spherical angles are measured. We write two unit vectors in the plane defined by $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ in the following way:

$$\vec{n}_k = \cos\theta_k \vec{x}^{(1)} + \sin\theta_k \vec{x}^{(2)}. \tag{3.2}$$

Here, the angle θ_k takes only two values:

$$\theta_1 = 0,\ \theta_2 = \frac{\pi}{2}. \tag{3.3}$$

We derive a necessary condition for the quantum expected value for the system in a pure spin-1/2 state lying in the x-y plane given in (3.1). We derive the possible values of the scalar product

$$\sum_{k=1}^{2} \left(E_{\mathrm{QM}}^k \times E_{\mathrm{QM}}^k\right) \equiv \|E_{\mathrm{QM}}\|^2. \tag{3.4}$$

E_{QM}^k is the quantum expected value given in (3.1). We see that

$$\|E_{\mathrm{QM}}\|^2 = \langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2. \tag{3.5}$$

We use the decomposition (3.2). We introduce simplified notations as

$$T_i = \mathrm{Tr}[\psi \vec{x}^{(i)} \cdot \vec{\sigma}] \tag{3.6}$$

and

$$(c_k^1, c_k^2,) = (\cos\theta_k, \sin\theta_k). \tag{3.7}$$

Then, we have

$$\|E_{\mathrm{QM}}\|^2 = \sum_{k=1}^{2} \left(\sum_{i=1}^{2} T_i c_k^i\right)^2 = \sum_{i=1}^{2} T_i^2 \le 1, \tag{3.8}$$

where we use the orthogonality relation

$$\sum_{k=1}^{2} c_k^\alpha c_k^\beta = \delta_{\alpha,\beta}. \tag{3.9}$$

From a proposition of the quantum theory, the Bloch sphere with the value of

$$\sum_{i=1}^{2} T_i^2 \tag{3.10}$$

is bounded as

$$\sum_{i=1}^{2} T_i^2 \le 1. \tag{3.11}$$

The reason of the condition (3.8) is the Bloch sphere

$$\sum_{i=1}^{3}(\mathrm{Tr}[\psi\vec{x}^{(i)}\cdot\vec{\sigma}])^2 \leq 1. \tag{3.12}$$

Thus we derive a proposition concerning a quantum expected value under an assumption of the existence of the Bloch sphere (in a spin-1/2 system). The proposition is

$$\|E_{\mathrm{QM}}\|^2 \leq 1. \tag{3.13}$$

This inequality is saturated iff ψ is a pure state lying in the x-y plane. That is,

$$\sum_{i=1}^{2}(\mathrm{Tr}[\psi\vec{x}^{(i)}\cdot\vec{\sigma}])^2 = 1. \tag{3.14}$$

Hence, we derive the following proposition concerning the existence of the Bloch sphere when the system is in a pure state lying in the x-y plane

$$\|E_{\mathrm{QM}}\|^2_{\max} = 1. \tag{3.15}$$

3.2 The existence of hidden measurement outcome which is ± 1

We assign the truth value "1" for Assumption M, Assumption E, and Assumption T. Let A_k be $\vec{n}_k\cdot\vec{\sigma}$. We assume four gedanken experiments in the same state ψ. The value of $\theta(\psi, A_1, t_1)$ is independent of $\theta(\psi, A_1, t_2)$. We note that the measurement time is different from each other. Here, we assume $t_1 \in \mathbf{N}_1$ and $t_2 \in \mathbf{N}_2$. The value of $\theta(\psi, A_2, t_3)$ is independent of $\theta(\psi, A_2, t_4)$. We note that the measurement time is different from each other. Here, we assume $t_3 \in \mathbf{N}_3$ and $t_4 \in \mathbf{N}_4$. The values of $\theta(\psi, A_1, t_1)$, $\theta(\psi, A_1, t_2)$, $\theta(\psi, A_2, t_3)$, and $\theta(\psi, A_2, t_4)$ are independent of each other. We note that the measurement time is different from each other. We assume that the number of each of quantum measurements is the countably infinite. We know that a sum of 'four' countably infinite is the countably infinite. We do not have to assign definite values to non-commuting observables in the same time.

From Assumption E and Assumption T, the quantum expected value in (3.1) ($k = 1$), which is the average of the results of measurements, is given by

$$E^1_{\mathrm{QM}} = \lim_{m_1\to\infty}\frac{\sum_{t_1=1}^{m_1}\theta(\psi, A_1, t_1)}{m_1}. \tag{3.16}$$

From Assumption M, the possible values of the measured result $\theta(\psi, A_1, t_1)$ are ± 1.

From Assumption T, the same quantum expected value is given by

$$E^1_{\mathrm{QM}} = \lim_{m_2\to\infty}\frac{\sum_{t_2=2}^{m_2}\theta(\psi, A_1, t_2)}{m_2}. \tag{3.17}$$

From Assumption M, the possible values of the measured result $\theta(\psi, A_1, t_2)$ are ± 1. From Assumption T, we see

$$\begin{aligned}&\|\{t_1|t_1\in\mathbf{N}_1\wedge\theta(\psi, A_1, t_1) = 1\}\| = \|\{t_2|t_2\in\mathbf{N}_2\wedge\theta(\psi, A_1, t_2) = 1\}\|,\\ &\|\{t_1|t_1\in\mathbf{N}_1\wedge\theta(\psi, A_1, t_1) = -1\}\| = \|\{t_2|t_2\in\mathbf{N}_2\wedge\theta(\psi, A_1, t_2) = -1\}\|.\end{aligned} \tag{3.18}$$

From Assumption E and Assumption T, the quantum expected value in (3.1) ($k = 2$), which is the average of the results of measurements, is given by

$$E_{\rm QM}^2 = \lim_{m_3\to\infty} \frac{\sum_{t_3=3}^{m_3} \theta(\psi, A_2, t_3)}{m_3}. \tag{3.19}$$

From Assumption M, the possible values of the measured result $\theta(\psi, A_2, t_3)$ are ± 1.

From Assumption T, the same quantum expected value is given by

$$E_{\rm QM}^2 = \lim_{m_4\to\infty} \frac{\sum_{t_4=4}^{m_4} \theta(\psi, A_2, t_4)}{m_4}. \tag{3.20}$$

From Assumption M, the possible values of the measured result $\theta(\psi, A_2, t_4)$ are ± 1. From Assumption T, we see

$$\begin{aligned}
&\|\{t_3 | t_3 \in \mathbf{N}_3 \wedge \theta(\psi, A_2, t_3) = 1\}\| = \|\{t_4 | t_4 \in \mathbf{N}_4 \wedge \theta(\psi, A_2, t_4) = 1\}\|,\\
&\|\{t_3 | t_3 \in \mathbf{N}_3 \wedge \theta(\psi, A_2, t_3) = -1\}\| = \|\{t_4 | t_4 \in \mathbf{N}_4 \wedge \theta(\psi, A_2, t_4) = -1\}\|.
\end{aligned} \tag{3.21}$$

We derive a necessary condition for the two quantum expected values for the system in a pure spin-1/2 state lying in the x-y plane given in (3.16) and (3.19). We derive the possible values of the scalar product $\|E_{\rm QM}\|^2$ of the two quantum expected values, $E_{\rm QM}^k$ given in (3.16) and (3.19).

We introduce an assumption that Sum rule and Product rule commute with each other [44]. We do not pursue the details of the assumption. To pursue the details is an interesting point. It is suitable to the next step of researches. We have

$$\begin{aligned}
&\|E_{\rm QM}\|^2\\
&= \left(\lim_{m_1\to\infty} \frac{\sum_{t_1=1}^{m_1} \theta(\psi, A_1, t_1)}{m_1} \times \lim_{m_2\to\infty} \frac{\sum_{t_2=2}^{m_2} \theta(\psi, A_1, t_2)}{m_2} \right)\\
&+ \left(\lim_{m_3\to\infty} \frac{\sum_{t_3=3}^{m_3} \theta(\psi, A_2, t_3)}{m_3} \times \lim_{m_4\to\infty} \frac{\sum_{t_4=4}^{m_4} \theta(\psi, A_2, t_4)}{m_4} \right)\\
&= \left(\lim_{m_1\to\infty} \frac{\sum_{t_1=1}^{m_1}}{m_1} \cdot \lim_{m_2\to\infty} \frac{\sum_{t_2=2}^{m_2}}{m_2} \theta(\psi, A_1, t_1)\theta(\psi, A_1, t_2) \right)\\
&+ \left(\lim_{m_3\to\infty} \frac{\sum_{t_3=3}^{m_3}}{m_3} \cdot \lim_{m_4\to\infty} \frac{\sum_{t_4=4}^{m_4}}{m_4} \theta(\psi, A_2, t_3)\theta(\psi, A_2, t_4) \right)\\
&\le \left(\lim_{m_1\to\infty} \frac{\sum_{t_1=1}^{m_1}}{m_1} \cdot \lim_{m_2\to\infty} \frac{\sum_{t_2=2}^{m_2}}{m_2} |\theta(\psi, A_1, t_1)\theta(\psi, A_1, t_2)| \right)\\
&+ \left(\lim_{m_3\to\infty} \frac{\sum_{t_3=3}^{m_3}}{m_3} \cdot \lim_{m_4\to\infty} \frac{\sum_{t_4=4}^{m_4}}{m_4} |\theta(\psi, A_2, t_3)\theta(\psi, A_2, t_4)| \right)\\
&= \left(\lim_{m_1\to\infty} \frac{\sum_{t_1=1}^{m_1}}{m_1} \cdot \lim_{m_2\to\infty} \frac{\sum_{t_2=2}^{m_2}}{m_2} \right) + \left(\lim_{m_3\to\infty} \frac{\sum_{t_3=3}^{m_3}}{m_3} \cdot \lim_{m_4\to\infty} \frac{\sum_{t_4=4}^{m_4}}{m_4} \right) = 2.
\end{aligned} \tag{3.22}$$

From Assumption M, we have

$$|\theta(\psi, A_1, t_1)\theta(\psi, A_1, t_2)| = +1,\ |\theta(\psi, A_2, t_3)\theta(\psi, A_2, t_4)| = +1. \tag{3.23}$$

The above inequality (3.22) is saturated when

$$\theta(\psi, A_1, t_1)\theta(\psi, A_1, t_2) = 1,\ \theta(\psi, A_2, t_3)\theta(\psi, A_2, t_4) = 1. \tag{3.24}$$

This implies

$$\theta(\psi, A_1, t_1) = \theta(\psi, A_1, t_2),\ \theta(\psi, A_2, t_3) = \theta(\psi, A_2, t_4). \tag{3.25}$$

The above condition (3.25) can be possible since, as we have said,

$$\begin{aligned}
&\|\{t_1|t_1 \in \mathbf{N}_1 \wedge \theta(\psi, A_1, t_1) = 1\}\| = \|\{t_2|t_2 \in \mathbf{N}_2 \wedge \theta(\psi, A_1, t_2) = 1\}\|,\\
&\|\{t_1|t_1 \in \mathbf{N}_1 \wedge \theta(\psi, A_1, t_1) = -1\}\| = \|\{t_2|t_2 \in \mathbf{N}_2 \wedge \theta(\psi, A_1, t_2) = -1\}\|,
\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}
&\|\{t_3|t_3 \in \mathbf{N}_3 \wedge \theta(\psi, A_2, t_3) = 1\}\| = \|\{t_4|t_4 \in \mathbf{N}_4 \wedge \theta(\psi, A_2, t_4) = 1\}\|,\\
&\|\{t_3|t_3 \in \mathbf{N}_3 \wedge \theta(\psi, A_2, t_3) = -1\}\| = \|\{t_4|t_4 \in \mathbf{N}_4 \wedge \theta(\psi, A_2, t_4) = -1\}\|.
\end{aligned} \tag{3.27}$$

Thus we derive a proposition concerning the two quantum expected values under an assumption that we assign the truth value "1" for Assumption M, Assumption E, and Assumption T (in a spin-1/2 system). The proposition is $\|E_{\rm QM}\|^2 \leq 2$. This inequality can be saturated. Hence we derive the following proposition concerning Assumption M, Assumption E, and Assumption T:

$$\|E_{\rm QM}\|^2_{\rm max} = 2. \tag{3.28}$$

3.3 Contradiction

We cannot assign the truth value "1" for two propositions (3.15) (concerning the existence of the Bloch sphere) and (3.28) (concerning Assumption M, Assumption E, and Assumption T), simultaneously, when the system is in a pure state lying in the x-y plane. Therefore, we are in the KS contradiction. We do not assign the truth value "1" for five assumptions

1. Assumption M
2. Assumption E
3. Assumption T
4. The existence of the Bloch sphere
5. Sum rule and Product rule commute with each other,

simultaneously.

4 High dimensional no-hidden-variables theorem of KS type

In this section, we derive a proposition concerning a quantum expected value under an assumption of the existence of the Bloch sphere in N spin-1/2 systems ($1 \leq N < +\infty$). This assumption intuitively depictures our physical world. However, the hidden variables theory (the result of measurements is ± 1) violates the proposition with a magnitude that grows exponentially with the number of particles. We have to give up either the existence of the Bloch sphere or the hidden variables theory. Therefore, the hidden variables theory cannot depicture our physical world with a violation factor that grows exponentially with the number of particles.

4.1 The existence of the Bloch sphere

Assume that we have a set of N spins $\frac{1}{2}$. Each of them is a spin-1/2 pure state lying in the x-y plane. Let us assume that one source of N uncorrelated spin-carrying particles emits them in a state, which can be described as a multi spin-1/2 pure uncorrelated state. Let us parameterize the settings of the jth observer with a unit vector $\vec{n}_j$ (its direction along which the spin component is measured) with $j = 1, \ldots, N$. One can introduce the 'hidden variables' correlation function, which is the average of the product of the hidden results of measurement

$$E_{\mathrm{HV}}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \langle r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) \rangle_{\mathrm{avg}}, \tag{4.1}$$

where r is hidden result. We assume the value of r is ± 1 (in $(\hbar/2)^N$ unit), which is obtained if the measurement directions are set at $\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N$.

Also one can introduce a quantum correlation function with the system in such a pure uncorrelated state

$$E_{\mathrm{QM}}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \mathrm{tr}[\rho \vec{n}_1 \cdot \vec{\sigma} \otimes \vec{n}_2 \cdot \vec{\sigma} \otimes \cdots \otimes \vec{n}_N \cdot \vec{\sigma}] \tag{4.2}$$

where $\otimes$ denotes the tensor product, $\cdot$ the scalar product in $\mathbf{R}^2$, $\vec{\sigma} = (\sigma_x, \sigma_y)$ is a vector of two Pauli operators, and ρ is the pure uncorrelated state,

$$\rho = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_N \tag{4.3}$$

with $\rho_j = |\Psi_j\rangle\langle\Psi_j|$ and $|\Psi_j\rangle$ is a spin-1/2 pure state lying in the x-y plane.

One can write the observable (unit) vector $\vec{n}_j$ in a plane coordinate system as follows:

$$\vec{n}_j(\theta_j^{k_j}) = \cos\theta_j^{k_j} \vec{x}_j^{(1)} + \sin\theta_j^{k_j} \vec{x}_j^{(2)}, \tag{4.4}$$

where $\vec{x}_j^{(1)} = \vec{x}$ and $\vec{x}_j^{(2)} = \vec{y}$ are the Cartesian axes. Here, the angle $\theta_j^{k_j}$ takes two values (two-setting model):

$$\theta_j^1 = 0,\ \theta_j^2 = \frac{\pi}{2}. \tag{4.5}$$

We derive a necessary condition to be satisfied by the quantum correlation function with the system in a pure uncorrelated state given in (4.2). In more detail, we derive the value of the product of the quantum correlation function,

$E_{\rm QM}$ given in (4.2), i.e., $\|E_{\rm QM}\|^2$. We use the decomposition (4.4). We introduce simplified notations as

$$T_{i_1 i_2 \dots i_N} = {\rm tr}[\rho \vec{x}_1^{(i_1)} \cdot \vec{\sigma} \otimes \vec{x}_2^{(i_2)} \cdot \vec{\sigma} \otimes \cdots \otimes \vec{x}_N^{(i_N)} \cdot \vec{\sigma}] \tag{4.6}$$

and

$$\vec{c}_j = (c_j^1, c_j^2) = (\cos\theta_j^{k_j}, \sin\theta_j^{k_j}). \tag{4.7}$$

Then, we have

$$\begin{aligned} & \|E_{\rm QM}\|^2 \\ & = \sum_{k_1=1}^{2} \cdots \sum_{k_N=1}^{2} \left(\sum_{i_1,\dots,i_N=1}^{2} T_{i_1 \dots i_N} c_1^{i_1} \cdots c_N^{i_N} \right)^2 \\ & = \sum_{i_1,\dots,i_N=1}^{2} T_{i_1 \dots i_N}^2 \le 1, \end{aligned} \tag{4.8}$$

where we use the orthogonality relation $\sum_{k_j=1}^{2} c_j^{\alpha} c_j^{\beta} = \delta_{\alpha,\beta}$. The value of $\sum_{i_1,\dots,i_N=1}^{2} T_{i_1 \dots i_N}^2$ is bounded as $\sum_{i_1,\dots,i_N=1}^{2} T_{i_1 \dots i_N}^2 \le 1$. We have

$$\prod_{j=1}^{N} \sum_{i_j=1}^{2} ({\rm tr}[\rho_j \vec{x}_j^{(i_j)} \cdot \vec{\sigma}])^2 \le 1. \tag{4.9}$$

From the convex argument, all quantum separable states must satisfy the inequality (4.8). Therefore, it is a separability inequality. It is important that the separability inequality (4.8) is saturated iff ρ is a multi spin-1/2 pure uncorrelated state such that, for every j, $|\Psi_j\rangle$ is a spin-1/2 pure state lying in the x-y plane. The reason of the inequality (4.8) is due to the following quantum inequality

$$\sum_{i_j=1}^{2} ({\rm tr}[\rho_j \vec{x}_j^{(i_j)} \cdot \vec{\sigma}])^2 \le 1. \tag{4.10}$$

The inequality (4.10) is saturated iff $\rho_j = |\Psi_j\rangle\langle\Psi_j|$ and $|\Psi_j\rangle$ is a spin-1/2 pure state lying in the x-y plane. The inequality (4.8) is saturated iff the inequality (4.10) is saturated for every j. Thus we have the maximal possible value of the scalar product as a quantum proposition concerning the existence of the Bloch sphere

$$\|E_{\rm QM}\|_{\max}^2 = 1 \tag{4.11}$$

when the system is in such a multi spin-1/2 pure uncorrelated state.

4.2 The hidden variables theory

On the other hand, a correlation function satisfies the hidden variables theory if it can be written as

$$E_{\rm HV}(\vec{n}_1, \vec{n}_2, \dots, \vec{n}_N) = \lim_{m \to \infty} \frac{\sum_{l=1}^{m} r(\vec{n}_1, \vec{n}_2, \dots, \vec{n}_N, l)}{m} \tag{4.12}$$

where l denotes a label and r is the result of measurement of the dichotomic observables parameterized by the directions of $\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N$.

Assume the quantum correlation function with the system in a pure uncorrelated state given in (4.2) admits the hidden variables theory. One has the following proposition concerning the hidden variables theory

$$E_{\mathrm{QM}}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \lim_{m\to\infty} \frac{\sum_{l=1}^{m} r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l)}{m}. \tag{4.13}$$

In what follows, we show that we cannot assign the truth value "1" for the proposition (4.13) concerning the hidden variables theory.

Assume the proposition (4.13) is true. By changing the label l into l', we have the same quantum expected value as follows

$$E_{\mathrm{QM}}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \lim_{m\to\infty} \frac{\sum_{l'=1}^{m} r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l')}{m}. \tag{4.14}$$

An important note here is that the value of the right-hand-side of (4.13) is equal to the value of the right-hand-side of (4.14) because we only change the label.

We abbreviate $r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l)$ to $r(l)$ and $r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l')$ to $r(l')$.

We introduce an assumption that Sum rule and Product rule commute with each other [44]. We have

$$\begin{aligned}
&\|E_{\mathrm{QM}}\|^2 \\
&= \sum_{k_1=1}^{2} \cdots \sum_{k_N=1}^{2} \left(\lim_{m\to\infty} \frac{\sum_{l=1}^{m} r(l)}{m} \times \lim_{m\to\infty} \frac{\sum_{l'=1}^{m} r(l')}{m} \right) \\
&= \sum_{k_1=1}^{2} \cdots \sum_{k_N=1}^{2} \left(\lim_{m\to\infty} \frac{\sum_{l=1}^{m}}{m} \cdot \lim_{m\to\infty} \frac{\sum_{l'=1}^{m}}{m} r(l) r(l') \right) \\
&\leq \sum_{k_1=1}^{2} \cdots \sum_{k_N=1}^{2} \left(\lim_{m\to\infty} \frac{\sum_{l=1}^{m}}{m} \cdot \lim_{m\to\infty} \frac{\sum_{l'=1}^{m}}{m} |r(l) r(l')| \right) \\
&= \sum_{k_1=1}^{2} \cdots \sum_{k_N=1}^{2} \left(\lim_{m\to\infty} \frac{\sum_{l=1}^{m}}{m} \cdot \lim_{m\to\infty} \frac{\sum_{l'=1}^{m}}{m} \right) = 2^N.
\end{aligned} \tag{4.15}$$

We use the following fact

$$|r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l) r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l')| = +1. \tag{4.16}$$

The inequality (4.15) is saturated since we have

$$\begin{aligned}
&\|\{l | r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l) = 1 \wedge l \in \mathbf{N}\}\| \\
&= \|\{l' | r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l') = 1 \wedge l' \in \mathbf{N}\}\|, \\
&\|\{l | r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l) = -1 \wedge l \in \mathbf{N}\}\| \\
&= \|\{l' | r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l') = -1 \wedge l' \in \mathbf{N}\}\|.
\end{aligned} \tag{4.17}$$

Hence one has the following proposition concerning the hidden variables theory.

$$\|E_{\mathrm{QM}}\|^2_{\max} = 2^N. \tag{4.18}$$

4.3 Contradiction

Clearly, we cannot assign the truth value "1" for two propositions (4.11) (concerning the existence of the Bloch sphere) and (4.18) (concerning the hidden variables theory), simultaneously, when the system is in a multiparticle pure uncorrelated state. Of course, each of them is a spin-1/2 pure state lying in the x-y plane. Therefore, we are in the KS contradiction when the system is in such a multiparticle pure uncorrelated state. Thus, we cannot accept the validity of the proposition (4.13) (concerning the hidden variables theory) if we assign the truth value "1" for the proposition (4.11) (concerning the existence of the Bloch sphere). In other words, the hidden variables theory does not reveal our physical world.

5 Two-dimensional no-hidden-variables theorem of the KS type

In this section, we consider the relation between the double-slit experiment and the hidden variables theory. We try to implement the double-slit experiment. There is a detector just after each slit. Thus interference figure does not appear, and we do not consider such a pattern. The possible values of the result of measurements are ± 1 (in $\hbar/2$ unit). If a particle passes one side slit, then the value of the result of measurement is $+1$. If a particle passes through another slit, then the value of the result of measurement is -1.

5.1 A wave function analysis

Let (σ_z, σ_x) be Pauli vector. We assume that a source of spin-carrying particles emits them in a state $|\psi\rangle$, which can be described as an eigenvector of a Pauli observable σ_z. We consider a quantum expected value $\langle \sigma_x \rangle$ as

$$\langle \sigma_x \rangle = \langle \psi | \sigma_x | \psi \rangle = 0. \tag{5.1}$$

The above quantum expected value is zero if we consider only a wave function analysis.

We derive a necessary condition for the quantum expected value for the system in the pure spin-1/2 state $|\psi\rangle$ given in (5.1). We derive the possible value of the product $\langle \sigma_x \rangle \times \langle \sigma_x \rangle = \langle \sigma_x \rangle^2$. $\langle \sigma_x \rangle$ is the quantum expected value given in (5.1). We derive the following proposition

$$\langle \sigma_x \rangle^2 = 0. \tag{5.2}$$

Hence we have

$$\langle \sigma_x \rangle^2 \leq 0. \tag{5.3}$$

Thus,

$$(\langle \sigma_x \rangle^2)_{\max} = 0. \tag{5.4}$$

5.2 The hidden variables theory

On the other hand, a mean value E admits the hidden variables theory if it can be written as

$$E = \frac{\sum_{l=1}^{m} r_l(\sigma_x)}{m} \tag{5.5}$$

where l denotes a label and r is the result of measurement of the Pauli observable σ_x. We assume the value of r is ± 1 (in $\hbar/2$ unit).

Assume the quantum mean value with the system in an eigenvector ($|\psi\rangle$) of the Pauli observable σ_z given in (5.1) admits the hidden variables theory. One has the following proposition concerning the hidden variables theory

$$\langle \sigma_x \rangle(m) = \frac{\sum_{l=1}^{m} r_l(\sigma_x)}{m}. \tag{5.6}$$

We can assume as follows by Strong Law of Large Numbers [45],

$$\langle \sigma_x \rangle(+\infty) = \langle \sigma_x \rangle = \langle \psi | \sigma_x | \psi \rangle. \tag{5.7}$$

In what follows, we show that we cannot assign the truth value "1" for the proposition (5.6) concerning the hidden variables theory.

Assume the proposition (5.6) is true. By changing the label l into l', we have the same quantum mean value as follows

$$\langle \sigma_x \rangle(m) = \frac{\sum_{l'=1}^{m} r_{l'}(\sigma_x)}{m}. \tag{5.8}$$

An important note here is that the value of the right-hand-side of (5.6) is equal to the value of the right-hand-side of (5.8) because we only change the label.

We introduce an assumption that Sum rule and Product rule commute with each other [44]. We have

$$\begin{aligned}
&\langle \sigma_x \rangle(m) \times \langle \sigma_x \rangle(m) \\
&= \frac{\sum_{l=1}^{m} r_l(\sigma_x)}{m} \times \frac{\sum_{l'=1}^{m} r_{l'}(\sigma_x)}{m} \\
&= \frac{\sum_{l=1}^{m}}{m} \cdot \frac{\sum_{l'=1}^{m}}{m} r_l(\sigma_x) r_{l'}(\sigma_x) \\
&\leq \frac{\sum_{l=1}^{m}}{m} \cdot \frac{\sum_{l'=1}^{m}}{m} |r_l(\sigma_x) r_{l'}(\sigma_x)| \\
&= \frac{\sum_{l=1}^{m}}{m} \cdot \frac{\sum_{l'=1}^{m}}{m} = 1.
\end{aligned} \tag{5.9}$$

We use the following fact

$$|r_l(\sigma_x) r_{l'}(\sigma_x)| = 1. \tag{5.10}$$

The inequality (5.9) is saturated since we have

$$\begin{aligned}
&\|\{l | r_l(\sigma_x) = 1 \wedge l \in \mathbf{N}\}\| = \|\{l' | r_{l'}(\sigma_x) = 1 \wedge l' \in \mathbf{N}\}\|, \\
&\|\{l | r_l(\sigma_x) = -1 \wedge l \in \mathbf{N}\}\| = \|\{l' | r_{l'}(\sigma_x) = -1 \wedge l' \in \mathbf{N}\}\|.
\end{aligned} \tag{5.11}$$

Thus we derive a proposition concerning the quantum mean value under an assumption that the hidden variables theory is true (in a spin-1/2 system), that is

$$(\langle\sigma_x\rangle(m) \times \langle\sigma_x\rangle(m))_{\max} = 1. \tag{5.12}$$

From Strong Law of Large Numbers, we have

$$(\langle\sigma_x\rangle \times \langle\sigma_x\rangle)_{\max} = 1. \tag{5.13}$$

Hence we derive the following proposition concerning the hidden variables theory

$$(\langle\sigma_x\rangle^2)_{\max} = 1. \tag{5.14}$$

5.3 Contradiction

We do not assign the truth value "1" for two propositions (5.4) (concerning a wave function analysis) and (5.14) (concerning the hidden variables theory), simultaneously. We are in the KS contradiction.

We cannot accept the validity of the proposition (5.6) (concerning the hidden variables theory) if we assign the truth value "1" for the proposition (5.4) (concerning a wave function analysis). In other words, the hidden variables theory does not meet the detector model for the spin observable σ_x.

6 Conclusions

In conclusion, we have derived new type of no-hidden-variables theorem based on the assumptions proposed by Kochen and Specker. We have considered N spin-1/2 systems. The hidden results of measurement have been either $+1$ or -1 (in $\hbar/2$ unit). We have derived some proposition concerning a quantum expected value under an assumption about the existence of the Bloch sphere in N spin-1/2 systems. However, the hidden variables theory has violated the proposition with a magnitude that grows exponentially with the number of particles. Therefore, we have to have given up either the existence of the Bloch sphere or the hidden variables theory. Also we have discussed two-dimensional no-hidden-variables theorem of the KS type. Especially, we have systematically described our assertion based on more mathematical analysis using raw data in a thoughtful experiment.

Acknowledgments

We would like to thank Professor Niizeki and Dr. Ren for valuable comments.

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